On alignment and collision avoidance models: from microscopic to mean field models

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Outline of the Talk

1. Self-alignment models (collab. C.-W. Shu, Brown Univ. Providence)
   - Microscopic models
   - Kinetic models
   - High order numerical methods
   - Numerical simulations for alignment

2. Collision avoidance (collab. C. Parzani, ENAC Toulouse)
   - Motivation
   - Microscopic models
   - Numerical experiments of the microscopic model
   - Mean field limit

Agent-based model of self-alignment with attraction-repulsion

The starting point of this study is an Individual-Based Model [1]

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ d\mathbf{v}_i = \mathbf{P}_{\mathbf{v}_i^\perp} \left(\overline{\mathbf{v}}_i \, dt + \sqrt{2d} \; d\mathbf{B}_t^i \right), \end{cases} \quad (1)$$

where $\mathbf{B}_t^i$ is a Brownian motion and d represents the noise intensity whereas $\mathbf{P}_{\mathbf{v}_i^\perp}$ is the projection matrix onto the normal plane to $\mathbf{v}_i$:

$$\mathbf{P}_{\mathbf{v}^\perp} = \mathrm{Id} - \mathbf{v} \otimes \mathbf{v}, \quad \overline{\mathbf{v}}_i = \frac{1}{|\mathbf{J}_i + \mathbf{R}_i|} (\mathbf{J}_i + \mathbf{R}_i),$$

with

$$\mathbf{J}_i = \sum_{j=1}^{N} k(|\mathbf{x}_j - \mathbf{x}_i|) \, \mathbf{v}_j, \qquad \mathbf{R}_i = -\sum_{j=1}^{N} \nabla_{\mathbf{x}_i} \phi(|\mathbf{x}_j - \mathbf{x}_i|), \quad (2)$$

[1] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, *PRL* (1995), I. Aoki, *BJSSF* (1982), I.D. Couzin, J. Krause, N.R. Franks and S.A. Levin, *Nature*, (2005)

Kinetic model of self-alignment

When the number of particles becomes large, that is $N \to \infty$, the unknown $f(t, \mathbf{x}, \mathbf{v})$, depending on the time t, the position $\mathbf{x}$, and the velocity $\mathbf{v}$, represents the distribution of particles in phase space for each species with $(\mathbf{x}, \mathbf{v}) \in \Omega \times \mathbb{S}^{d-1}$, $d = 1, .., 3$, where $\Omega \subset \mathbb{R}^d$.
Its behaviour is given by the Vlasov equation [2],

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = -\nabla_{\mathbf{v}} \cdot [\mathbf{P}_{\mathbf{v}^{\perp}} \mathbf{v}_f f] + \alpha \Delta_{\mathbf{v}} f, \tag{3}$$

where $\alpha > 0$ and

$$\begin{cases} \mathbf{v}_f = \dfrac{1}{|\mathbf{J}_f + \mathbf{R}_f|} (\mathbf{J}_f + \mathbf{R}_f), \\ \mathbf{J}_f = \displaystyle\int_{\Omega \times \mathbb{S}^{d-1}} k(|\mathbf{x}' - \mathbf{x}|) \mathbf{v}' f(t, \mathbf{x}', \mathbf{v}') \, d\mathbf{x}' d\mathbf{v}', \\ \mathbf{R}_f = -\nabla_{\mathbf{x}} \displaystyle\int_{\Omega \times \mathbb{S}^{d-1}} \phi(|\mathbf{x}' - \mathbf{x}|) f(t, \mathbf{x}', \mathbf{v}') \, d\mathbf{x}' d\mathbf{v}'. \end{cases} \tag{4}$$

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[2] F. Bolley, J. A. Cañizo and J. A. Carrillo *M3AS* 2011, Degond and Motsch *M3AS* 2008, P. Degond, J.G. Liu, S. Motsch and V. Panferov *MAA* 2013.

Discontinuous Galerkin method

We look for $(f_h, \mathbf{q}_h) \in \mathcal{G}_h^k \times \mathcal{U}_h^k$, $\mathbf{v}_h \in \mathcal{U}_h^r$, such that for all $g \in \mathcal{G}_h^k$,

$$\int_K \frac{\partial f_h}{\partial t} g \, d\mathbf{x} d\mathbf{v} - \int_K f_h \mathbf{v} \cdot \nabla_{\mathbf{x}} g \, d\mathbf{x} d\mathbf{v} - \int_K (\mathbf{P}_{\mathbf{v}^\perp} \mathbf{v}_h \, f_h - \alpha \, \mathbf{q}_h) \cdot \nabla_{\mathbf{v}} g \, d\mathbf{x} d\mathbf{v}$$

$$+ \int_{K_{\mathbf{v}}} \int_{\partial K_{\mathbf{x}}} \widehat{f_h \mathbf{v}} \cdot \mathbf{n}_x \, g^- \, d\sigma_{\mathbf{x}} \, d\mathbf{v} + \int_{K_{\mathbf{x}}} \int_{\partial K_{\mathbf{v}}} \left(\widehat{f_h \mathbf{P}_{\mathbf{v}^\perp} \mathbf{v}_h} - \alpha \, \widehat{\mathbf{q}_h} \right) \cdot \mathbf{n}_{\mathbf{v}} \, g^- \, d\sigma_{\mathbf{v}} d\mathbf{x} = 0,$$

and for all $\mathbf{u} \in \mathcal{U}_h^k$,

$$\int_K \mathbf{q}_h \cdot \mathbf{u} \, d\mathbf{x} \, d\mathbf{v} + \int_K f_h \nabla_{\mathbf{v}} \cdot \mathbf{u} \, d\mathbf{x} d\mathbf{v} - \int_K \int_{\partial K_{\mathbf{v}}} \widehat{f_h} \, \mathbf{n}_{\mathbf{v}} \cdot \mathbf{u} \, d\mathbf{x} \, d\sigma_{\mathbf{v}} = 0,$$

where $\mathbf{n}_{\mathbf{x}}$ and $\mathbf{n}_{\mathbf{v}}$ are outward unit normals of $\partial K_{\mathbf{x}}$ and $\partial K_{\mathbf{v}}$, respectively. Furthermore, the velocity $\mathbf{v}_h \in L^\infty(\Omega)$ with $\|\mathbf{v}_h\| = 1$, and

$$\mathbf{v}_h(t, \mathbf{x}) = \frac{1}{\|\mathbf{J}_h(t, \mathbf{x}) + \mathbf{R}_h(t, \mathbf{x})\|} \left(\mathbf{J}_h(t, \mathbf{x}) + \mathbf{R}_h(t, \mathbf{x}) \right),$$

Discontinuous Galerkin method

Lemma (Mass conservation)

The numerical solution $(f_h, \mathbf{q}_h) \in \mathcal{G}_h^k \times \mathcal{U}_h^k$ *with* $k \geq 0$ *given by the DGM satisfies*

$$\frac{d}{dt} \int_{\Omega \times \mathbb{S}^{d-1}} f_h d\mathbf{x} d\mathbf{v} = 0, \tag{5}$$

Equivalently, for $\rho_h(\mathbf{x}, t)$, *for any* $t > 0$, *the following holds:*

$$\int_\Omega \rho_h(t, \mathbf{x}) d\mathbf{x} = \int_\Omega \rho_h(0, \mathbf{x}) d\mathbf{x}.$$

Lemma (L^2-stability of f_h)

Assume that the initial data $f_h(0)$ *is uniformly bounded in* $L^2(\Omega \times \mathbb{S}^{d-1})$. *Then for* $k \geq 0$, *the numerical solution* $(f_h, \mathbf{q}_h) \in \mathcal{G}_h^k \times \mathcal{U}_h^k$ *given by the DGM satisfies for any* $t \geq 0$

$$\|f_h(t)\|_{L^2}^2 + 2\alpha \int_0^t \|\mathbf{q}_h(s)\|_{L^2}^2 ds \leq \|f_h(0)\|_{L^2}^2 \left(1 + e^t\right).$$

Discontinuous Galerkin method

Consider that k and ϕ are nonnegative functions which satisfy

$$k, \phi \in \mathcal{C}_c^p([0,\infty)), \quad \text{with } p \geq 2.$$

and for periodic boundary conditions in space, we have

$$\mathbf{J}_f(t,\mathbf{x}) = \int_{\mathrm{supp}(k)} k(|\mathbf{y}|)\,\rho\,\mathbf{u}(t,\mathbf{x}+\mathbf{y})\,d\mathbf{y}, \quad \mathbf{R}_f(t,\mathbf{x}) = \int_{\mathrm{supp}(\phi)} \nabla_{\mathbf{y}}\phi(|\mathbf{y}|)\,\rho(t,\mathbf{x}+\mathbf{y})\,d\mathbf{y}.$$

Assume that the solution $f \in H^{k+1}$ with $\mathbf{J}_f$ and $\mathbf{R}_f$ such that for any $T > 0$, there exists a constant $\xi_T > 0$ such that for all $(t,\mathbf{x}) \in [0,T] \times \Omega$

$$|\mathbf{J}_f(t,\mathbf{x}) + \mathbf{R}_f(t,\mathbf{x})| \geq \xi_T.$$

We denote $\mathbf{q} = \nabla_{\mathbf{v}} f$ and the error functions by

$$\varepsilon_1 = f - f_h, \quad \varepsilon_2 = \mathbf{q} - \mathbf{q}_h$$

Using the standard interpolation theory [3], we obtain

$$\frac{d}{dt}\|\varepsilon_{1,h}\|_{L^2}^2 + \alpha\|\varepsilon_{2,h}\|_{L^2}^2 \leq C\,\|\varepsilon_{1,h}\|_{L^2}^2 + Ch^{2k+1},$$

[3] P.-G. Ciarlet, The finite element methods for elliptic problems, North-Holland, Amsterdam (1975).

Discontinuous Galerkin method

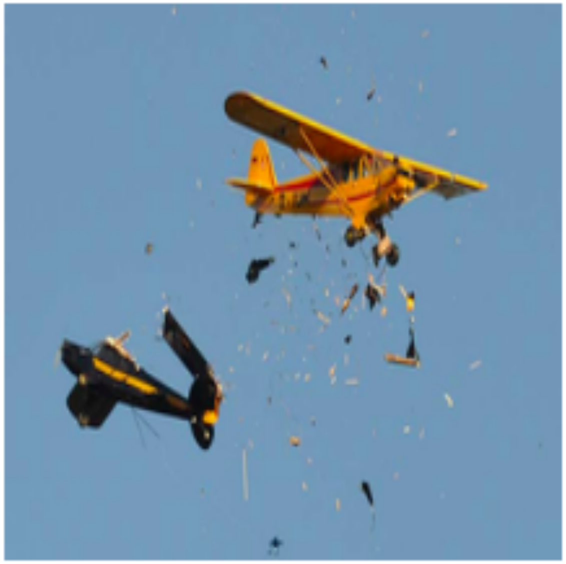
We consider $\Omega = (-1,1)^2$ and $\phi \equiv 0$ (without repulsion) and the initial data as

$$f_0(x,y,\theta) = \frac{1}{2}\left(1 + \frac{1}{2}\cos(\theta)\,\sin(\theta)\right)\,\left(1 + \frac{3}{5}\sin(k_x\,x)\cos(k_y\,y)\right).$$

Discontinuous Galerkin method

Motivation of this work

About Unmanned Aerial Vehicles (UAV) :



- Development of multiple autonomous UAV for missions like search that are more efficiently done by a group rather than a single UAV alone.
- The use of sophisticated decentralized and cooperative control algorithms.
- Coordinating hundreds or thousands of UAVs present a variety of new exciting challenges.

About swarming of birds or bats

Try to understand collective behavior from the mechanical properties of the individual

- vision, sensors
- ability to brake,
- ability to change its direction, etc

Collision avoidance models

- Collision avoidance in robotics with obstacles : require the knowledge of the path of the obstacle, possibility to stop or to change suddently of directions. Investigate all possible trajectories [4]...
- Collision avoidance based on collision cone approach. The algorithm is not decentralized as a UAV implementing this algorithm requires information of all other UAVs[5].
- Collision avoidance in traffic management : based on flight plan sharing between aircraft. It is all right for low density traffic.
- Collision avoidance using artificial potential based methods: individuals are treated as charged particles. The artificial potential methods are susceptible to local minima and require breaking forces, and therefore is not widely in UAV collision avoidance.

Aim :

Our goal is to develop a dynamical approach in 3D based on particle interactions and perform a mean field limit to replace self-interactions between particles by self-consistent fields (easier to compute).

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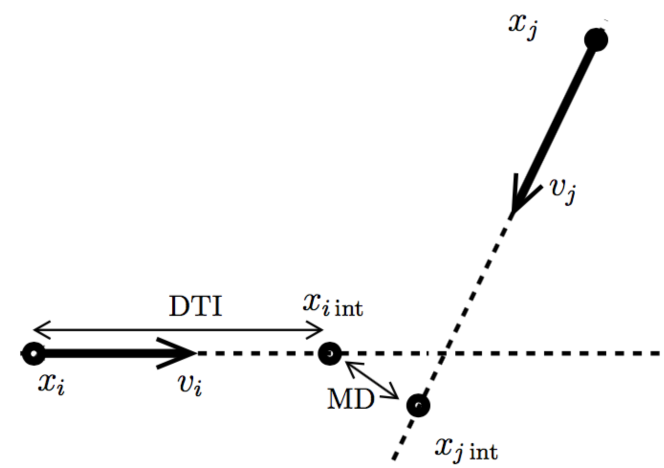
[4] Gomez and Fraichard (2009)

[5] Chakravarthy and Ghose (1998)

Agent-based model for collision avoidance

From the works on pedestrians[6], individuals follow a rule composed of two phases: a perception phase and a decision-making phase

- Perception phase : the key observables are the distance-to-interaction (DTI), the time-to-interaction (TTI) and the minimal distance (MD)
- Decision-making phase : it consists in changing the current cruising direction $\mathbf{v}$ to a new cruising direction $\mathbf{v}'$



The Minimal Distance is this minimal distance between the subject and his collision partner.

[6] P. Degond, C. Appert-Rolland, M. Moussaid, J. Pettre, G. Theraulaz *KRM* (2014); M. Moussaid, D. Helbing and G. Theraulaz, *Proc. Nat. Acad. Sci.* (2011)

Perception phase

We set[7] $D_{i,j}(t)$ the distance beween two particles at time $t \geq 0$,

$$D^2_{i,j}(t) \;=\; |(\mathbf{x}_j + \mathbf{v}_j\, t) \;-\; (\mathbf{x}_i + \mathbf{v}_i\, t)|^2$$

From this, we deduce for the particle i, the time to interaction $\tau_{\rm int}$, the distance to interaction $d_{\rm int}$ and the minimal distance d_{ij} (mininum of $D^(t)$)

$$\begin{cases} \tau_{\rm int} = -\dfrac{(\mathbf{x}_i - \mathbf{x}_j)\cdot \mathbf{v}_i - \mathbf{v}_j)}{|\mathbf{v}_i - \mathbf{v}_j|^2}. \\[2ex] d_{{\rm int},i} = -\dfrac{(\mathbf{x}_i - \mathbf{x}_j)\cdot \mathbf{v}_i - \mathbf{v}_j)}{|\mathbf{v}_i - \mathbf{v}_j|^2}|v_i|. \\[2ex] d_{ij} \;=\; \left(|\mathbf{x}_i - \mathbf{x}_j|^2 - \left(\dfrac{(\mathbf{x}_i - \mathbf{x}_j)\cdot \mathbf{v}_i - \mathbf{v}_j)}{|\mathbf{v}_i - \mathbf{v}_j|} \right)^2 \right)^{1/2}. \end{cases}$$

Collision avoidance will occur when $d_{ij} \leq R$ and $\tau_{\rm int} > 0$. Furthermore, we can add some restrictions according to the perception sensitivity of the individual (vision, sensors, etc) by defining an interaction region.

[7] P. Degond, C. Appert-Rolland, M. Moussaid, J. Pettre, G. Theraulaz *KRM* (2014); M. Moussaid, D. Helbing and G. Theraulaz, *Proc. Nat. Acad. Sci.* (2011)

Decision making phase

Remark

The situation here is quite different from the 2D case (collision avoidance for pedestrians or robots) : particles cannot suddently stop or brake! Here we will only consider rotations to avoid collision.

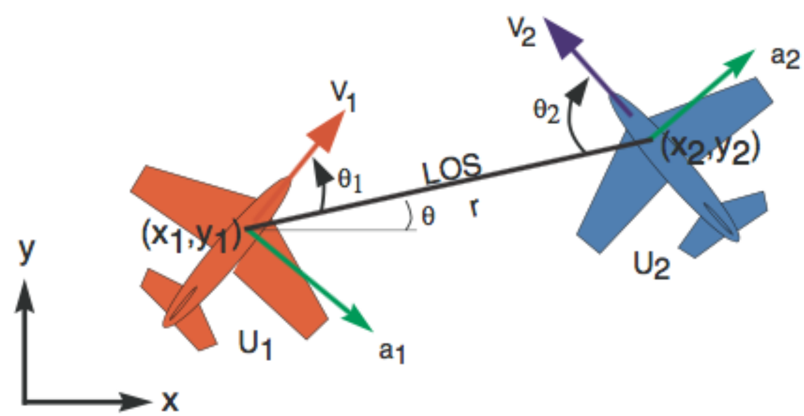
Consider a test particle i interacting with another particle j such that they are in an the same interaction region and $d_{i,j} \leq R$, $\tau_i \geq 0$, then

- The particle i will rotate along the axis $(\mathbf{x}_i - \mathbf{x}_j) \wedge (\mathbf{v}_i - \mathbf{v}_j)$.
- The frequency of rotation is proportional to

$$\frac{M}{|\mathbf{v}_i - \mathbf{v}_j| \, |\mathbf{x}_i - \mathbf{x}_j|^\gamma},$$

where $M > 0$ and $\gamma \geq 1$.

- A friction term may also act for instance when $d_{i,j} \ll 1$.



Collision avoidance in the plane $(0, (\mathbf{x}_i - \mathbf{x}_j), (\mathbf{v}_i - \mathbf{v}_j))$

Agent-based model for collision avoidance

Finally from these requirements we get the following model

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\[2ex] \dfrac{d\mathbf{v}_i}{dt} = \mathbf{v}_i \wedge \mathbf{R}_i + \mathbf{F}_{\text{ext}}(\mathbf{x}_i, \mathbf{v}_i) - \Sigma\, \mathbf{v}_i, \end{cases} \tag{6}$$

where the operator $\mathbf{R}_i$ describes the inteactions between particles

$$\mathbf{R}_i = \frac{1}{\#S_i(t)} \sum_{j \in S_i} M \left[\delta \mathbf{e}_z + \frac{1}{|\mathbf{v}_i - \mathbf{v}_j|\, |\mathbf{x}_i - \mathbf{x}_j|^\gamma} (\mathbf{x}_i - \mathbf{x}_j) \wedge (\mathbf{v}_i - \mathbf{v}_j) \right].$$

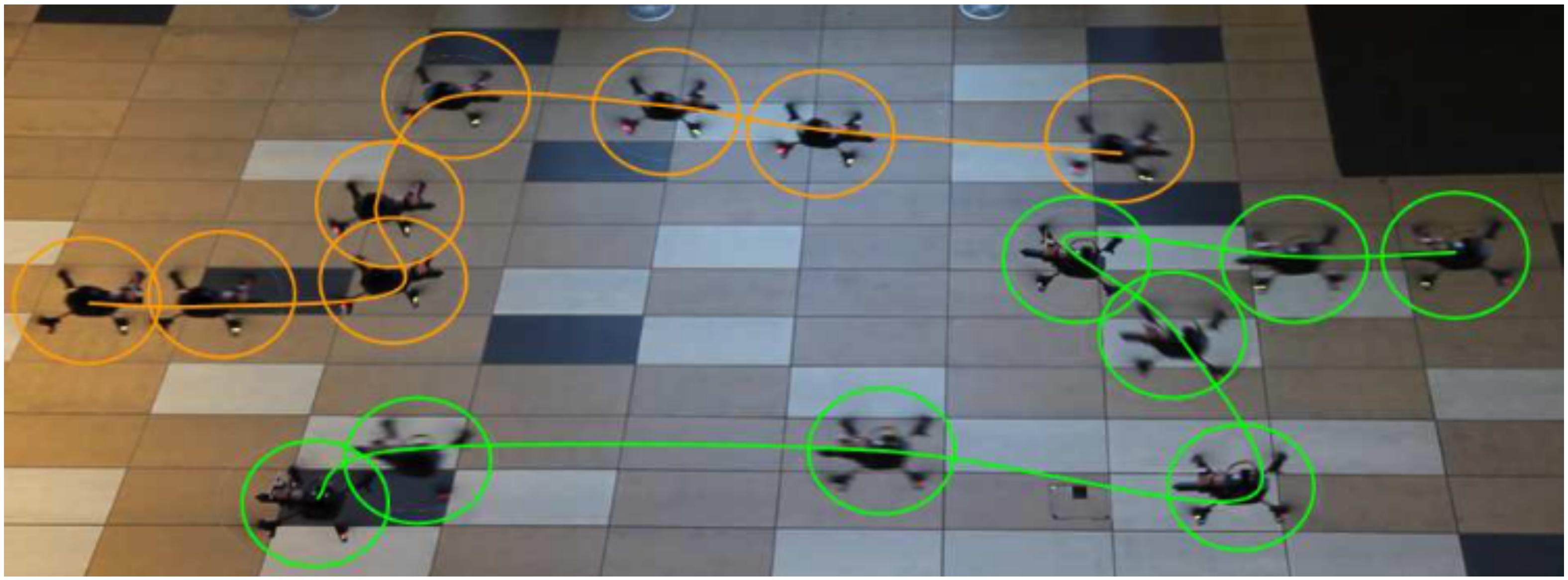
whereas $\mathbf{F}_{\text{ext}}$ takes into account the target (confinement potential), obstacles (rotation around the obstacle), gravity....

Remark

Observe that in the collision avoidance operator, we take a weighted average of forces acting on the particle i and not the minimum...

Experiments on Unmanned Aerial Vehicle (UAV)

We first consider two crossing UAV : the first image represents what is called a “reciprocal dance” with a non smooth trajectory[8]



The second one represents a smooth “collision free trajectory” case

[8] Parker Conroy, Daman Bareiss, Matt Beall and Jur van den Berg (2014)

Influence of the interaction point

We first consider only two particles

- The goal for each particle is to go at the opposite location $-\mathbf{x}(0)$ of the initial position.
- The initial velocity is pointed to 0.

collision avoidance with $|\mathbf{x}_i - \mathbf{x}_j| \leq R$ collision avoidance with $|d_{i,j}| \leq R$

Influence of the vision

We first consider only two particles

- The goal for each particle is to go at $(6, 0, 0)$.
- The initial velocity is pointed to the target

Numerical experiments of the microscopic model

Of course the same result occur in 3D and with more particles. For the same initial configuration, we get

Numerical experiments of the microscopic model

We then consider 10 particles with the same configuration as before.

Interaction with an obstacle

We then consider particles moving around an obstacle.

Mean field kinetic model

Aim :

Our motivation is twofold

- *practical purpose : when the number of interacting individual is large, we cannot distinguish each individual, but the interaction occur with a cloud*
- *numerical purpose: the interacting term is costly to compute $O(N^2)$. We want to apply Particle-In-Cell like methods...*

Instead of using exact position and velocity, we rather describe the dynamics of the probabilty distribtuion $f(t, \mathbf{x}, \mathbf{v}) \geq 0$, which satisfied a Vlasov type equation

$$\partial_t f \,+\, \mathbf{v} \cdot \nabla_{\mathbf{x}} f \,+\, \mathrm{div}_{\mathbf{v}} \left(\left(\mathbf{v} \wedge \mathbf{R}_f(t, \mathbf{x}, \mathbf{v}) \,-\, \Sigma \mathbf{v}\right) \, f\right) - \nabla_{\mathbf{x}} \phi(\mathbf{x}) \cdot \nabla_{\mathbf{v}} f \,=\, 0,$$

where ϕ is an external potential (attracting protential) and $\mathbf{R}_f$ is defined as

$$\mathbf{R}_f(t, \mathbf{x}, \mathbf{v}) \;=\; \frac{M}{\displaystyle\int_{S(\mathbf{x},\mathbf{v})} f(t, \mathbf{x}+\mathbf{y}, \mathbf{v}+\mathbf{w}) d\mathbf{y}\, d\mathbf{w}} \int_{S(\mathbf{x},\mathbf{v})} \mathbf{y} \wedge \mathbf{w}\, f(t, \mathbf{x}+\mathbf{y}, \mathbf{v}+\mathbf{w}) d\mathbf{y}\, d\mathbf{w},$$

with the interacting region $S(\mathbf{x}, \mathbf{v})$.

Properties of the mean field model

Proposition

For smooth and nonnegative initial data f_0, the solution to the kinetic model satisfies

- *for all $t \geq 0$, we have $f(t) \geq 0$;*
- *conservation of mass*

$$\int_{\mathbb{R}^6} f(t,\mathbf{x},\mathbf{v}) d\mathbf{x}\, d\mathbf{v} = \int_{\mathbb{R}^6} f_0(\mathbf{x},\mathbf{v}) d\mathbf{x}\, d\mathbf{v};$$

- *energy dissipation*

$$\frac{d}{dt}\int_{\mathbb{R}^6} f(t,\mathbf{x},\mathbf{v}) \left(\frac{|\mathbf{v}|^2}{2} + \phi(\mathbf{x}) \right) d\mathbf{x}\, d\mathbf{v} \leq -\sigma \int_{\mathbb{R}^6} f(t,\mathbf{x},\mathbf{v}) \frac{|\mathbf{v}|^2}{2}\, d\mathbf{x}\, d\mathbf{v}$$

This last property indicates that the solution may concentrate in velocity and around the target.

Numerical simulation of the mean field model

Conclusion and perspectives

Inspired by various works on pedestrian motion, traffic flow management, we have developed a microscopic model for collision avoidance in 3D

- time is continuous and changes of direction are not instantaneous (we modify the acceleration term)
- the model is based on the ability of the individual to predict an interaction point
- this model is sensitive to the ability to rotate, friction effects

Passing to the limit $N \to \infty$, we can construct a mean field model where the forces take into account self-interactions

- It is possible to obtain a macroscopic model by considering a mono-kinetic approximation

$$f(t, \mathbf{x}, \mathbf{v}) = \rho(t, \mathbf{x})\, \delta(\mathbf{v} - \mathbf{U}(t, \mathbf{x})).$$

- we get some alignments of the particle trajectory in 2D, whereas in 3D it requires more careful computation.

Improvement of the microscopic model

The microscopic model can be improved by considering the “body frame” dynamics. We consider

- Position and velocity $(\mathbf{x}, \mathbf{v})$ in the reference frame
- Rotation matrix o the quadrator **R** which defines the orientation of the quadrator in the body frame.

It satisfies the following system

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ m\dfrac{d\mathbf{v}_i}{dt} = \mathbf{R}_i\mathbf{f} + \mathbf{F}_{\text{ext}}(\mathbf{x}_i, \mathbf{v}_i), \\ \dfrac{d\mathbf{R}_i}{dt} = \mathbf{R}_i\Omega_i, \\ \dfrac{d\omega_i}{dt} = -\mathbf{J}^{-1}\Omega_i\mathbf{J}\omega_i, \end{cases} \quad (7)$$

where **f** is the force generated by the rotors, **J** is the inertia matrix of the rotor, Ω_i is the tensor form of ω_i

$$\Omega_i = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$