Phase transition in a model for territorial development

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Introduction to Gangs

- Gangs are responsible for much of the violent crimes in the U.S. and worldwide
- One of the main concerns of a gang is its territory
- This territory is marked and defended
- Graffiti (tagging) is often used to claim or maintain territory

Figure: Graffiti gang war in Los Angeles, California. Figure adopted from www.workhorsevisuals.com

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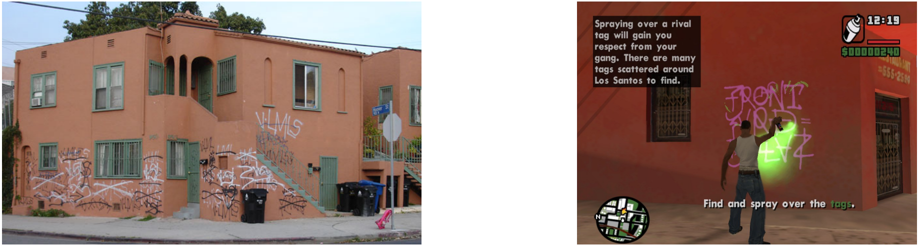

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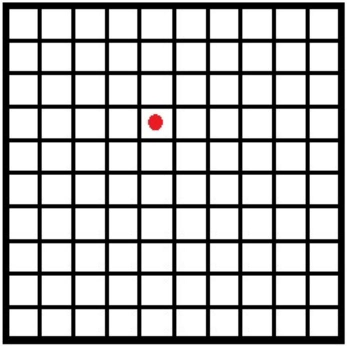
Lattice Model Summary

- We consider two gangs, let's say red and blue
- Initially, agents are uniformly distributed over a toroidal lattice
- Agents have some probability of tagging the lattice site which they currently occupy
- Agents then are forced to move to one of the four neighboring sites
  - $\Rightarrow$ The total number of agents in each gang is conserved



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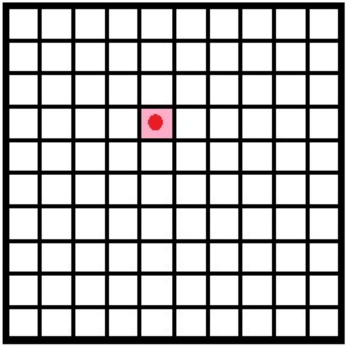

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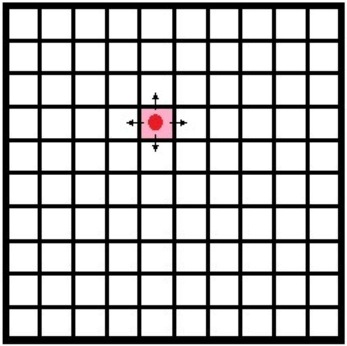
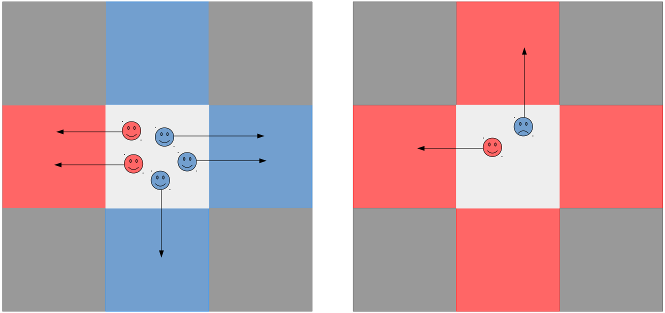


Our Discrete Model

- We base our model on the assumption that gangs claim territory by tagging
- Each gang preferentially avoids areas marked by the other gang
- Gang members do not interact directly
    - Multiple agents may occupy the same site
    - Agents interact only through the graffiti field



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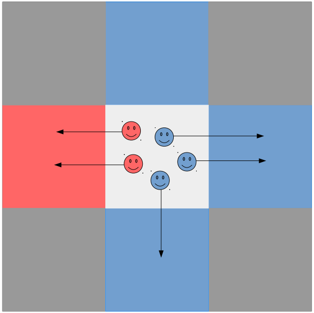

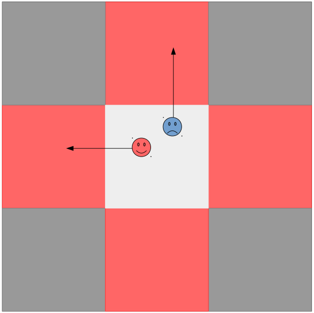

Discrete Dynamics

- The densitity of agents of gangs A and B at site (x, y) at time t are denoted by $\rho_A(x, y, t)$ and $\rho_B(x, y, t)$
- The density of graffiti of gang A and B at site (x, y) at time t are $\xi_A(x, y, t)$ and $\xi_B(x, y, t)$
- The probability of an agent from gang A to move to a neighbouring site is

$$M_A(x_1 \to x_2, y_1 \to y_2, t) := \frac{e^{-\beta \xi_B(x_2, y_2, t)}}{\sum_{(\tilde{x}, \tilde{y}) \sim (x_1, y_1)} e^{-\beta \xi_B(\tilde{x}, \tilde{y}, t)}}$$

- β is parameter that symbolizes the avoidance of graffiti belonging to the other gang.

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Discrete Equations

According to our discrete model, employing this notation, we find:

- The expected density of gang A agents at site $(x, y) \in S$ at time $t + \delta t$ is

$$\begin{aligned}\rho_A(x, y, t+\delta t) =& \rho_A(x, y, t) + \sum_{(\tilde{x},\tilde{y})\sim(x,y)} \rho_A(\tilde{x}, \tilde{y}, t) M_A(\tilde{x} \to x, \tilde{y} \to y, t) \\ & - \rho_A(x, y, t) \sum_{(\tilde{x},\tilde{y})\sim(x,y)} M_A(x \to \tilde{x}, y \to \tilde{y}, t)\end{aligned}$$

- The graffiti evolution is given by

$$\xi_A(x, y, t+\delta t) = \xi_A(x, y, t) - \delta t \cdot \lambda \cdot \xi_A(x, y, t) + \delta t \cdot \gamma \cdot \rho_A(x, y, t),$$

Simulations of Gang Dynamics: Well-Mixed Phase

- $\beta = 1 \times 10^{-6}$:

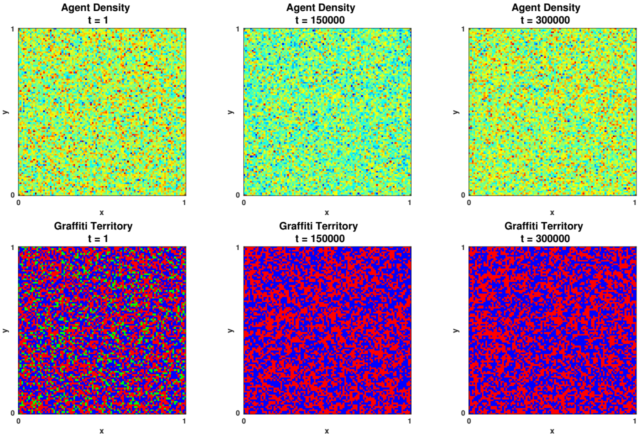


Figure: Temporal evolution of the densities for a well-mixed phase.

Segregated Phase: Agent Density

- $\beta = 2 \times 10^{-5}$:

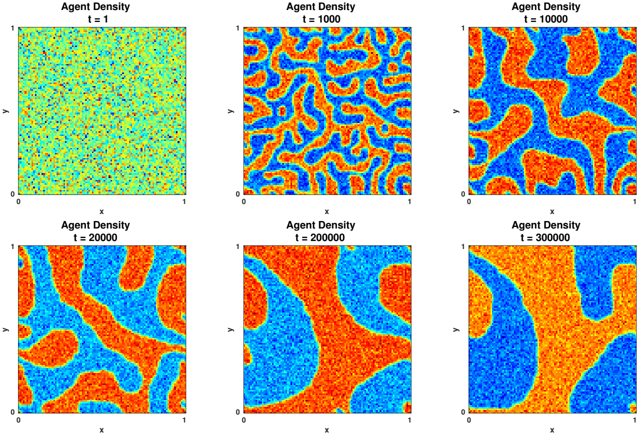


Figure: Temporal evolution of the densities for a segregated phase.

Segregated Phase: Graffiti Field

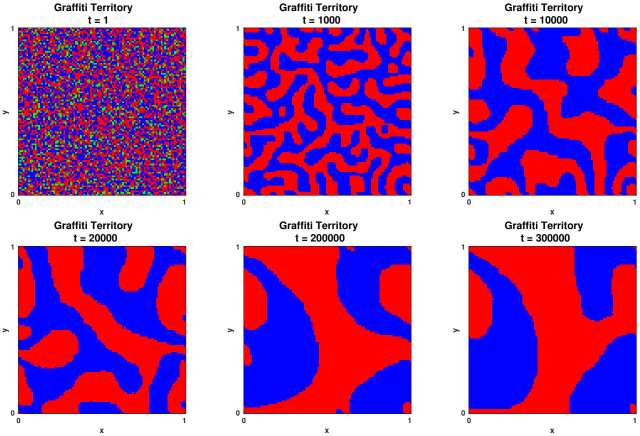


Figure: Temporal evolution of the densities for a segregated phase.

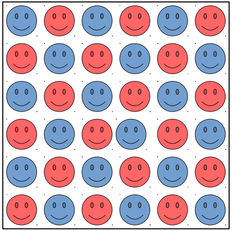
Probability of Movement for Different Beta Values

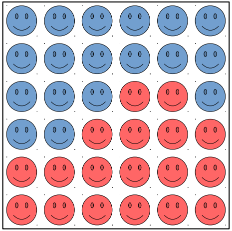
β	$\xi_B = 1 \times 10^5$, M_{left}	$\xi_B = 0.55 \times 10^5$, M_{right}	$\xi_B = 0.5 \times 10^5$, M_{up}	$\xi_B = 0.2 \times 10^5$, M_{down}
1×10^{-6}	0.2392	0.2502	0.2514	0.2591
6.5×10^{-6}	0.1849	0.2478	0.2560	0.3111
2×10^{-5}	0.0898	0.2209	0.2442	0.4449

Table: Probabilities of an agent from gang A moving to a neighbouring site for different β values.

Phase Transitions

- Phase transitions can be observed in many models for collective dynamics
  - Different macroscopic behaviors depending on the parameter values
- Think of solid, liquid, and gas phases from physics
- Order parameters, Hamiltonians, and energies can be defined to help track the phase transition





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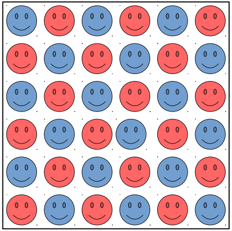

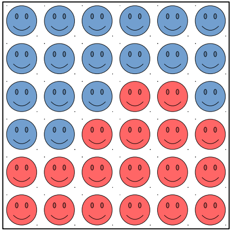

Phase Transitions in Kinetic Models

- In the context of kinetic models, phase transitions come up frequently
- These models often exhibit collective dynamics in one parameter regime, but not in another
- The collective dynamics are then considered as a phase
- Consider flocking models:
  - In one regime (e.g. high noise regime), there is very little order in the way the agents are moving
  - In another regime (e.g. low noise), the agents align and move together in an organized way
  - Polarity can be useful as an order parameter to track the phase

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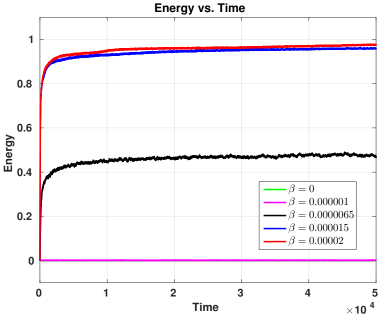
Phase Transition in the Discrete Model

We use an "'energy function" to examine the system phases.

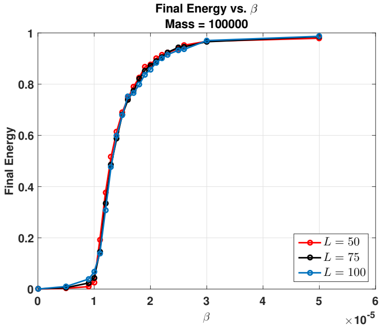
- The energy at time t is defined as

$$\mathcal{E}(t) = \frac{1}{4}\left(\frac{1}{LN}\right)^2 \sum_{(x,y)\in S}\ \sum_{(\tilde{x},\tilde{y})\sim(x,y)} (\rho_A - \rho_B)(\tilde{\rho}_A - \tilde{\rho}_B).$$



- The system has high energy in segregated phase.
- The system has low energy in well-mixed phase.

Phase Transitions for Different Mass Values



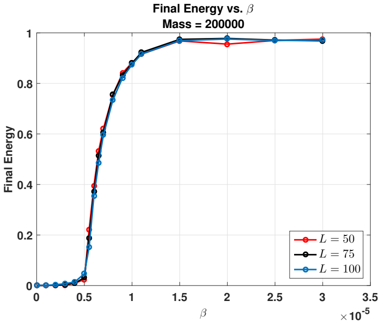


Figure: The energy at the final time step against β for different lattice sizes and number of agents.

Phase Transitions for Different Ratio Values

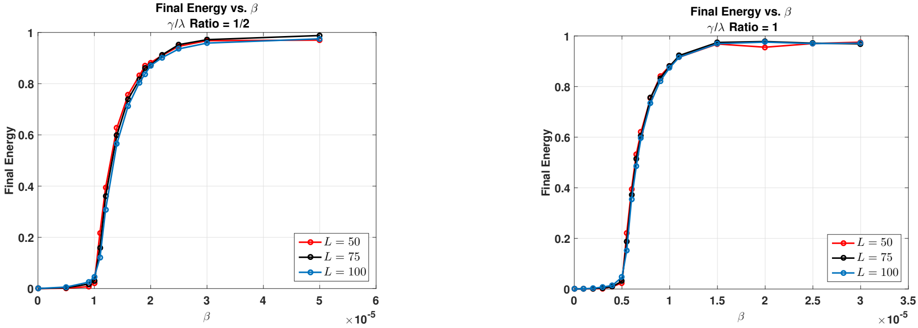


Figure: The energy at the final time step against β for different lattice sizes and $\frac{\gamma}{\lambda}$ ratios.

Deriving the Macroscopic Model from the Microscopic

- Recall the probability of movement from site x_1 to site x_2:

$$M_A(x_1 \to x_2, y_1 \to y_2, t) := \frac{e^{-\beta \xi_B(x_2, y_2, t)}}{\sum_{(\tilde{x},\tilde{y})\sim(x_1,y_1)} e^{-\beta \xi_B(\tilde{x},\tilde{y},t)}}.$$

- We know the density of agents at site (x, y) at time $t + \delta t$:

$$\begin{aligned}\rho_A(x,y,t+\delta t) =& \rho_A(x,y,t) + \sum_{(\tilde{x},\tilde{y})\sim(x,y)} \rho_A(\tilde{x},\tilde{y},t) M_A(\tilde{x} \to x, \tilde{y} \to y, t) \\ & - \rho_A(x,y,t) \sum_{(\tilde{x},\tilde{y})\sim(x,y)} M_A(x \to \tilde{x}, y \to \tilde{y}, t)\end{aligned}$$

- We also have the density of the graffiti for each gang at site (x, y) at time $t + \delta t$:

$$\xi_A(x,y,t+\delta t) = \xi_A(x,y,t) - \delta t \cdot \lambda \cdot \xi_A(x,y,t) + \delta t \cdot \gamma \cdot \rho_A(x,y,t)$$

- Our goal now is to find macroscopic equations which govern the evolution of these four quantities over time.

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Deriving a Macroscopic Version of the Graffiti Density

- Evolution of the graffiti density:

$$\xi_A(x, y, t + \delta t) = \xi_A(x, y, t) - \delta t \cdot \lambda \cdot \xi_A(x, y, t) + \delta t \cdot \gamma \cdot \rho_A(x, y, t)$$

- Obtain a discrete derivative on the left-hand side:

$$\frac{\xi_A(x, y, t + \delta t) - \xi_A(x, y, t)}{\delta t} = -\lambda \xi_A(x, y, t) + \gamma \rho_A(x, y, t)$$

- Simply taking $\delta t \to 0$, we have our two macroscopic equations:

$$\frac{\partial \xi_A}{\partial t}(x, y, t) = \gamma \rho_A(x, y, t) - \lambda \xi_A(x, y, t)$$
$$\frac{\partial \xi_B}{\partial t}(x, y, t) = \gamma \rho_B(x, y, t) - \lambda \xi_B(x, y, t)$$

Deriving a Macroscopic Version of the Agent Density

- The derivation of the equations for the agent density are more complicated
- Considering the discrete equations:

$$M_A(x_1 \to x_2, y_1 \to y_2, t) := \frac{e^{-\beta \xi_B(x_2, y_2, t)}}{\sum_{(\tilde{x},\tilde{y}) \sim (x_1, y_1)} e^{-\beta \xi_B(\tilde{x},\tilde{y},t)}}$$

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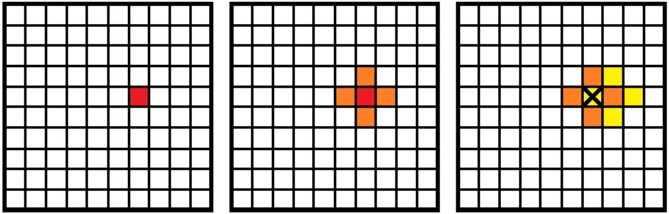
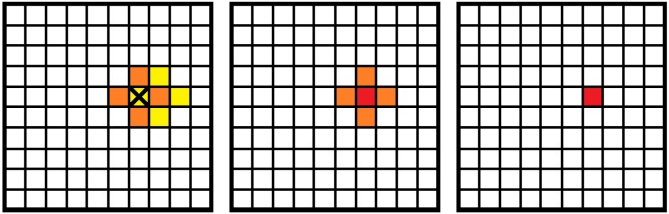


Deriving a Macroscopic Version of the Agent Density

- We formally derive a macroscopic version
  - Taylor expansion
- To circumnavigate the complication arising from the neighbors' neighbors, we employ the discrete spatial Laplacian in two dimensions:

$$\Delta f(x,y,t) = \frac{1}{l^2}\left[\sum_{(\tilde{x},\tilde{y})\sim(x,y)} f(\tilde{x},\tilde{y},t) - 4f(x,y,t)\right] + \mathcal{O}(l^2),$$

- We apply the discrete Laplacian several times to get rid of the dependence on the neighboring points.



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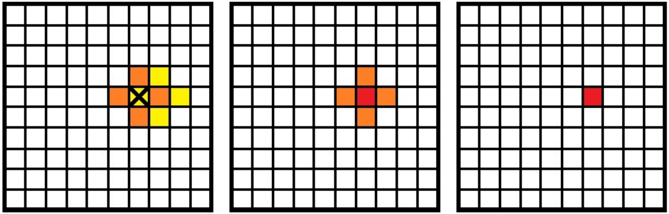

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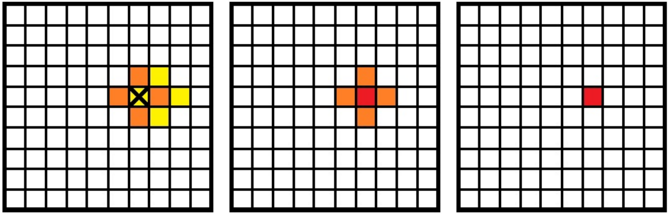


Continuum Equations

The full system of continuum equations is

$$\frac{\partial \xi_A}{\partial t}(x,y,t) = \gamma \rho_A(x,y,t) - \lambda \xi_A(x,y,t)$$
$$\frac{\partial \xi_B}{\partial t}(x,y,t) = \gamma \rho_B(x,y,t) - \lambda \xi_B(x,y,t)$$
$$\frac{\partial \rho_A}{\partial t}(x,y,t) = \frac{D}{4}\nabla \cdot \left[\nabla \rho_A(x,y,t) + 2\beta\big(\rho_A(x,y,t)\nabla \xi_B(x,y,t)\big)\right]$$
$$\frac{\partial \rho_B}{\partial t}(x,y,t) = \frac{D}{4}\nabla \cdot \left[\nabla \rho_B(x,y,t) + 2\beta\big(\rho_B(x,y,t)\nabla \xi_A(x,y,t)\big)\right]$$

Linear Stability Analysis

- We linearize our model by considering a perturbation of the equilibrium solution

$$
\begin{aligned}
\xi_A &= \bar{\xi_A} + \delta_{\xi_A} e^{\alpha t} e^{ikx} \\
\xi_B &= \bar{\xi_B} + \delta_{\xi_B} e^{\alpha t} e^{ikx} \\
\rho_A &= \bar{\rho_A} + \delta_{\rho_A} e^{\alpha t} e^{ikx} \\
\rho_B &= \bar{\rho_B} + \delta_{\rho_B} e^{\alpha t} e^{ikx},
\end{aligned}
$$

Solving for eigenvalue α, we find the following four eigenvalues:

$$
\begin{aligned}
\alpha_{1,2} &= -\frac{1}{8}\left(4\lambda + D|k|^2 \pm \sqrt{16\lambda^2 - 8D(\lambda + 4\beta\gamma\sqrt{\bar{\rho_A}\bar{\rho_B}})|k|^2 + D^2|k|^4}\right) \\
\alpha_{3,4} &= -\frac{1}{8}\left(4\lambda + D|k|^2 \pm \sqrt{16\lambda^2 - 8D(\lambda - 4\beta\gamma\sqrt{\bar{\rho_A}\bar{\rho_B}})|k|^2 + D^2|k|^4}\right).
\end{aligned}
$$

Critical β for Continuum Model

- α_1, α_2, and α_3 are always negative
- However, α_4 is positive for

$$\beta \geq \frac{1}{2(\frac{\gamma}{\lambda})\sqrt{\bar{\rho}_A \bar{\rho}_B}},$$

- This defines a critical β for the continuum system

Comparing Discrete and Continuous Critical β

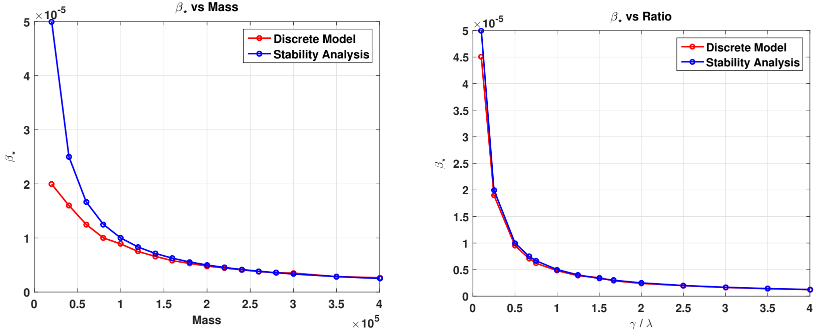
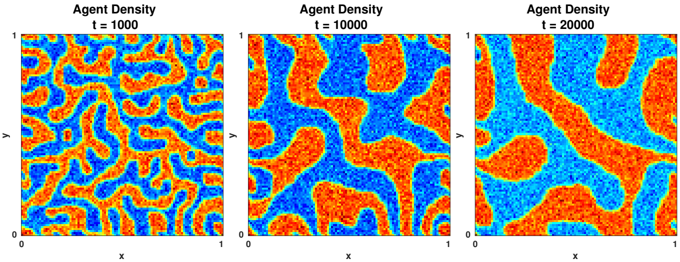


Figure: Critical β comparision.

Future Work

- Numerical solution of the coupled system:
  - how does it compare to the discrete model's evolution?
- Analysis of coupled system:
  - energy, stability, coarsening rates, etc.



Thank you for your attention!