Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous dilute polymers

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joint work with
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To Eitan Tadmor, on the occasion of his 60th birthday.

Viscoelastic fluids

Gareth McKinley's Non-Newtonian Fluid Dynamics Group, MIT

Jonathan Rothstein's Non-Newtonian Fluids Dynamics Lab, University of Massachusetts

Statement of the model

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$: bounded open Lipschitz domain,
- T: length of the time interval of interest, and
- $Q := \Omega \times (0, T)$: the associated space-time domain.

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Consider the following system of nonlinear PDEs:

$$
\begin{aligned}
\rho\,(\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})) - \operatorname{div} \mathbf{T} &= \rho\, \boldsymbol{f} && \text{in } Q, \\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } Q, \\
\boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}_0(\cdot) && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

and the boundary condition

$$
\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega \times (0, T).
$$

We assume that the *Cauchy stress* T is decomposed as

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  $\mathbf{S}_v$ and $\mathbf{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ are assumed to be related via a maximal monotone graph described by the implicit relation:

  $$\boxed{\mathbf{G}(\mathbf{S}_v, \mathbf{D}(\boldsymbol{u})) = \mathbf{0},} \qquad (2)$$

  where $\mathbf{G} : \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is a continuous mapping.

- $\mathbf{S}_e : Q \to \mathbb{R}^{d\times d}_{sym}$ is the elastic part of the deviatoric stress.

Examples of $\mathbf{G}(\mathbf{S}_v, \mathbf{D}(\boldsymbol{u})) = \mathbf{0}$

- Newtonian (Navier–Stokes) fluids: $\mathbf{S}_v = 2\mu_* \mathbf{D}(\boldsymbol{u})$, with $\mu_* > 0$;

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- Activated fluids, such as Bingham and Herschel–Bulkley fluids:

$$|\mathbf{S}_\nu| \le \tau_* \Leftrightarrow \mathbf{D}(\boldsymbol{u}) = \mathbf{0} \text{ and } |\mathbf{S}_\nu| > \tau_* \Leftrightarrow \mathbf{S}_\nu = \frac{\tau_* \mathbf{D}(\boldsymbol{u})}{|\mathbf{D}(\boldsymbol{u})|} + 2\nu(|\mathbf{D}(\boldsymbol{u})|^2) \mathbf{D}(\boldsymbol{u}).$$

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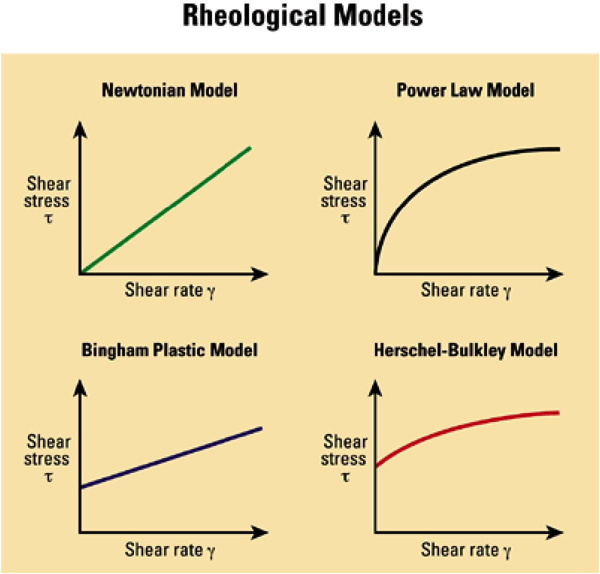
$$\text{i.e.} \quad 2\nu(|\mathbf{D}(\boldsymbol{u})|^2)(\tau_* + (|\mathbf{S}_\nu| - \tau_*)_+) \mathbf{D}(\boldsymbol{u}) = (|\mathbf{S}_\nu| - \tau_*)_+ \mathbf{S}_\nu, \quad \tau_* > 0.$$

Examples of $\mathbf{S}_\nu (= \tau)$ vs. $\mathbf{D}(\boldsymbol{u}) (= \gamma)$



We identify the implicit relation (2) with a graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$, i.e.,

$$\mathbf{G}(\mathbf{S},\mathbf{D}) = \mathbf{0} \iff (\mathbf{D},\mathbf{S}) \in \mathcal{A}.$$

We assume that, for some $r \in (1,\infty)$, $\mathcal{A}$ is a *maximal monotone r-graph*:

(A1) *$\mathcal{A}$ includes the origin;* i.e., $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$;

(A2) *$\mathcal{A}$ is a monotone graph;* i.e.,

$$(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) \geq 0 \text{ for all } (\mathbf{D}_1,\mathbf{S}_1),(\mathbf{D}_2,\mathbf{S}_2) \in \mathcal{A};$$

(A3) *$\mathcal{A}$ is a maximal monotone graph;* i.e., for any $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$,

$$\text{if } (\tilde{\mathbf{S}} - \mathbf{S})\cdot(\tilde{\mathbf{D}} - \mathbf{D}) \geq 0 \text{ for all } (\tilde{\mathbf{D}},\tilde{\mathbf{S}}) \in \mathcal{A}, \text{ then } (\mathbf{D},\mathbf{S}) \in \mathcal{A};$$

(A4) *$\mathcal{A}$ is an r-graph;* i.e., there exist positive constants C_1, C_2 such that

$$\mathbf{S}\cdot\mathbf{D} \geq C_1(|\mathbf{D}|^r + |\mathbf{S}|^{r'}) - C_2 \text{ for all } (\mathbf{D},\mathbf{S}) \in \mathcal{A}.$$

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Definition of S_e: kinetic theory of polymers

Large number of internal degrees of freedom $\longrightarrow$ statistical physics.

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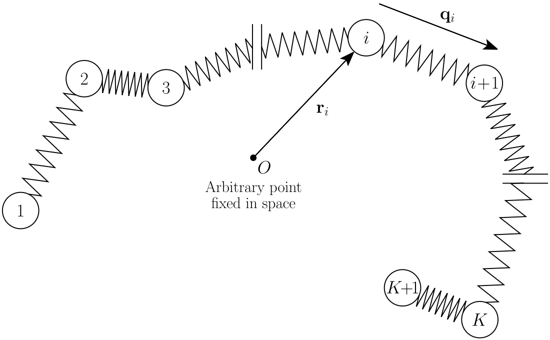

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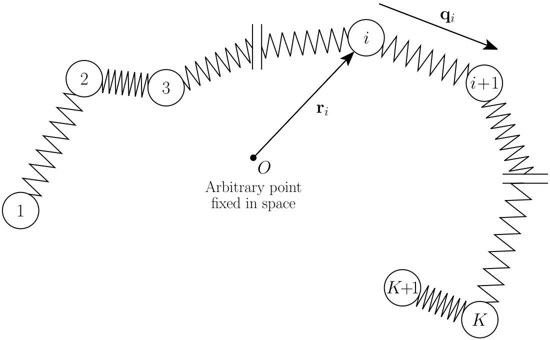

Let $D_i \subset \mathbb{R}^d$, $i = 1, \dots, K$, be bounded open balls centred at $\mathbf{0}$.

Consider the *Maxwellian* $M(\boldsymbol{q}) := M_1(\boldsymbol{q}_1) \cdots M_K(\boldsymbol{q}_K)$, with $\boldsymbol{q}_i \in D_i$, where

$$M_i(\boldsymbol{q}_i) := \frac{\mathrm{e}^{-U_i(\frac{1}{2}|\boldsymbol{q}_i|^2)}}{\int_{D_i} \mathrm{e}^{-U_i(\frac{1}{2}|\boldsymbol{p}_i|^2)} \mathrm{d}\boldsymbol{p}_i}, \qquad i = 1, \dots, K.$$

S_e is defined by the *Kramers expression*:

$$\mathsf{S}_e(x,t) := k_B\,\mathtt{T}\sum_{i=1}^{K}\int_D M(\boldsymbol{q})\,\nabla_{\boldsymbol{q}_i}\widehat{\psi}(x,\boldsymbol{q},t)\otimes\boldsymbol{q}_i\,\mathrm{d}\boldsymbol{q},$$

where $\boldsymbol{q} = (\boldsymbol{q}_1^{\mathrm{T}},\ldots,\boldsymbol{q}_K^{\mathrm{T}})^{\mathrm{T}} \in D_1\times\cdots\times D_K =: D$ and

$$\widehat{\psi} := \psi/M$$

is the normalized probability density function, that is the solution of a Fokker–Planck equation.

Fokker–Planck equation

The function $\widehat{\psi} = \psi/M$ satisfies the *Fokker–Planck equation*:

$$(M\widehat{\psi})_t + \operatorname{div}(M\widehat{\psi}\boldsymbol{u}) + \operatorname{div}_{\boldsymbol{q}}(M\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}) = \triangle(M\widehat{\psi}) + \operatorname{div}_{\boldsymbol{q}}\boldsymbol{A}(M\nabla_{\boldsymbol{q}}\widehat{\psi}) \qquad (3)$$

in $O \times (0,T)$, with $O := \Omega \times D$, subject to the boundary conditions:

$$M\nabla\widehat{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T),$$
$$(M\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}_i - \boldsymbol{A}_i(M\nabla_{\boldsymbol{q}}\widehat{\psi}))\cdot\boldsymbol{n}_i = 0 \qquad \text{on } \Omega \times \partial\bar{D}_i \times (0,T),$$

for all $i = 1,\ldots,K$, and the initial condition

$$\widehat{\psi}(x,\boldsymbol{q},0) = \widehat{\psi}_0(x,\boldsymbol{q}) \qquad \text{in } O.$$

$\boldsymbol{A} \in \mathbb{R}^{K\times K}_{\text{symm}}$: *Rouse matrix* (symmetric, positive definite).

J.W. Barrett & E. Süli (M3AS, 21 (2011), 1211–1289):
Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains

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M. Bulíček, J. Málek & E. Süli (Communications in PDE, 38 (2013), 882–924):
Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous flows of dilute polymers
DOI: 10.1080/03605302.2012.742104

Assumptions on the data

For the Maxwellian M we assume that

$$M \in C_0(\overline{D}) \cap C^{0,1}_{\mathrm{loc}}(D), \quad \text{and } M > 0 \text{ on } D. \tag{4}$$

For the initial velocity $\boldsymbol{u}_0$ we assume that

$$\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega). \tag{5}$$

For $\widehat{\psi}_0 := \psi_0/M$ we assume, with $O := \Omega \times D$, that

$$\widehat{\psi}_0 \geq 0 \;\text{ a.e. in } O, \quad \widehat{\psi}_0 \log \widehat{\psi}_0 \in L^1_M(O), \tag{6}$$

and that the initial marginal probability density function

$$\int_D M(\boldsymbol{q})\, \widehat{\psi}_0(\cdot, \boldsymbol{q})\, \mathrm{d}\boldsymbol{q} \in L^\infty(\Omega). \tag{7}$$

Theorem

For $d \in \{2,3\}$ *let* $D_i \subset \mathbb{R}^d$, $i = 1,\dots,K$, *be bounded open balls centred at the origin in* $\mathbb{R}^d$, *let* $\Omega \subset \mathbb{R}^d$ *be a bounded open Lipschitz domain and suppose* $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}}(\Omega))$, $r \in (1,\infty)$. *Assume that* $\mathcal{A}$, *given by* $\mathbf{G}$, *is a maximal monotone* r*-graph satisfying* **(A1) – (A4)**, *the Maxwellian* $M : D \to \mathbb{R}$ *satisfies* (4), *and* $(\boldsymbol{u}_0, \widehat{\psi}_0)$ *satisfy* (5)–(7).

Then, there exist $(\boldsymbol{u}, \mathbf{S}_v, \mathbf{S}_e, \widehat{\psi})$ *such that*

$$\boldsymbol{u} \in L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega)^d) \cap L^r(0,T;W^{1,r}_0(\Omega)^d) \cap W^{1,r^*}(0,T;W^{-1,r^*}_{0,\mathrm{div}}(\Omega)),$$

$$\mathbf{S}_v \in L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), \qquad \mathbf{S}_e \in L^2(0,T;L^2(\Omega)^{d\times d}),$$

$$\widehat{\psi} \in L^\infty(Q;L^1_M(D)) \cap L^2(0,T;W^{1,1}_M(O)), \qquad \widehat{\psi} \geq 0 \ \text{a.e. in } O \times (0,T),$$

$$M\widehat{\psi} \in W^{1,1}(0,T;W^{-1,1}(O)), \qquad \widehat{\psi}\log\widehat{\psi} \in L^\infty(0,T;L^1_M(O)),$$

where

$$r^* := \min\left\{r', 2, \left(1 + \tfrac{2}{d}\right)r\right\} \qquad \text{and} \qquad r' := \tfrac{r}{r-1}.$$

Theorem (Continued...)

Moreover, (1) *is satisfied in the following sense:*

$$\int_0^T \langle \boldsymbol{u}_t, \boldsymbol{w} \rangle \, \mathrm{d}t + \int_0^T \big(-(\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{w}) + (\mathbf{S}_v, \nabla \boldsymbol{w}) \big) \mathrm{d}t$$
$$= \int_0^T \big(-(\mathbf{S}_e, \nabla \boldsymbol{w}) + \langle \boldsymbol{f}, \boldsymbol{w} \rangle \big) \mathrm{d}t \qquad \text{for all } \boldsymbol{w} \in L^\infty(0,T; W^{1,\infty}_{0,\mathrm{div}}(\Omega)),$$

where

$$(\mathbf{S}_v(x,t), \mathbf{D}(\boldsymbol{u}(x,t))) \in \mathcal{A} \qquad \text{for a.e. } (x,t) \in Q,$$

and $\mathbf{S}_e$ *is given by the Kramers expression*

$$\mathbf{S}_e(x,t) = k_B \, \mathrm{T} \sum_{i=1}^K \int_D M \, \nabla_{\boldsymbol{q}_i} \widehat{\psi}(x, \boldsymbol{q}, t) \otimes \boldsymbol{q}_i \, \mathrm{d}\boldsymbol{q} \qquad \text{for a.e. } (x,t) \in Q.$$

Theorem (Continued...)

In addition, the Fokker–Planck eqn (3) is satisfied in the following sense:

$$\int_0^T \left[\langle (M\widehat{\psi})_t, \varphi\rangle - (M\boldsymbol{u}\widehat{\psi}, \nabla\varphi)_{\mathcal{O}} - (M\widehat{\psi}(\nabla\boldsymbol{u})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}\right] \mathrm{d}t$$

$$+ \int_0^T \left[(M\nabla\widehat{\psi}, \nabla\varphi)_{\mathcal{O}} + (M\boldsymbol{A}\nabla_{\boldsymbol{q}}\widehat{\psi}, \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}\right] \mathrm{d}t = 0$$

$$\textit{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})),$$

and the initial data are attained strongly in $L^2(\Omega)^d \times L^1_M(\mathcal{O})$, *i.e.,*

$$\lim_{t\to 0_+} \|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0(\cdot)\|_2^2 + \|\widehat{\psi}(\cdot,t) - \widehat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})} = 0.$$

Theorem (Continued...)

Further, for $t \in (0,T)$ the following energy inequality holds in a weak sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(k\int_O M\,\widehat{\psi}\log\widehat{\psi}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\tfrac{1}{2}\|\boldsymbol{u}\|_2^2\right)+(\mathsf{S}_v,\mathsf{D}(\boldsymbol{u}))+4k\left(M\nabla\sqrt{\widehat{\psi}},\nabla\sqrt{\widehat{\psi}}\right)_O$$

$$+4k\left(M\,\boldsymbol{A}\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}},\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}}\right)_O\le\langle\boldsymbol{f},\boldsymbol{u}\rangle,\qquad \textit{with } k:=k_B\mathtt{T}.$$

Proof

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STEP 5. We require strongly convergent sequences for passage to limit in ℓ in the various nonlinear terms. This is the most difficult step to realize.

weak convergence $\longrightarrow$ strong convergence

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(k\int_{\mathcal{O}} M\,\widehat{\psi}^{\ell}\log\widehat{\psi}^{\ell}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\tfrac{1}{2}\|\boldsymbol{u}^{\ell}\|_2^2\right)+(\mathsf{S}_v^{\ell},\mathsf{D}(\boldsymbol{u}^{\ell}))+4k\left(M\,\nabla\sqrt{\widehat{\psi}^{\ell}},\nabla\sqrt{\widehat{\psi}^{\ell}}\,\right)_{\mathcal{O}}$$
$$+4k\left(M\,\boldsymbol{A}\,\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}^{\ell}},\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}^{\ell}}\,\right)_{\mathcal{O}}\leq\langle\boldsymbol{f},\boldsymbol{u}^{\ell}\rangle,\qquad \text{with } k:=k_B\mathtt{T}.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(k\int_{O} M\,\widehat{\psi}^{\ell}\log\widehat{\psi}^{\ell}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\tfrac{1}{2}\|\boldsymbol{u}^{\ell}\|_2^2\right)+(\mathsf{S}_{\nu}^{\ell},\mathsf{D}(\boldsymbol{u}^{\ell}))+4k\left(M\nabla\sqrt{\widehat{\psi}^{\ell}},\nabla\sqrt{\widehat{\psi}^{\ell}}\right)_{O}$$
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- Velocity:
  strong convergence immediate by Aubin–Lions–Simon compactness theorem.

weak convergence $\longrightarrow$ strong convergence

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  - Uniform interior estimates on $\Omega \times D \times (0,T)$, obtained by function space interpolation from the energy inequality.

STEP 6. *Identification of* S_e: the sequence of truncated Kramers expressions S_e^ℓ converges to S_e strongly in $L^q(0,T;L^q(\Omega)^{d\times d})$, $q\in[1,2)$.

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STEP 7. The initial data are attained strongly in $L^2(\Omega)^d \times L^1_M(O)$, i.e.,

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STEP 8. *Identification of* $\mathbf{S}_v$: by a parabolic Acerbi–Fusco type Lipschitz-truncation of Diening, Ružička & Wolf (2010) and STEP 6:

$$\lim_{\ell\to\infty}\int_Q\left|(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{u})))\cdot\mathbf{D}(\boldsymbol{u}^\ell-\boldsymbol{u})\right|^\alpha\,\mathrm{d}x\,\mathrm{d}t=0\qquad\forall\alpha\in(0,1).$$

$\mathbf{S}^*$ is a measurable selection such that for any $\mathbf{D}$ we have $(\mathbf{S}^*(\mathbf{D}),\mathbf{D})\in\mathcal{A}$.

Thus, for a subsequence,

$$(\mathsf{S}_\nu^\ell - \mathsf{S}^*(\mathsf{D}(\boldsymbol{u}))) \cdot \mathsf{D}(\boldsymbol{u}^\ell - \boldsymbol{u}) \to 0 \qquad \text{almost everywhere in } Q.$$

Moreover, using the energy inequality, we see that

$$\int_Q \left| (\mathsf{S}_\nu^\ell - \mathsf{S}^*(\mathsf{D}(\boldsymbol{u}))) \cdot \mathsf{D}(\boldsymbol{u}^\ell - \boldsymbol{u}) \right| \mathrm{d}x\,\mathrm{d}t \le C.$$

We apply Chacon's Biting Lemma to find a nondecreasing countable sequence of measurable sets $Q_1 \subset \cdots \subset Q_k \subset Q_{k+1} \subset \cdots \subset Q$ such that

$$\lim_{k\to\infty} |Q \setminus Q_k| \to 0$$

and such that for any k there is a subsequence such that

$$(\mathsf{S}_\nu^\ell - \mathsf{S}^*(\mathsf{D}(\boldsymbol{u}))) \cdot \mathsf{D}(\boldsymbol{u}^\ell - \boldsymbol{u}) \qquad \text{converges weakly in } L^1(Q_k).$$

By Vitali's theorem we then deduce that

$$(\mathsf{S}_\nu^\ell - \mathsf{S}^*(\mathsf{D}(\boldsymbol{u}))) \cdot \mathsf{D}(\boldsymbol{u}^\ell - \boldsymbol{u}) \to 0 \qquad \text{strongly in } L^1(Q_k).$$

The weak convergence of (S_ν^ℓ) to S_ν and $(\mathsf{D}(\boldsymbol{u}^\ell))$ to $\mathsf{D}(\boldsymbol{u})$ implies that

$$\lim_{\ell\to\infty}(\mathsf{S}_\nu^\ell, \mathsf{D}(\boldsymbol{u}^\ell))_{Q_k} = (\mathsf{S}_\nu, \mathsf{D}(\boldsymbol{u}))_{Q_k}.$$

The assumption that $\mathcal{A}$ is a maximal monotone r-graph then implies that

$$(\mathsf{S}_\nu, \mathsf{D}(\boldsymbol{u})) \in \mathcal{A} \qquad \text{a.e. in } Q_k,\ k = 1, 2, \dots.$$

Finally, by a diagonal procedure and $\lim_{k\to\infty}|Q \setminus Q_k| \to 0$ we deduce that

$$(\mathsf{S}_\nu, \mathsf{D}(\boldsymbol{u})) \in \mathcal{A} \qquad \text{a.e. in } Q = \Omega \times (0, T).$$

□

Open problems

- The extension of these results to implicitly constituted kinetic models with variable density, density-dependent viscosity and drag is open.

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  For Navier–Stokes–Fokker–Planck systems with variable density and density-dependent dynamic viscosity and drag the existence of global weak solutions was shown in
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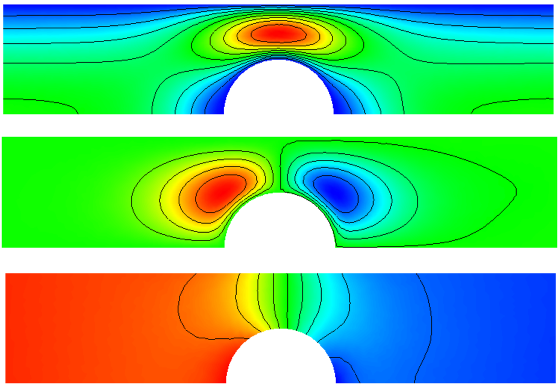
  - Barrett & Süli (Journal of Differential Equations, 2012).

- The numerical analysis of implicitly constituted kinetic models of polymers is open.

  Special cases:
  - Barrett & Süli (M2AN, 2012)
  - Diening, Kreuzer & Süli (SIAM J. Numer. Anal., 2013)
  - Kreuzer & Süli (In preparation, 2014).

2D/4D: Flow around a cylinder

- Standard benchmark problem: flow around a cylinder
- Assume Stokes flow, parabolic inflow BCs on u_x, no-slip on stationary walls and cylinder
- Steady state solution (computed on 8 processors):



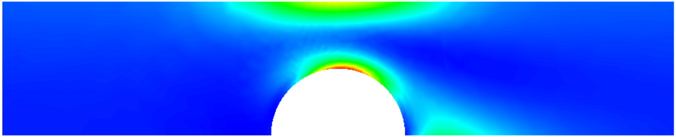
u_x :

u_y :

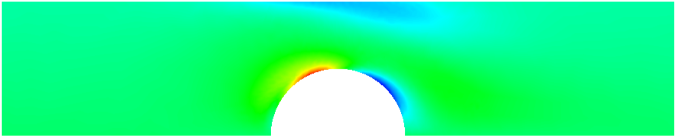
p :

2D/4D: Flow around a cylinder: extra stress tensor

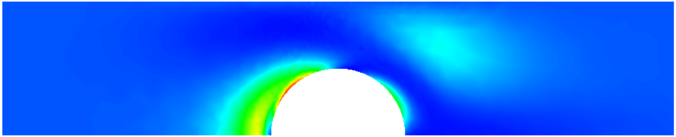
$(\mathsf{S}_e)_{11}$:



$(\mathsf{S}_e)_{12}$:



$(\mathsf{S}_e)_{22}$:



2D/4D: Flow around a cylinder: probability density fn.

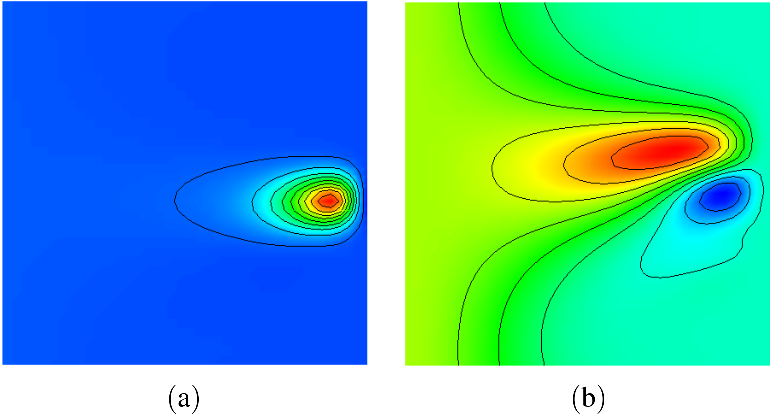
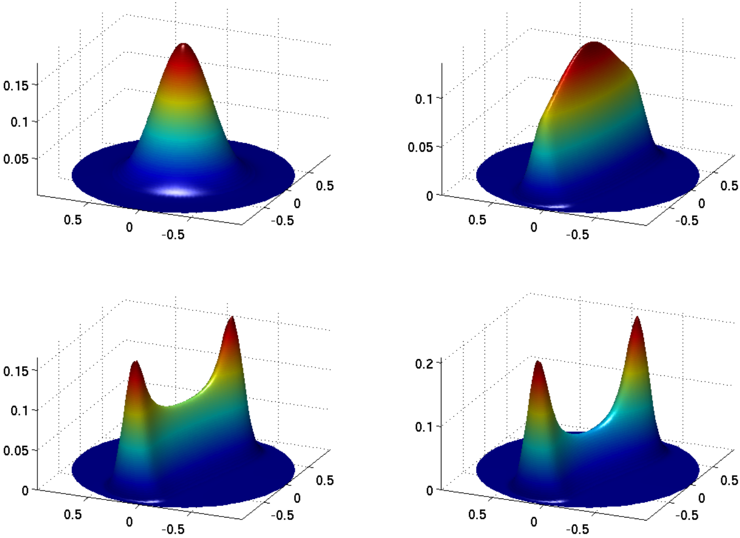


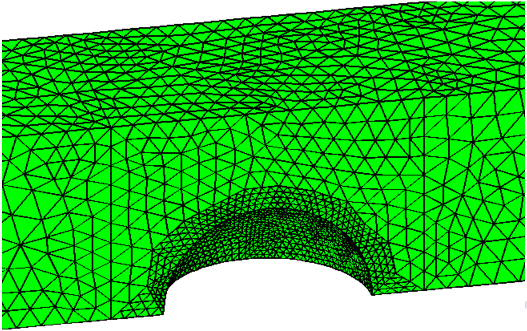
Figure : Configuration space cross-sections of ψ for x in (r, θ)-coordinates: (a) wake of cylinder, and (b) between cylinder and wall.

2D/4D: Flow around a cylinder: probability density fn.

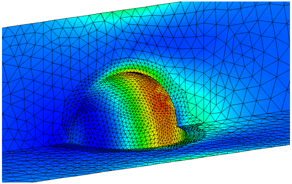


3D/6D: Flow past a ball in a channel

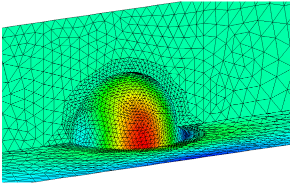
- Pressure-drop-driven flow past a ball in hexahedral channel.
- P_2/P_1 mixed FEM for (Navier–)Stokes equation on a mesh with 3045 tetrahedral elements and 51989 Gaussian quadrature points.
- Fokker–Planck equation solved using heterogenous ADI method in 6D domain $\Omega \times D$. 51989 3D solves per time step in $\mathbf{q} = (q_1, q_2, q_3) \in D$ and 1800 3D solves per time-step in $\mathbf{x} = (x, y, z) \in \Omega$.
- Computed using 120 processors; 45s/time step; 10 time steps; $\Delta t = 0.05$; $\lambda = 0.5$.



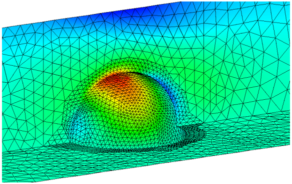
3D/6D: Flow past a ball in a channel: elastic part of the Cauchy stress



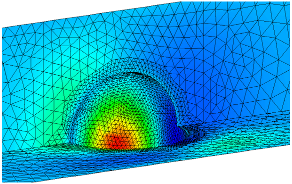
$(\mathbf{S}_e)_{11}$



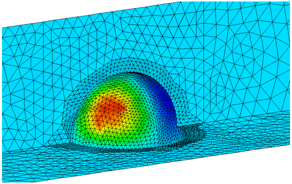
$(\mathbf{S}_e)_{12}$



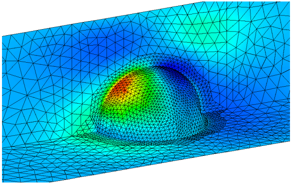
$(\mathbf{S}_e)_{13}$



$(\mathbf{S}_e)_{22}$



$(\mathbf{S}_e)_{23}$



$(\mathbf{S}_e)_{33}$


HAPPY BIRTHDAY, EITAN!