Pedestrian macroscopic models: game-theoretical versus mechanistic viewpoints

Pierre Degond

Institut de Mathématiques de Toulouse
CNRS and Université Paul Sabatier

pierre.degond@math.univ-toulouse.fr (see http://sites.google.com/site/degond/)
Toulouse Math. Institute (IMT)

P. Degond, J. Fehrenbach, J. Hua
S. Motsch (now at Univ. Maryland)

Animal Cognition Lab, Toulouse (CRCA)

M. Moussaid, G. Theraulaz, M. Moreau

Theoretical Physics Lab, Orsay (LPT)

C. Appert-Rolland, A. Jelic

INRIA project BUNRAKU, Rennes

S. Donikian, S. Lemercier, J. Pettré
1. Issues & context

2. The Heuristic-Based Model (HBM)

3. Mean-field models

4. Macroscopic model

5. Relation to game theory

6. Conclusion
1. Issues & context
Issues

Safety

Avoid crowd disasters
  e.g. Duisburg love parade
  Cambodia water festival

Demonstration control

Design, comfort, efficiency

Terminals, shopping malls, etc.
Pedestrian models

Individual-Based Models (IBM)
Each individual followed in time

Social force model [Helbing & Molnar, Phys. Rev. E51, 1995]
Analogy with physics:
Attractive/repulsive forces
others ...

Cellular automata
[Burstedde et al, Physica A 295, 2001]
Pedestrian models

Macroscopic models

Inspired by gas kinetics

[Henderson, Transp. Res. 8, 1974]

Static/dynamic field (∼ chemotaxis)


Inspired from road traffic

[Colombo et al, MMAS 28, 2005]
2. The Heuristic-Based Model (HBM)
Heuristic-Based Model (HBM)

[Moussaïd, Helbing, Theraulaz, PNAS 2011]
Perception phase

Pedestrians have constant speed

Evaluation assumes pedestrians move on straight lines

Distance to Interaction (DTI)

Minimal Distance (MD)

In case of multiple encounters

Take the minimal DTI
Nash equilibrium: Discrete time step

New cruising direction $u'$ chosen such that

Estimation $X_E(u')$ minimizes distance to target $X_T$

$$\|X_E(w) - X_T\|^2$$ among test directions $w$
Experiments

Motion capture system

- Sensors reflect infra-red light
- Reflection point camera recorded
- Triangulation $\rightarrow$ coordinates

Circular arena

- Avoids boundary effects
Experiments vs model

Lane formation

Lane definition

by clustering method

Cluster lifetime statistics

\[ p(t)dt = \text{probability that lifetime } \in [t, t + dt] \]

Stretched exponential \( p(t) = p_0 e^{at^k}, \ k = 0.4 \)

In insert: results of model (See [Moussaid et al, PlosCB 2012])
$N$ Particles (pedestrians) $i = 1, \ldots, N$

Position $x_i(t)$, velocity $u_i(t)$, Target direction $a_i(t)$

with $|u_i(t)| = 1$, $|a_i(t)| = 1$, i.e. $u_i, a_i \in \mathbb{S}^1$

$$\dot{x}_i = cu_i,$$

$$du_i = F_i \, dt + P_{u_i^\perp}(\sqrt{2d} \circ dB_i(t))$$

Speed $c$, noise intensity $d$, Stratonowich sense $\circ$

Force $F_i^\perp u_i$, $P_{u_i^\perp}$ maintains $|u_i| = 1$
Test velocity directions $w \in S^1 \rightarrow$ Potential $\Phi_i(w, t)$

$$\Phi_i(w, t) = \frac{k}{2} |D_i(w)w - La_i|^2$$

Reaction rate $k$, horizon $L$

$D_i(w)$ maximal walkable distance in direction $w$

Force $F_i(t)$ defined by steepest descent of $\Phi_i$

$$F_i(t) = -\nabla_w \Phi_i(u_i(t), t)$$
Maximal walkable distance $D_i(w)$

DTI of 'i' against 'j' when 'i' walks in direction $w$: $D_{ij}(w)$

$$D_i(w) = "\min_j" D_{ij}(w)$$

For continuum model, replace 'min' by average 

- e.g. harmonic average in some interaction region
3. The Mean-Field Model
Mean-Field Model

Distribution function $f(x, u, a, t) \quad x \in \mathbb{R}^2, \quad u, a \in S^1$

Probability to find pedestrians at $x$

with velocity $u$ and target velocity $a$ at time $t$

$$\partial_t f + \nabla_x \cdot (cu f) + \nabla_u \cdot (F f) = d \Delta_u f$$

$$F = -\nabla_w \Phi(x, a, t)(u)$$

$$\Phi(x, a, t)(w) = \frac{k}{2} |D(x, t)(w) w - La|^2$$

$D(x, t)(w)$ walkable distance of subject at $x$

in direction $w$: functional of $f$
Case: local interactions / no blind zone

Supposes interaction region "very small"

\[ D^{-1}_{(x,t)}(w) = \frac{\int_{(v,b) \in \mathbb{T}^2} \Delta(|v - w|) f(x, v, b, t) \, dv \, db}{\int_{(v,b) \in \mathbb{T}^2} f(x, v, b, t) \, dv \, db} \]

where \( \Delta \) is analytically known (related to the DTI)

If blind zone, \( \Delta = \Delta(u, |v - w|) \)

Then \( D = D_{(x,u,t)}(x) \) and \( \Phi = \Phi_{(x,u,a,t)}(w) \)

Dependence of \( \Phi \) on \( u \) problematic

Subsequent macroscopic theory cannot be developed

Other closures can be done
4. Macroscopic model
Let $D(u)$ be arbitrary and define

$$Q_D(f) = -\nabla_u \cdot (F_D f) + d\nabla_u f$$

$$F_D(u, a) = -\nabla_u \Phi_D(u, a), \quad \Phi_D(u, a) = \frac{k}{2}|D(u) u - L a|^2$$

For $f(u, a)$ arbitrary, define

$$D_f^{-1}(u) = \frac{\int_{(v,b)\in\mathbb{T}^2} \Delta(|v - u|) f(v, b) \, dv \, db}{\int_{(v,b)\in\mathbb{T}^2} f(x, v, b, t) \, dv \, db}$$

Then mean-field model can be written

$$\partial_t f + \nabla_x \cdot (c u f) = \frac{1}{\varepsilon} Q_{D_f}(f)$$
'Generalized' Von-Mises (GVM) distributions

For given $D(u)$, solutions $f$ of $Q_D(f) = 0$ are of the form

$$f(u, a) = \rho(a) \ M_D(u, a)$$

with $\rho(a)$ arbitrary and

$$M_D(u, a) = \frac{1}{Z_D(a)} \exp \left( - \frac{\Phi_D(u, a)}{d} \right)$$

where $Z_D(a)$ is s.t.

$$\int M_D(u, a) \ du = 1$$
Equilibria

Solutions $f$ of $Q_{D_f}(f) = 0$:

are GVM $f = \rho(a) M_D(u, a)$

such that $D = D_{\rho M_D}$

Leads to a fixed point equation

$$D^{-1}(u) = \frac{\int_{(v,b) \in T^2} \Delta(|v - u|) \rho(b) M_D(v, b) \, dv \, db}{\int_{(v,b) \in S^1} \rho(b) \, db}$$

Mathematical theory open

Here we assume that for any function $\rho(a)$:

there exists a 'distinguished' solution $D_{\rho}$
When $\varepsilon \to 0$, formally we have

$$f^\varepsilon \to \rho(x,t)(a) \ M_D\rho(x,t)(u, a)$$

where $\rho(x,t)(a)$ satisfies the continuity eq.

$$\partial_t \rho(x,t)(a) + \nabla_x \cdot (c\rho(x,t)(a)U_{\rho(x,t)}(a)) = 0$$

and $U_{\rho(x,t)}(a)$ is the mean equilibrium velocity

$$U_{\rho}(a) = \int_{u \in S^1} M_{D\rho}(u, a) \ u \ du$$
5. Relation to game theory
Spatially homogeneous case:

For probability \( f(u, a) \), introduce the 'cost function'

\[
\mu_f(u, a) = \Phi_{D_f}(u, a) + d \ln f(u, a)
\]

Non-cooperative anonymous game with a continuum of players (aka 'Mean-Field Game [Lasry & Lions])

each pedestrian (player) tries to minimize its cost by acting on its own decision variable \( u \)
$f_{\text{NE}}$ is a Nash Equilibrium if

No player can reduce its cost by acting on its control variable $u$

$f_{\text{NE}}$ is a Nash Equilibrium iff $\exists K$ s.t.

\[
\mu_{f_{\text{NE}}}(u, a) = K, \quad \forall (u, a) \in \text{Supp}(f_{\text{NE}})
\]

\[
\mu_{f_{\text{NE}}}(u, a) \geq K, \quad \forall (u, a) \in \mathbb{T}^2
\]

The following statements are equivalent:

$f$ is an equilibrium of the kinetic model

and is therefore a GVM distribution

$f$ is a Nash Equilibrium for the Mean-Field Game

defined by cost function $\mu_f$
Hydrodynamic model

Spatially inhomogeneous case

Hydrodynamic model is obtained by

- Taking the continuity equation (i.e. taking the first moment of kinetic eq. wrt $u$)
- Closing the model by taking the local Nash Equilibrium

See a general framework for

- Kinetic models coupled with Mean-Field Games in [D., Liu, Ringhofer, A Nash equilibrium macroscopic closure for kinetic models coupled with Mean-Field Games, arXiv:1212.6130]
6. Conclusion
Summary

Heuristic-Based model of Moussaid, Helbing Theraulaz

Takes into account following behavior

Derivation of

Time continuous IBM
potential minimization replaced by steepest descent

Mean-Field Model
approximation of local interactions, no blind zone

Hydrodynamic Model

Equilibria can be interpreted as

Nash equilibria of a Mean-Field Game
View of pedestrians as fully rational agents
≠ mechanistic view of most models