On the Dynamics of Social and Political Conflicts Looking for the Black Swan

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Lecture 1 - Index

Lecture 1 - Reasoning on Complex Systems

1.1 A brief historical excursus on complex systems

E. Kant, (1790), *Critique de la raison pure, Traduction Francaise, Press Univ. de France, 1967*
Living systems: Special structures organized and with the ability to chase a purpose.

E. Schrödinger, (1943), *What is Life?*
Living systems have the ability to extract entropy to keep their own at low levels.

From www.mathaware.org (2011) *We live in a complex world. Many familiar examples of complex systems might be found in very different entities at very different scales: from power grids, to transportation systems, from financial markets, to the Internet, and even in the underlying environment to cells in our bodies. Mathematics and statistics can guide us in unveiling, defining and understanding these systems,in order to enhance their reliability and improve their performance.*

Lecture 1 - Reasonings on Complex Systems

- E. Mayr, **What Evolution Is**, (Basic Books, New York, 2001).
- A.L. Barabasi **Linked. The New Science of Networks**, (Perseus Publishing, Cambridge Massachusetts, 2002.)
- F. Schweitzer, **Brownian Agents and Active Particles**, (Springer, Berlin, 2003).
- M. Talagrand, **Spin Glasses, a Challenge to Mathematicians**, Springer, (2003).
- M.A. Nowak, **Evolutionary Dynamics**, Princeton Univ. Press, (2006).
- N.N. Taleb, **The Black Swan: The Impact of the Highly Improbable**, 2007.
- N.Bellomo, **Modelling Complex Living Systems. A Kinetic Theory and Stochastic Game Approach**, (Birkhauser-Springer, Boston, 2008).
- K. Sigmund, **The Calculus of Selfishness**, Princeton Univ. Press, (2011).

1.2 Do Living, hence complex, systems exhibit common features?

10 Selected Common Features

- ***1. Ability to express a strategy:*** Living entities are capable to develop specific *strategies* related to their *organization ability* depending on the state and entities in their surrounding environment. These can be expressed without the application of any principle imposed by the outer environment.

- ***2. Heterogeneity:*** The said ability is not the same for all entities. Indeed, *heterogeneous behaviors* characterize a great part of living systems. Namely, the characteristics of interacting entities can even differ from an entity to another belonging to the same structure.

- ***3. Large number of components:*** *Complexity in living systems is induced by a large number of variables*, which are needed to describe their overall state. Therefore, the number of equations needed for the modeling approach may be too large to be practically treated.

1 - Reasoning on Complex Systems

- ***4. Interactions:*** Interactions are nonlinearly additive and involve immediate neighbors, but in some cases also distant particles, as living systems have the ability to communicate and may possibly choose different observation paths. In some cases, the topological distribution of a fixed number of neighbors can play a prominent role in the development of the strategy and interactions.

- ***5. Stochastic games:*** Living entities at each interaction *play a game* with an output that is technically related to their strategy often related to surviving and adaptation ability, namely to a personal search of fitness. The output of the game is not generally deterministic even when a causality principle is identified.

- ***6. Learning ability:*** Living systems have the *ability to learn from past experience*. Therefore their strategic ability and the characteristics of interactions among living entities evolve in time.

- ***7. Darwinian selection and time as a key variable:*** All living systems are evolutionary. For instance birth processes can generate individuals more fitted to the environment, who in turn generate new individuals again more fitted to the outer environment.

1 - Reasoning on Complex Systems

• ***8. Multiscale aspects:*** The study of complex living systems always needs a *multiscale approach*. For instance in biology, the dynamics of a cell at the molecular (genetic) level determines the ability of cells to express specific functions.

• ***9. Emerging behaviors:*** Large living systems *show collective emerging behaviors* that are not directly related to the dynamics of a few interacting entities. Generally, emerging behaviors are reproduced only at a qualitative level.

• ***10. Large deviations:*** The observation of living systems should focus on the emerging behaviors that appear in non-equilibrium conditions. Very similar input conditions reproduce, in several cases, the qualitative behaviors. However, large deviations can be observed corresponding to small changes in the input conditions.

1.3 What is the Black Swan?

It is worth detailing a little more the expression *Black Swan*, introduced in the specialized literature for indicating unpredictable events, which are far away from those generally observed by repeated empirical evidence. According to the definition by Taleb a Black Swan is specifically characterized as follows:

"A Black Swan is a highly improbable event with three principal characteristics: It is unpredictable; it carries a massive impact; and, after the fact, we concoct an explanation that makes it appear less random, and more predictable, than it was."

N. N. Taleb, **The Black Swan: The Impact of the Highly Improbable**, Random House, New York City, 2007.

Since it is very difficult to predict directly the onset of a black swan, it is useful looking for *the presence of early signals*

M. Scheffer, J. Bascompte, W. A. Brock, V. Brovkin, S. R. Carpenter, V. Dakos, H. Held, E. H. van Nes, M. Rietkerk, and G. Sugihara, Early-warning signals for critical transitions, *Nature*, 461:53–59, 2009.

1.4 The Metamorphosis by Escher

Workgroup on Complex Dynamical Systems

Complexity and Mathematics

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"To those who do not know mathematics, it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in". (Richard Feynman)

1.4 The Metamorphosis by Escher

Let us imagine, with the help of our fantasy, the time is on the x-axis. Some free (imaginative) interpretations are as follows:

i) The landscape evolves in time going first from a geometrical village made of similarly looking houses (looking like boxes) to a real village with heterogeneous distribution of the shape of houses;

ii) The evolution is related to interactions. Definitely multiple ones; but are they nonlinear?

iii) Is the evolution selective in some weak Darwinian sense?

iv) Does the tower exists in reality?

v) Suddenly the landscape changes from a village it becomes a chess plate, the only connection is a Bridge and a Tower. Can this sudden change be interpreted as a “Black Swan”?

Lecture 2 - Index

2 - Mathematical Tools and Sources of Nonlinearity

2.1 Functional Subsystems and Representation

Hallmarks of the kinetic theory of active particles:

- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- The state of each functional subsystem is defined by a suitable, time dependent, *probability distribution over the activity variable*;
- Interactions are modeled by games, more precisely stochastic games, where the state of the interacting particles and the output of the interactions are known in probability;
- Interactions are delocalized and nonlinearly additive;
- The evolution of the probability distribution is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.

2 - Mathematical Tools and Sources of Nonlinearity

2.1 Functional Subsystems and Representation

Consider active particles in a node for functional subsystems labeled by the subscript i. The description of the system is delivered by the ***generalized one-particle distribution functions***

$$f_i = f_i(t, \mathbf{x}, \mathbf{v}, u) \; : \quad [0, T] \times \Omega \times D_{\mathbf{v}} \times D_u \; \to_+ ,$$

such that $f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}\, du$ denotes the number of active particles whose state, at time t, is in the interval $[\mathbf{w}, \mathbf{w} + d\mathbf{w}]$. where $\mathbf{w} = \mathbf{x}, \mathbf{v}, u$ is an element of the ***space of the microscopic states***. Marginal densities are computed as follows:

$$f_i^m(t, \mathbf{x}, \mathbf{v}) = \int_{D_u} f_i(t, \mathbf{x}, \mathbf{v}, u)\, du\,, \qquad f_i^a(t, \mathbf{x}, u) = \int_{D_{\mathbf{v}}} f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}.$$

$$f_i^b(t, u) = \int_{D_{\mathbf{x}} \times D_{\mathbf{v}}} f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}.$$

$$\nu[f_i](t, \mathbf{x}) = \int_{D_{\mathbf{v}} \times D_u} f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du = \int_{D_{\mathbf{v}}} f_i^m(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}.$$

2 - Mathematical Tools and Sources of Nonlinearity

First order moments provide either ***linear mechanical macroscopic*** quantities, or ***linear activity macroscopic*** quantities.

- The ***flux of particles***, at the time t in the position $\mathbf{x}$, is given by

$$\mathbf{Q}[f_i](t,\mathbf{x}) = \int_{D_{\mathbf{v}}\times D_u} \mathbf{v}\, f_i(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}\, du = \int_{D_{\mathbf{v}}} f_i^m(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}.$$

- The ***mass velocity*** of particles, at the time t in the position $\mathbf{x}$, is given by

$$\mathbf{U}[f_i](t,\mathbf{x}) = \frac{1}{\nu_i[f_i](t,\mathbf{x})} \int_{D_{\mathbf{v}}\times D_u} \mathbf{v}\, f_i(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}\, du.$$

- The **activity** terms are computed as follows:

$$a[f_i](t,\mathbf{x}) = \int_{D_{\mathbf{v}}\times D_u} u f_i(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}\, du\,,$$

while the ***local activation density*** is given by:

$$a^d[f_i](t,\mathbf{x}) = \frac{a_j[f_i](t,\mathbf{x})}{\nu_i[f_i](t,\mathbf{x})} = \frac{1}{\nu_i[f_i](t,\mathbf{x})} \int_{D_{\mathbf{v}}\times D_u} u f_i(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}\, du.$$

2 - Mathematical Tools and Sources of Nonlinearity

2.2 Stochastic Games

- ***Test*** particles with microscopic state u, at the time t: $f_{ij} = f_{ij}(t, u)$.
- ***Field*** particles with microscopic state, u_* at the time t: $f_{ij} = f_{ij}(t, u_*)$.
- ***Candidate*** particles with microscopic state, u^* at the time t: $f_{ij} = f_{ij}(t, u^*)$.

The interaction rule is as follows:

i) The ***candidate*** particle interacts with ***field*** particles and acquires, in probability, the state of the ***test*** particle. ***Test*** particles interact with field particles and lose their state.

ii) Interactions can: modify the microscopic state of particles; generate proliferation or destruction of particles in their microscopic state; and can also generate a particle in a new functional subsystem.

Loss of determinism: Modeling of systems of the inert matter is developed within the framework of deterministic causality principles, which does not any longer holds in the case of the living matter.

2.2 Stochastic Games

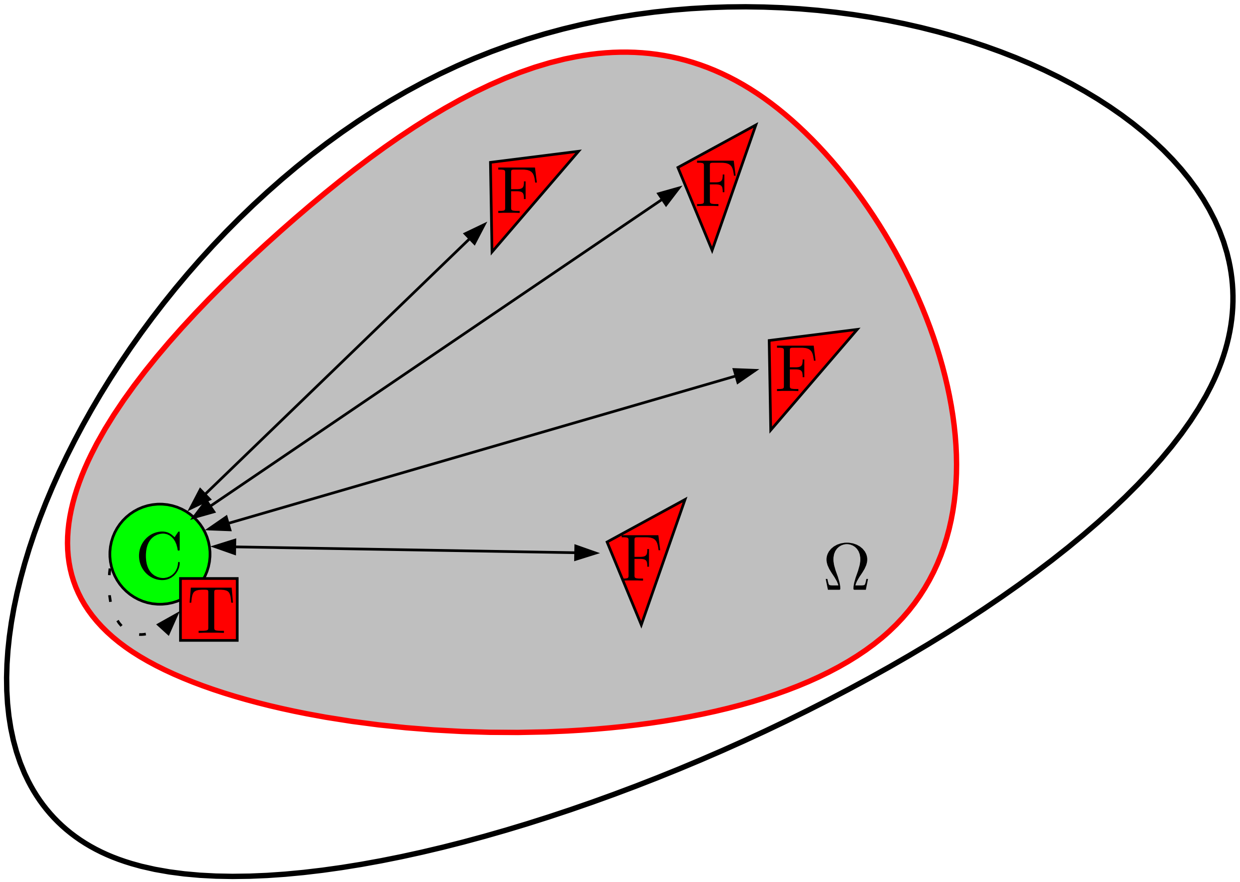


Figure 1: – Active particles interact with other particles in their action domain

2 Mathematical Tools and Sources of Nonlinearity

Interactions with modification of activity and transition: Generation of particles into a new functional subsystem occurs through pathways. Different paths can be chosen according to the dynamics at the lower scale.

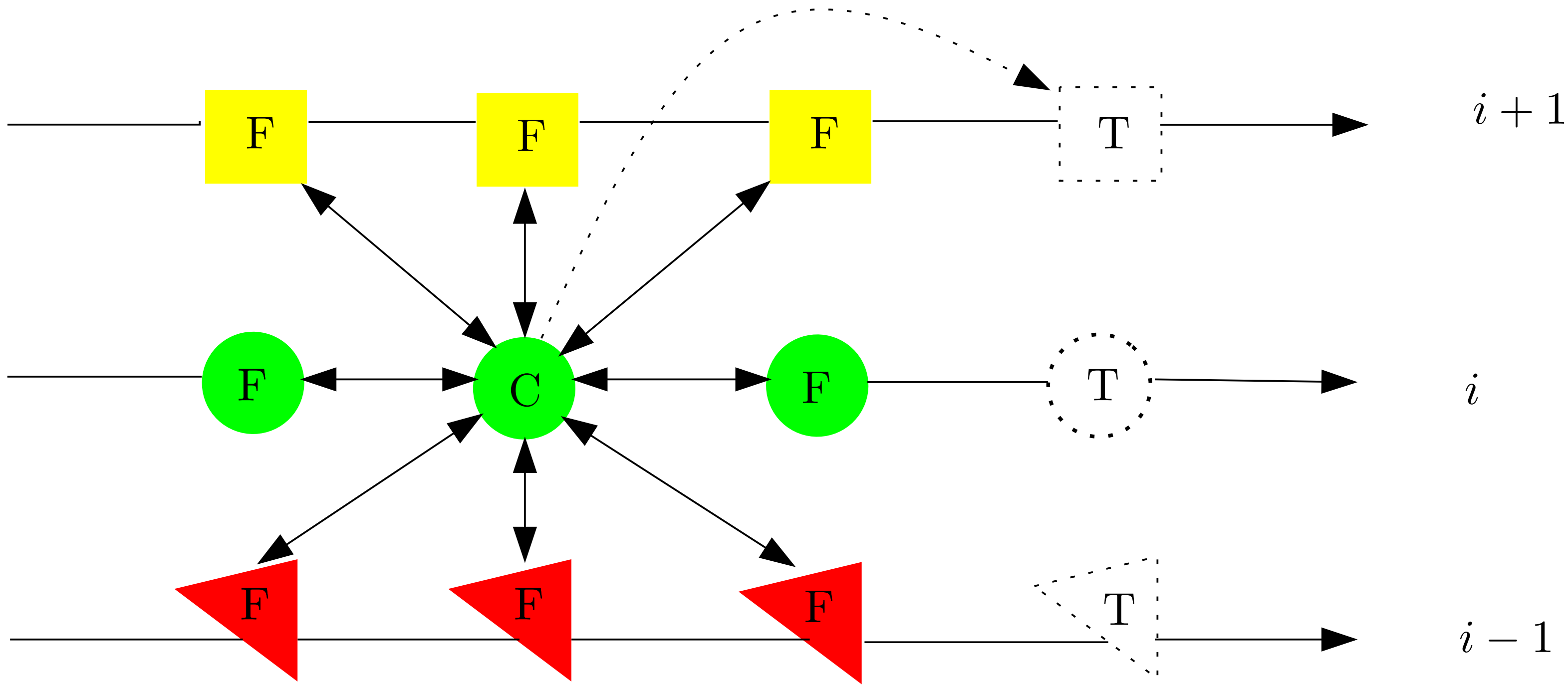


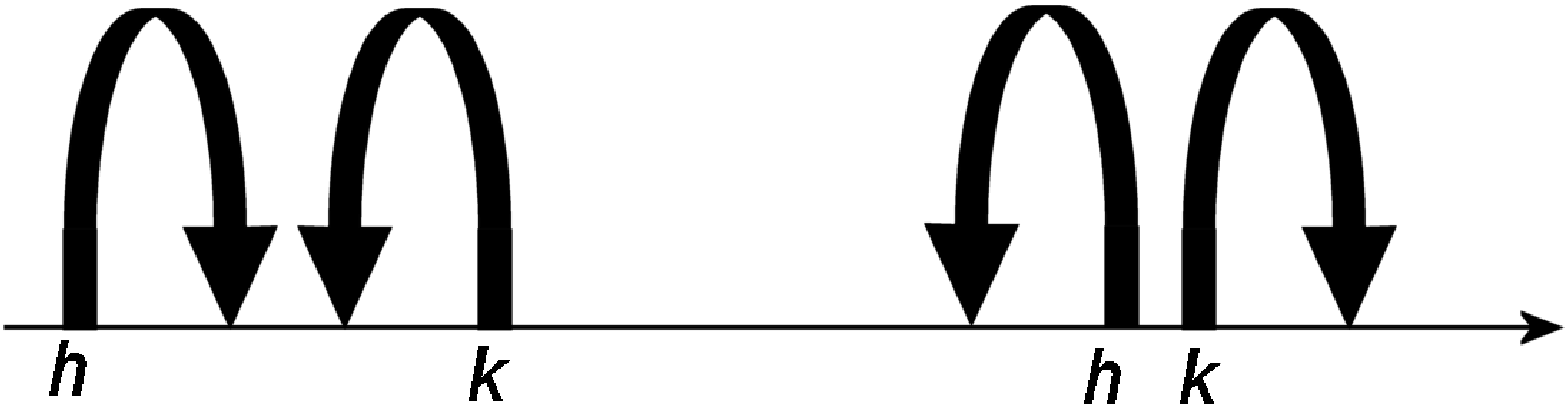
Figure 2: – Active particles during proliferation move from one functional subsystem to the other through pathways.

2.2 Stochastic Games

Cooperative/Competitive Games

Cooperative behavior The active particle with state $h < k$ improves its state by gaining from the particle k , which cooperates loosing part of its state.

Competitive behavior The active particle with state $h < k$ decreases its state by contributing to the particle k, which due to competition increases part of its state.



2.3 Mathematical Structures

Balance within the space of microscopic states

VARIATION RATE OF THE NUMBER OF CELLS IN THE ELEMENTARY VOLUME OF THE STATE SPACE

=

INLET FLUX DUE TO CONSERVATIVE INTERACTIONS

OUTLET FLUX DUE TO CONSERVATIVE INTERACTIONS

+

FLUX DUE TO PROLIFERATIVE INTERACTIONS

−

FLUX DUE TO DESTRUCTIVE INTERACTIONS

+

NET FLUX DUE TO INTERACTIONS WITH MUTATIONS

2.3 Mathematical Structures

H.1. Candidate or test particles in $\mathbf{x}$, interact with the field particles in $\mathbf{x}^* \in \Omega$ located in the interaction domain Ω. Interactions are weighted by the ***interaction rate*** $\eta_{hk}[\mathbf{f}](\mathbf{x}^*)$ supposed to depend on the local density in the position of the field particles.

H.2. Candidate particle modifies its state according to the term defined as follows: $\mathcal{A}_{hk}[\mathbf{f}](\mathbf{v}_* \to \mathbf{v},\, u_* \to u|\mathbf{v}_*, \mathbf{v}^*, u_*, u^*)$, which denotes the probability density that a candidate particles of the h-subsystems with state $\mathbf{v}_*, u_*$ reaches the state $\mathbf{v}, u$ after an interaction with the field particles k-subsystems with state $\mathbf{v}^*, u^*$.

H.3. Candidate particle, in $\mathbf{x}$, can proliferate, due to encounters with field particles in $\mathbf{x}^*$, with rate μ^i_{hk}, which denotes the proliferation rate into the functional subsystem i, due the encounter of particles belonging the functional subsystems h and k. Destructive events can occur only within the same functional subsystem.

2.3 Mathematical Structures

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\; f_i(t, \mathbf{x}, \mathbf{v}, u) = \left[\sum_{j=1}^{n} \mathcal{C}_{ij}[\mathbf{f}] + \sum_{h=1}^{n} \sum_{k=1}^{n} \mathcal{S}^i_{hk}[\mathbf{f}] \right] (t, \mathbf{x}, \mathbf{v}, u)\,,$$

where

$$\begin{aligned} \mathcal{C}_{ij}[\mathbf{f}] &= \int_{\Omega \times D_u^2 \times D_{\mathbf{v}}^2} \eta_{ij}[\mathbf{f}](\mathbf{x}^*)\, \mathcal{A}_{ij}[\mathbf{f}](\mathbf{v}_* \to \mathbf{v},\, u_* \to u | \mathbf{v}_*, \mathbf{v}^*, u_*, u^*) \\ &\times \quad f_i(t, \mathbf{x}, \mathbf{v}_*, u_*) f_j(t, \mathbf{x}^*, \mathbf{v}^*, u^*)\, d\mathbf{v}_*\, d\mathbf{v}^*\, du_*\, du^*\, d\mathbf{x}^*\,, \\ &- \quad f_i(t, \mathbf{x}, \mathbf{v}) \int_{\Omega \times D_u \times D_{\mathbf{v}}} \eta_{ij}[\mathbf{f}](\mathbf{x}^*)\, f_j(t, \mathbf{x}^*, \mathbf{v}^*, u^*)\, d\mathbf{v}^*\, du^*\, d\mathbf{x}^* \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}^i_{hk}[\mathbf{f}] &= \int_{\Omega \times D_u^2 \times D_{\mathbf{v}}} \eta_{hk}[\mathbf{f}](\mathbf{x}^*)\, \mu^i_{hk}[\mathbf{f}](u_*, u^*) \\ &\times \quad f_h(t, \mathbf{x}, \mathbf{v}, u_*) f_k(t, \mathbf{x}^*, \mathbf{v}^*, u^*)\, d\mathbf{v}^*\, du_*\, du^*\, d\mathbf{x}^*. \end{aligned}$$

2.3 Mathematical Structures

The Spatially Homogeneous Case

The evolution equation follows from a balance equation for net flow of particles in the elementary volume of the space of the microscopic state by transport and interactions:

$$\partial_t f_i(t,u) + \mathcal{F}_i(t)\,\partial_u f_i(t,u) = \mathcal{C}_i[\mathbf{f}](t,u) + \mathcal{S}_i[\mathbf{f}](t,u),$$

where

$$\begin{aligned} C_i[\mathbf{f}] &= \sum_{j=1}^{n} \int_{D_u \times D_u} \psi_{ij}[\mathbf{f}]\mathcal{B}_{ij}[\mathbf{f}]\,(u_* \to u | u_*, u^*) \\ &- f_i(t,u) \sum_{j=1}^{n} \int_{D_u} \psi_{ij}[\mathbf{f}]\, f_j(t,u^*)\, du^*. \end{aligned}$$

and

$$\mathcal{S}_i[\mathbf{f}] = \sum_{h=1}^{n} \sum_{k=1}^{n} \int_{D_u} \int_{D_u} \psi_{hk}[\mathbf{f}]\, \mu_{hk}^{i}[\mathbf{f}](u_*, u^*) f_h(t,u^*) f_k(t,u^*)\, du^*.$$

2.4 Sources of Nonlinearity

Hierarchic distance: The rate of interactions within the same functional subsystem depends on the distance of the interacting microstates. However, when two active particles belong to different functional subsystems, the rate differs depending on the distance between the functional states.

Affinity distance: Two systems with closed distribution, $d_{ad} = ||f_h - f_k||$, are *affine* and hence they interact with higher frequency.

Domain of influence: The communication is effective only within a certain domain of influence of the micro-state. Topological rather than metric distance.

Output of the interaction: An active particle interacts with all particles in its interaction domain. The output of the interaction depends on $\mathbf{f}$ within the interaction domain. For instance it can depend on the moments computed for the particles within the interaction domain as shown again on the modeling of swarms.

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1. Reasoning on Complex Systems

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3. From Welfare Policy to Social Conflicts and Political Competition

3.1 Preliminary Reasonings

3.2 Modeling of the Dynamics of Welfare Distribution

3.3 Simulations

4. Looking at the Beautiful Shapes of Swarms

3 From Welfare Policy to Social Conflicts and Political Competition

3.1. Preliminary Reasonings

A.-L. Barabási, R. Albert, and H. Jeong, Mean-field theory for scale-free random networks, *Physica A*, 272(1–2):173–187, 1999.

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G. Ajmone Marsan *New Paradigms and Mathematical Methods for Complex Systems in Behavioral Economics*. PhD thesis, IMT Lucca, Italy, 2009.

3 - From Welfare Policy to Social Conflicts and Political Competition

3.1. Preliminary Reasonings

- The dynamics of social and economic systems are necessarily based on individual behaviors, by which single subjects express, either consciously or unconsciously, a particular strategy, which is heterogeneously distributed. The latter is often based not only on their own individual purposes, but also on those they attribute to other agents.

- In the last few years, a radical philosophical change has been undertaken in social and economic disciplines. An interplay among Economics, Psychology, and Sociology has taken place, thanks to a new cognitive approach no longer grounded on the traditional assumption of rational socio-economic behavior. Starting from the concept of bounded rationality, the idea of Economics as a subject highly affected by individual (rational or irrational) behaviors, reactions, and interactions has begun to impose itself.

- A key experimental feature of such systems is that interaction among heterogeneous individuals often produces unexpected outcomes, which were absent at the individual level, and are commonly termed emergent behaviors.

3 - From Welfare Policy to Social Conflicts and Political Competition

• The new point of view promoted the image of Economics as an evolving complex system, where interactions among heterogeneous individuals produce unpredictable emerging outcomes. In this context, setting up a mathematical description able to capture the evolving features of socio-economic systems is a challenging, however difficult, task, which calls for a proper interaction between mathematics and social sciences.

• **Mathematical models might focus, in particular, on the prediction of the so called *Black Swan*.** The latter is defined to be a rare event, showing up as an irrational collective trend generated by possibly rational individual behaviors.

• Socio-economic systems can be described as ensembles of several living entities able to develop **behavioral strategies**, by which they interact with each other. For this reason, such entities are also called **active particles**. Typically, their strategies are **heterogeneously distributed** and **change in time** in consequence of the interactions, since active particles can update them by **learning from past experiences**.

3 - From Welfare Policy to Social Conflicts and Political Competition

3.2 Modeling of the Dynamics of Welfare Distribution

Let u be a variable denoting the strategy expressed by active particles, that in the present context can be identified with their wealth status. We are interested in it is customary to speak of *wealth classes*, which implies that the strategy u is a discrete variable taking values in a lattice $I_u = \{u_1, \ldots, u_i, \ldots, u_n\}$.
The number of individuals that, at a certain time t, are expressing the strategy u_i, or, in other words, that belong to the i-th wealth class, is given by a distribution function

$$f_i = f_i(t) : [0, T_{\max}] \to_{+}, \quad i = 1, \ldots, n,$$

$T_{\max} > 0$ being a certain final time (possibly $+\infty$). Notice that, up to normalizing with respect to the total number of active particles, $\mathbf{f} = (f_1, \ldots, f_i, \ldots, f_n)$ is a time-evolving discrete probability distribution over the lattice I_u.

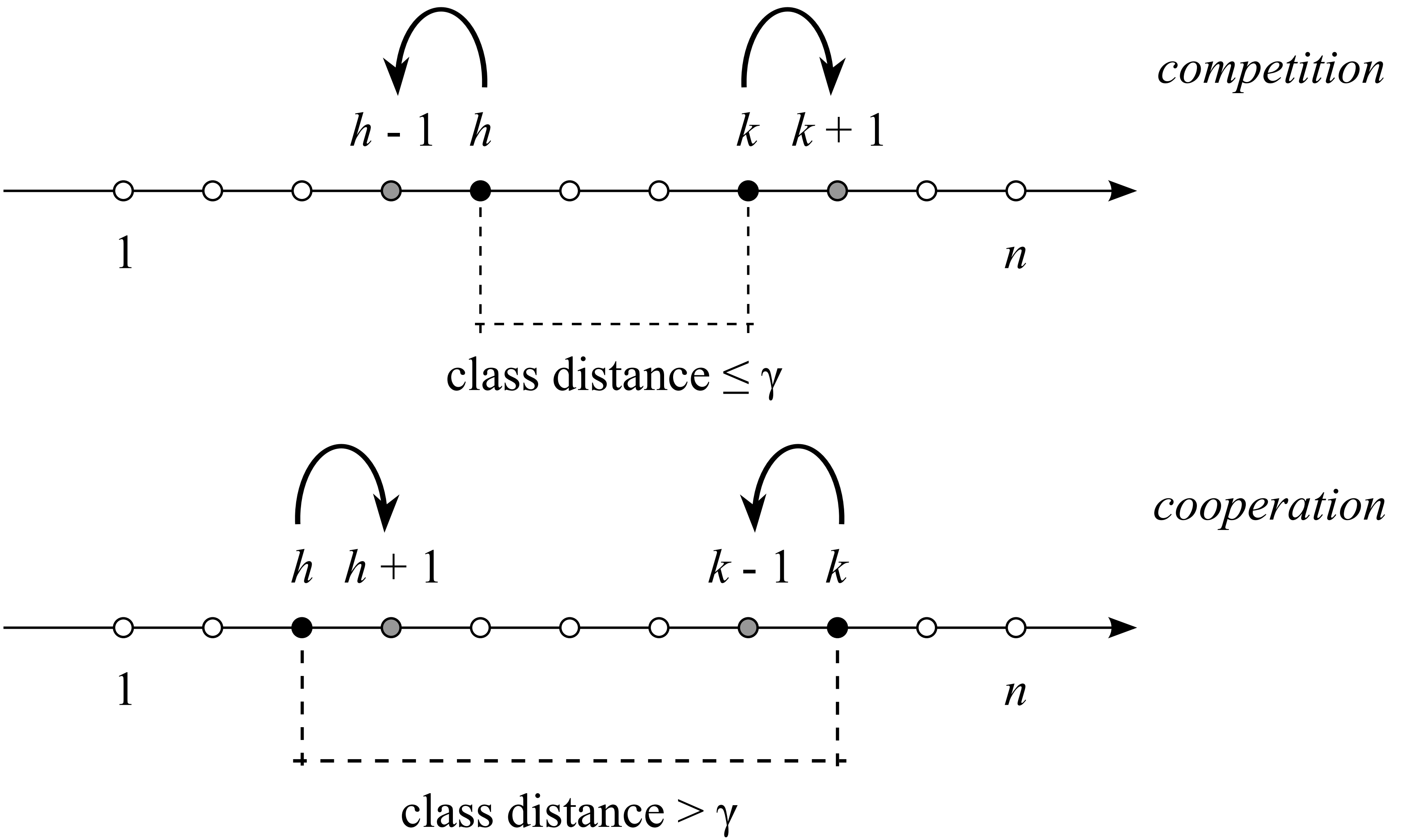
$$\frac{df_i}{dt} = \sum_{h,\,k=1}^{n} \eta_{hk} \mathbf{B}_{hk}(i) f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k, \quad i = 1, \ldots, n.$$

3 - From Welfare Policy to Social Conflicts and Political Competition

- *Consensus* – The candidate particle sees its state either increased, by profiting from a field particle with a higher state, or decreased, by pandering to a field particle with a lower state. After mutual interaction, the states of the particles become closer than before the interaction.
- *Dissensus* – The candidate particle sees its state either further decreased, by facing a field particle with a higher state, or further increased, by facing a field particle with a lower state. After mutual interaction, the states of the particles become farther than before the interaction.

A critical distance triggers either cooperation or competition among the classes. If the distance is lower than the critical one then a competition takes place, which causes a further enrichment of the wealthier class and a further impoverishment of the poorer one. Conversely, if the actual distance is greater than the critical one then the social organization forces cooperation.

Modeling Consesus/dissensus Games



3 - From Welfare Policy to Social Conflicts and Political Competition

The characteristics of the present framework are summarized as follows.

- *Interaction rate*. Two different rates of interaction are considered, corresponding to competitive and cooperative interactions, respectively.
- *Strategy leading to the transition probabilities*. When interacting with other particles, each active particle plays a game with stochastic output. If the difference of wealth class between the interacting particles is lower than a critical distance $\gamma[\mathbf{f}]$ (where, here and henceforth, square brackets indicate a functional dependence on the probability distribution $\mathbf{f}$) then the particles compete in such a way that those with higher wealth increase their state against those with lower wealth. Conversely, if the difference of wealth class is higher than $\gamma[\mathbf{f}]$ then the opposite occurs. The critical distance evolves in time according to the global wealth distribution over wealthy and poor particles.

3 - From Welfare Policy to Social Conflicts and Political Competition

1. **Interaction between equally wealthy particles**, viz. $h = k$:

$$\begin{cases} \mathbf{B}_{hh}(h) = 1 \\ \mathbf{B}_{hh}(i) = 0, \; \forall\, i \neq h \end{cases} \tag{1}$$

2. **Cooperative interaction**, viz. $h \neq k$ and $|h - k| > \gamma[\mathbf{f}]$:

$$h < k \begin{cases} \mathbf{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathbf{B}_{hk}(h+1) = \alpha_{hk} \\ \mathbf{B}_{hk}(i) = 0, \; \forall\, i \neq h,\; h+1 \end{cases}$$

$$h > k \begin{cases} \mathbf{B}_{hk}(h-1) = \alpha_{hk} \\ \mathbf{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathbf{B}_{hk}(i) = 0, \; \forall\, i \neq h-1,\; h, \end{cases} \tag{2}$$

where it is assumed that interactions within the same class produce no effect.

3 - From Welfare Policy to Social Conflicts and Political Competition

3. **Competitive interaction,** viz. $h \neq k$ and $|h-k| \leq \gamma[\mathbf{f}]$:

$$h = 1,\, n \begin{cases} \mathbf{B}_{hk}(h) = 1 \\ \mathbf{B}_{hk}(i) = 0,\ \forall\, i \neq h \end{cases}$$

$$h \neq 1,\, n \begin{cases} h < k \begin{cases} k < n \begin{cases} \mathbf{B}_{hk}(h-1) = \alpha_{hk} \\ \mathbf{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathbf{B}_{hk}(i) = 0,\ \forall\, i \neq h-1,\, h \end{cases} \\ k = n \begin{cases} \mathbf{B}_{hn}(h) = 1 \\ \mathbf{B}_{hn}(i) = 0,\ \forall\, i \neq h \end{cases} \end{cases} \\ h > k \begin{cases} k = 1 \begin{cases} \mathbf{B}_{h1}(h) = 1 \\ \mathbf{B}_{h1}(i) = 0,\ \forall\, i \neq h \end{cases} \\ k > 1 \begin{cases} \mathbf{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathbf{B}_{hk}(h+1) = \alpha_{hk} \\ \mathbf{B}_{hk}(i) = 0,\ \forall\, i \neq h,\, h+1 \end{cases} \end{cases} \end{cases} \tag{3}$$

3 - From Welfare Policy to Social Conflicts and Political Competition

The parameter $\alpha_{hk} \in [0,\ 1]$ has the following meaning:

- In case of competition, it is the probability that the candidate particle further increases (or decreases) its wealth if it is richer (or poorer) than the field particle;
- In case of cooperation, it is the probability that the candidate particle gains (or transfers) part of its wealth if it is poorer (or richer) than the field particle.

This probability may be constant or may depend on the wealth classes, e.g.,

$$\alpha_{hk} = \frac{|k-h|}{n-1},$$

in such a way that the larger the distance between the interacting classes the more stressed the effect of cooperation or competition. The **interaction rate** $\eta_{hk} \geq 0$ is assumed to be piecewise constant over the wealth classes:

$$\eta_{hk} = \begin{cases} \eta_0 & \text{if } |k-h| \leq \gamma[\mathbf{f}] \text{ (competition)}, \\ \mu\eta_0 & \text{if } |k-h| > \gamma[\mathbf{f}] \text{ (cooperation)}, \end{cases}$$

where $\eta_0 > 0$ is a constant to be hidden in the time scale and $0 < \mu \leq 1$.

3 - From Welfare Policy to Social Conflicts and Political Competition

The **critical distance** $\gamma[\mathbf{f}]$ is here assumed to depend on the instantaneous distribution of the active particles over the wealth classes, such that the time evolution of $\gamma[\mathbf{f}]$ should translate the following phenomenology of social competition:

- in general, $\gamma[\mathbf{f}]$ grows with the number of poor active particles, thus causing larger and larger gaps of social competition. Few wealthy active particles insist on maintaining, and possibly improving, their benefits;
- in a population constituted almost exclusively by poor active particles $\gamma[\mathbf{f}]$ attains a value such that cooperation is inhibited, for individuals tend to be involved in a "battle of the have-nots";
- conversely, in a population constituted almost exclusively by wealthy active particles $\gamma[\mathbf{f}]$ attains a value such that competition is inhibited, because individuals tend preferentially to cooperate for preserving their common benefits.

3 - From Welfare Policy to Social Conflicts and Political Competition

$$S[\mathbf{f}] := N^-[\mathbf{f}] - N^+[\mathbf{f}] = \sum_{i=1}^{\frac{n-1}{2}} f_i(t) - \sum_{i=\frac{n+3}{2}}^{n} f_i(t).$$

- $S[\mathbf{f}] = S_0 \Rightarrow \gamma[\mathbf{f}] = \gamma_0$, where S_0, γ_0 are a reference social gap and the corresponding reference critical distance, respectively;
- $S[\mathbf{f}] = 1 \Rightarrow \gamma[\mathbf{f}] = n$, which implies that when the population is composed by poor particles only ($N^- = 1$, $N^+ = 0$) the socio-economic dynamics are of full competition;
- $S[\mathbf{f}] = -1 \Rightarrow \gamma[\mathbf{f}] = 0$, which implies that, conversely, when the population is composed by wealthy particles only ($N^- = 0$, $N^+ = 1$) the socio-economic dynamics are of full cooperation.

$$\gamma[\mathbf{f}] = \frac{2\gamma_0(S[\mathbf{f}]^2 - 1) - n(S_0 + 1)(S[\mathbf{f}]^2 - S_0)}{2(S_0^2 - 1)} + \frac{n}{2} S[\mathbf{f}],$$

where $\cdot$ denotes integer part (floor).

3 - From Welfare Policy to Social Conflicts and Political Competition

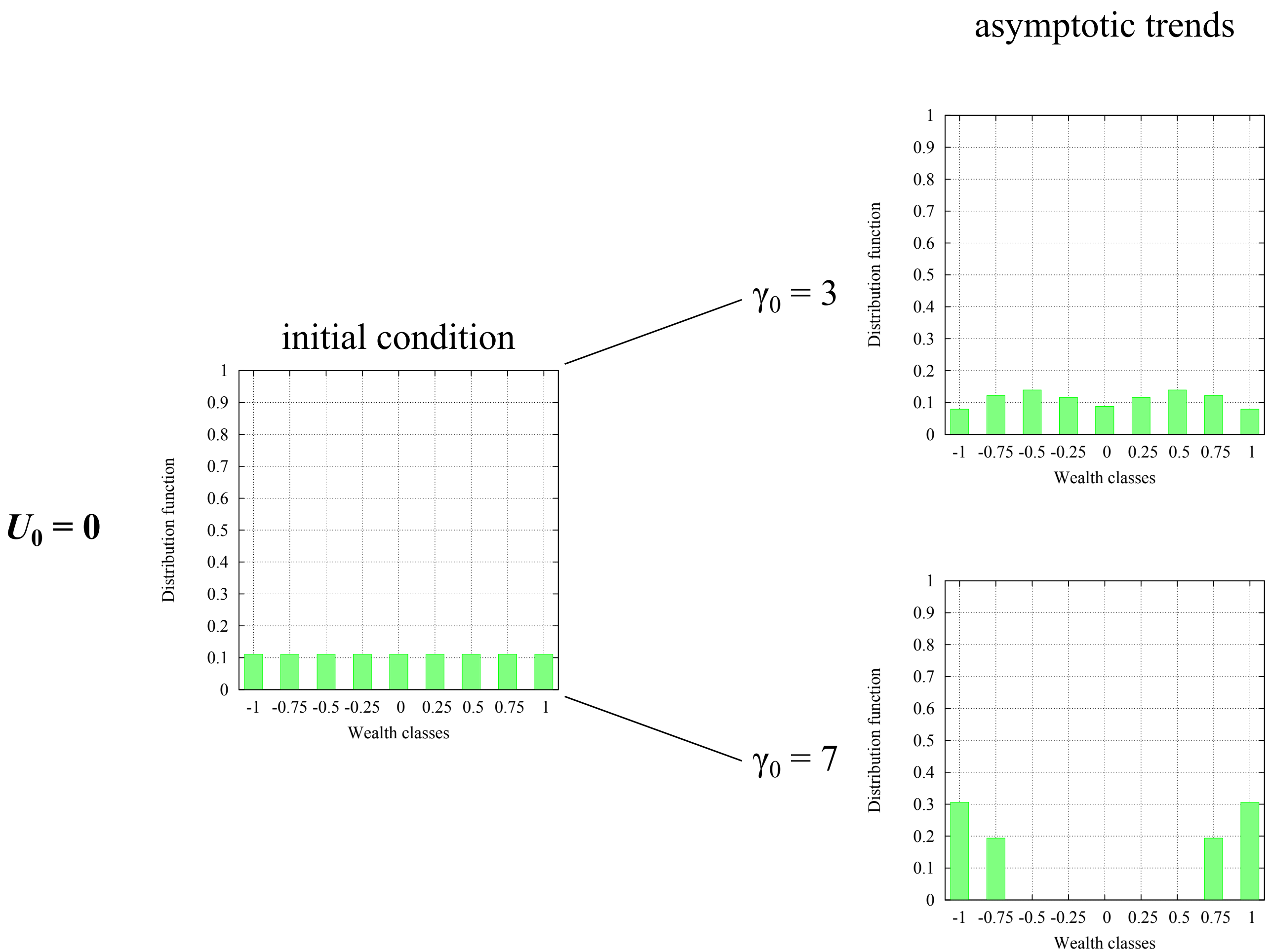
Simulations

The evolution of the system predicted by the model depends essentially on the four parameters n (the number of wealth classes), μ (the relative encounter rate for cooperation, U_0 (the average wealth of the population), and γ_0 (the reference critical distance). In more detail:

- $n = 9$ and $\mu = 0.3$ are selected;
- two case studies for U_0 are addressed, namely $U_0 = -0.4 < 0$ and $U_0 = 0$, in order to compare, respectively, the economic dynamics of a society in which poor classes dominate with those of a society in which the initial distribution of active particles encompasses uniformly poor and rich classes;
- in addition, in each of the case studies above the asymptotic configurations for both constant and variable $\gamma[\mathbf{f}]$ are investigated, assuming, for duly comparison, that in the former the critical distance coincides with γ_0. Particularly, $\gamma_0 = 3$, corresponding to a mainly cooperative attitude, and $\gamma_0 = 7$, corresponding instead to a strongly competitive attitude, are chosen.

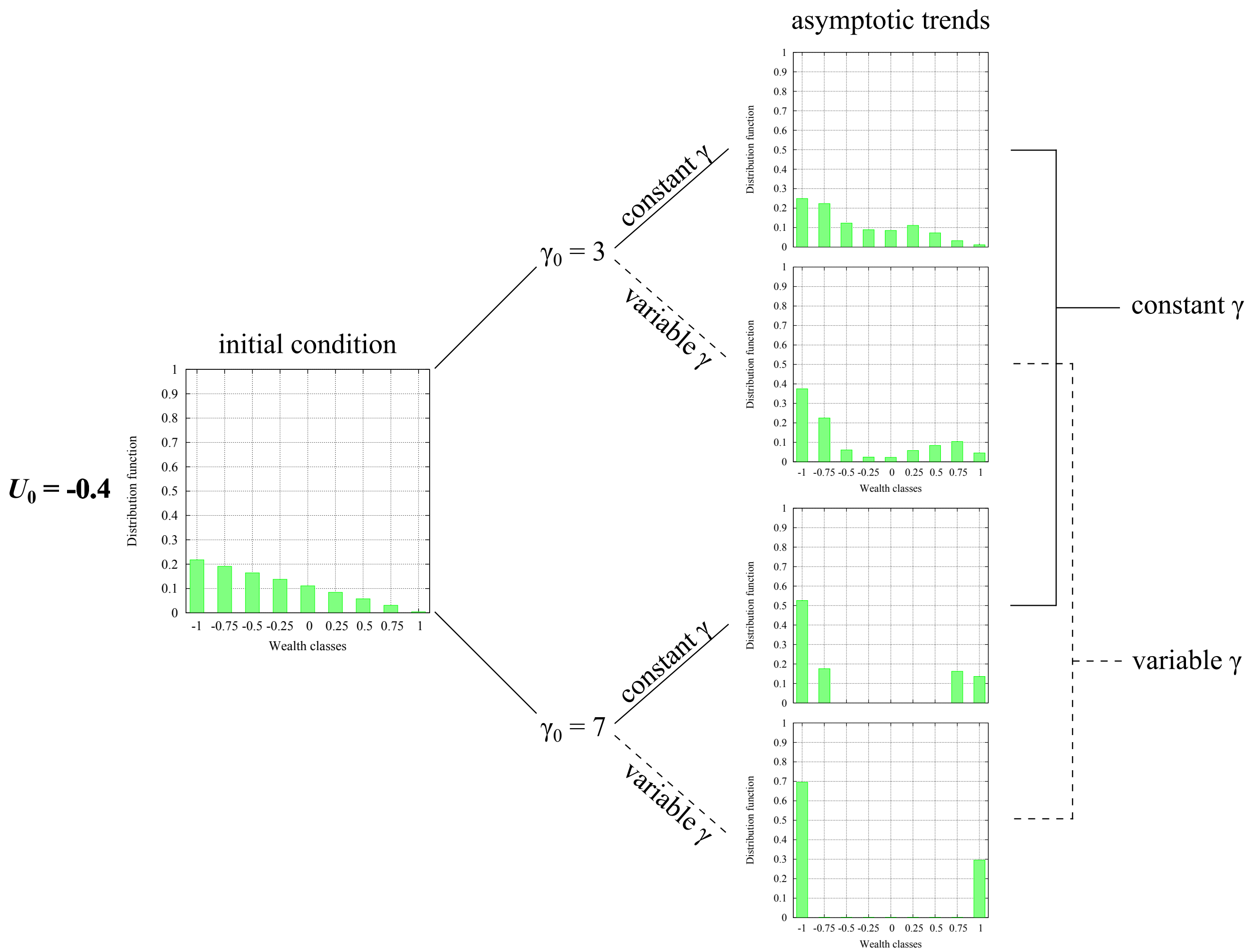
3 - From Welfare Policy to Social Conflicts and Political Competition

Simulations



3 - From Welfare Policy to Social Conflicts and Political Competition

Simulations



3 - Modeling Social Conflicts and Political Competition

Welfare Policy and Political Competition

The mathematical framework presented does not account for a possible interplay between different interaction dynamics due to concomitant social and economic issues. Let us consider the case of a multiple strategy (u, v), with u representing the wealth status of the active particles and v their level of support/opposition to a Government policy. Let us investigate how the welfare dynamics can induce changes of personal opinions in terms of support/opposition to a certain political regime.

Assume that also v is a discrete variable taking values in a lattice

$$I_v = \{v_1 = -1, \ldots, v_{\frac{m+1}{2}} = 0, \ldots, v_m = 1\},$$

$$v_r = \frac{2}{m-1} r - \frac{m+1}{m-1}, \quad r = 1, \ldots, m,$$

agreeing that $v_1 = -1$ corresponds to the strongest opposition whereas $v_m = 1$ to the maximum support.

3 - Modeling Social Conflicts and Political Competition

The mathematical structure is as follows:

$$\frac{df_i^r}{dt} = \sum_{p,\,q=1}^{m} \sum_{h,\,k=1}^{n} \eta_{hk}^{pq} \mathbf{B}_{hk}^{pq}(i,\, r) f_h^p f_k^q - f_i^r \sum_{q=1}^{m} \sum_{k=1}^{n} \eta_{ik}^{rq} f_k^q,$$

for $i = 1, \ldots, n$ and $r = 1, \ldots, m$, and where

$$f_i^r = f_i^r(t) : [0,\, T_{\text{max}}] \to \mathbb{R}_+$$

is the number of individuals that, at time t, are expressing the strategy $(u_i,\, v_r)$, or, in other words, that belong to the wealth class u_i and to the opinion class v_r. In addition:

• η_{hk}^{pq} is the **interaction rate** between candidate particles with strategy $(u_h,\, v_p)$ and field particles with strategy $(u_k,\, v_q)$. The same model fur the welfare policy is used, according to the assumption that interactions among active particle are mainly driven by the wealth status rather than by the difference of political opinion. Thus η_{hk}^{pq} is independent of the opinion classes that candidate and field particles belong to, $\eta_{hk}^{pq} = \eta_{hk}$.

3 - Modeling Social Conflicts and Political Competition

• $\mathbf{B}_{hk}^{pq}(i,\, r)$ is the **probability density** that a candidate particle changes its strategy to the test one $(u_i,\, v_r)$ after interacting with a field particle. Analogously to the one dimensional case, it is required to fulfill the probability density property:

$$\sum_{r=1}^{m}\sum_{i=1}^{n}\mathbf{B}_{hk}^{pq}(i,\, r) = 1, \quad \forall\, h,\, k = 1,\, \dots,\, n, \quad \forall\, p,\, q = 1,\, \dots,\, m.$$

The following factorization on the output test state $(u_i,\, v_r)$ is proposed, relying simply on intuition: $\mathbf{B}_{hk}^{pq}(i,\, r) = \bar{\mathbf{B}}_{hk}^{pq}(i)\hat{\mathbf{B}}_{hk}^{pq}(r)$,

- $\bar{\mathbf{B}}_{hk}^{pq}(i)$ encodes the transitions of wealth class, which are further supposed to be independent of the political feelings of the interacting pairs: $\bar{\mathbf{B}}_{hk}^{pq}(i) = \bar{\mathbf{B}}_{hk}(i)$.
- $\hat{\mathbf{B}}_{hk}^{pq}(r)$ encodes the changes of political opinion resulting from interactions. Coherently with the observation made above that political persuasion is neglected, so that political feelings originate in the individuals in consequence of their own wealth condition, this term is assumed to depend on the wealth status and political opinion of the candidate particle only: $\hat{\mathbf{B}}_{hk}^{pq}(r) = \hat{\mathbf{B}}_{h}^{p}(r)$.

3 - Modeling Social Conflicts and Political Competition

1. Poor individual ($u_h < 0$) in a poor society ($U_0 < 0$):

$$p = 1 \begin{cases} \hat{\mathbf{B}}_h^1(1) = 1 \\ \hat{\mathbf{B}}_h^1(r) = 0, \; \forall\, r \neq 1 \end{cases}$$

$$p > 1 \begin{cases} \hat{\mathbf{B}}_h^p(p-1) = 2\beta \\ \hat{\mathbf{B}}_h^p(p) = 1 - 2\beta \\ \hat{\mathbf{B}}_h^p(r) = 0, \; \forall\, r \neq p-1, \, p \end{cases}$$

2. Wealthy individual ($u_h \geq 0$) in a wealthy society ($U_0 \geq 0$):

$$p < m \begin{cases} \hat{\mathbf{B}}_h^p(p) = 1 - 2\beta \\ \hat{\mathbf{B}}_h^p(p+1) = 2\beta \\ \hat{\mathbf{B}}_h^p(r) = 0, \; \forall\, r \neq p, \, p+1 \end{cases}$$

$$p = m \begin{cases} \hat{\mathbf{B}}_h^m(m) = 1 \\ \hat{\mathbf{B}}_h^m(r) = 0 \, \forall\, r \neq m, \end{cases}$$

3 - Modeling Social Conflicts and Political Competition

3. Wealthy individual ($u_h \geq 0$) in a poor society ($U_0 < 0$) or poor individual ($u_h < 0$) in a wealthy society ($U_0 \geq 0$):

$$p = 1 \quad \begin{cases} \hat{\mathbf{B}}_h^1(1) = 1 - \beta \\ \hat{\mathbf{B}}_h^1(2) = \beta \\ \hat{\mathbf{B}}_h^1(r) = 0, \; \forall\, r \neq 1,\, 2 \end{cases}$$

$$p \neq 1,\, m \begin{cases} \hat{\mathbf{B}}_h^p(p-1) = \beta \\ \hat{\mathbf{B}}_h^p(p) = 1 - 2\beta \\ \hat{\mathbf{B}}_h^p(p+1) = \beta \\ \hat{\mathbf{B}}_h^p(r) = 0, \; \forall\, r \neq p-1,\, p,\, p+1 \end{cases}$$

$$p = m \quad \begin{cases} \hat{\mathbf{B}}_h^m(m-1) = \beta \\ \hat{\mathbf{B}}_h^m(m) = 1 - \beta \\ \hat{\mathbf{B}}_h^m(r) = 0, \; \forall\, r \neq m-1,\, m \end{cases}$$

3 - Modeling Social Conflicts and Political Competition

Welfare Policy and Political Competition

Therefore, changes of political opinion are triggered jointly by the individual wealth status, and the average collective one of the population:

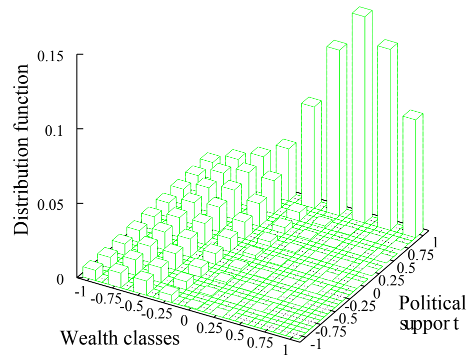
- poor individuals in a poor society tend to distrust markedly the Government policy, sticking in the limit at the strongest opposition;
- wealthy individuals in a poor society and poor individuals in a wealthy society exhibit, in general, the most random behavior. In fact, they may trust the Government policy either because of their own wealthiness, regardless of the possibly poor general condition, or because of the collective affluence, in spite of their own poor economic status. On the other hand, they may also distrust the Government policy either because of the poor general condition, in spite of their individual wealthiness, or because of their own poor economic status, regardless of the collective affluence;
- wealthy individuals in a wealthy society tend instead to trust earnestly the Government policy, sticking in the limit at the maximum support.

3 - Modeling Social Conflicts and Political Competition

Simulations

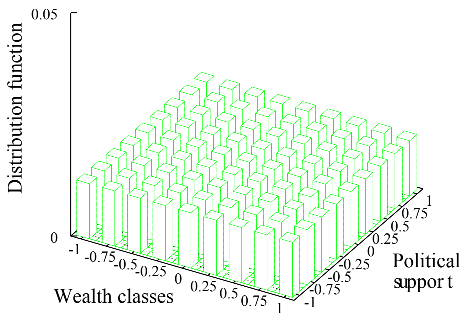
asymptotic trends

$\gamma_0 = 3$

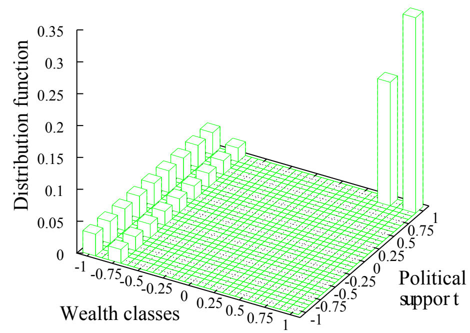


initial condition

$U_0 = 0$



$\gamma_0 = 7$



Simulations

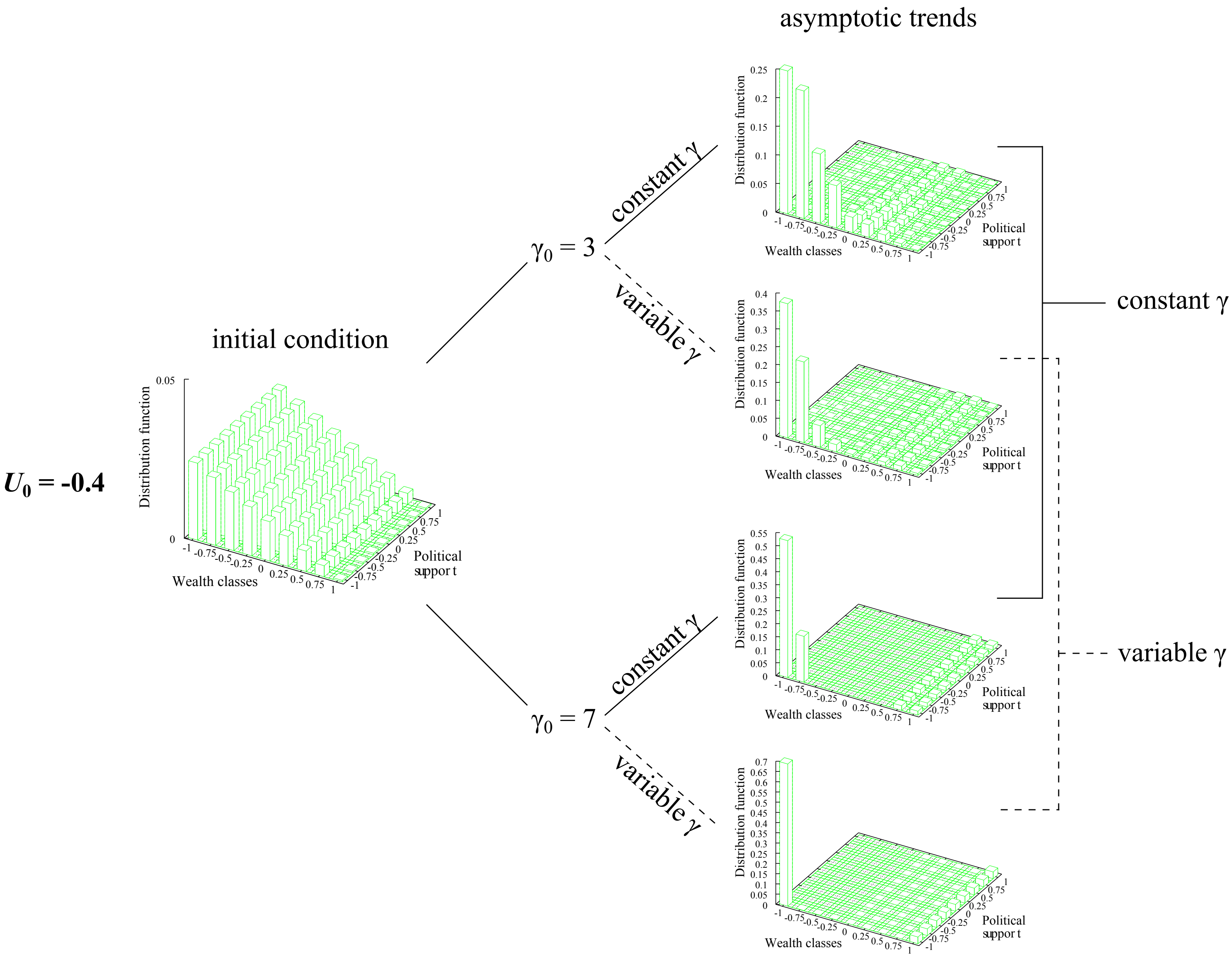
This Figure refers to the case $U_0 = 0$, and shows that:

- In an economically neutral society with uniform wealth distribution not only do wealthy classes stick at an earnest support to the Government policy, but also poor ones do not completely distrust them, especially in a context of prevalent cooperation among the classes ($\gamma_0 = 3$).

- Therefore, this example does not suggest the development of significant polarization in that society, although a greater polarization is observed for higher values of γ.

3 - Modeling Social Conflicts and Political Competition

Simulations



3 - Modeling Social Conflicts and Political Competition

Simulations

The Figure refers to the case $U_0 = -4$, and shows that:

- A strong radicalization of the opposition. The model predicts indeed that, in such a poor society, poor classes stick asymptotically at the strongest opposition, whereas wealthy classes spread over the whole range of political orientations, however with a mild tendency toward opposition for the moderately rich ones (say, $u_5 = 0$, $u_6 = 0.25$, and $u_7 = 0.5$.

- The growth of political aversion is especially emphasized under variable critical distance $\gamma[\mathbf{f}]$, when the marked clustering of the population in the lowest wealth classes, due to a more competitive spontaneous attitude, entails in turn a clustering in the highest distrust of the regime.

3 - Modeling Social Conflicts and Political Competition

Early Signals of a Black Swan: Let us assume that a specific model has a trend to an asymptotic configuration described by stationary distributions $\{\bar{f}_i^r\}_{\substack{i=1,\,\dots,\,n\\ r=1,\,\dots,\,m}}$:

$$\lim_{t \to +\infty} \|\bar{f}_\cdot^r - f_\cdot^r(t)\| = 0, \quad \forall\, r = 1, \ldots, m,$$

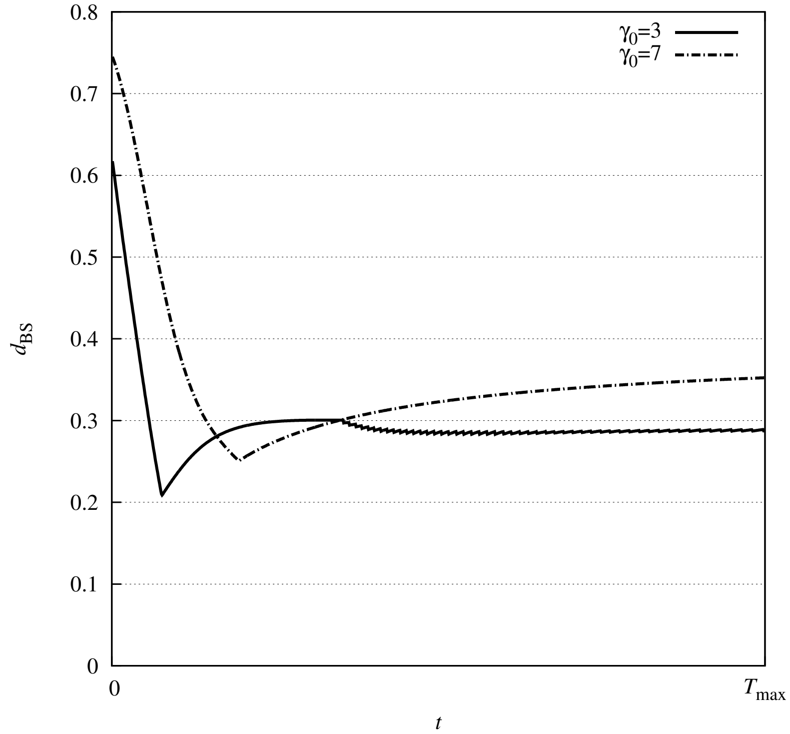
where $\|\cdot\|$ is a suitable norm in ^{n} over the activity $u \in I_u$. In addition, let us assume that the modeled system is expected to exhibit a stationary trend described by some phenomenologically guessed distributions $\{\tilde{f}_i^r\}_{\substack{i=1,\,\dots,\,n\\ r=1,\,\dots,\,m}}$.

Accordingly, we define the following time-evolving distance d_{BS} (the subscript “BS” standing for Black Swan):

$$d_{\mathrm{BS}}(t) := \max_{r=1,\,\dots,\,m} \|\tilde{f}_\cdot^r - f_\cdot^r(t)\|,$$

which, however, will generally not approach zero as time goes by for the heuristic asymptotic distribution does not translate the actual trend of the system. This function can be possibly regarded as one of the *early-warning signals* for the emergence of critical transitions to rare events, because it may highlight the onset of strong deviations from expectations.

Early Signals of the Black Swan



The mapping $t \mapsto d_{\mathrm{BS}}(t)$ computed in the case studies with variable γ, taking as phenomenological guess the corresponding asymptotic distributions obtained with constant γ.

Lecture 4 - Index

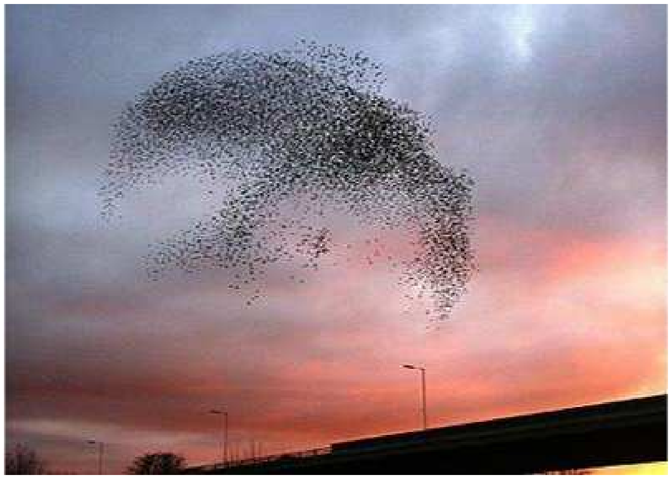
1. **Reasonings on Complex Systems**

2. **2. Mathematical Tools and Sources of Nonlinearity**

3. **Modeling Welfare Policy**

3. **From Welfare Policy to Social Conflicts and Political Competition**

4. **Looking at the Beautiful Shapes of Swarms**



Lecture 4 - Swarms

Selected bibliography on swarms

• M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, Proc. Nat. Acad. Sci., 105(4) (2008), 1232-1237.

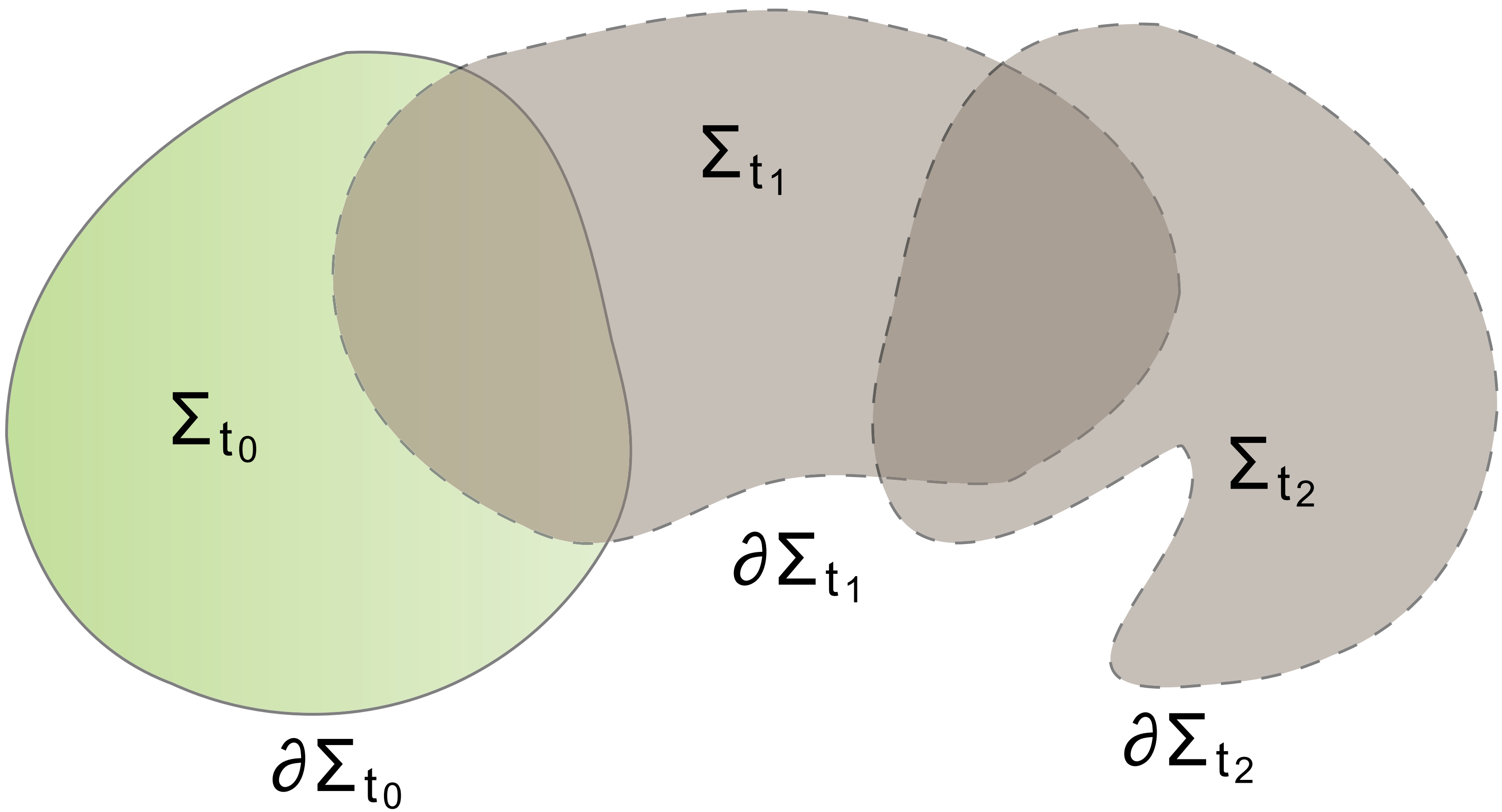
• A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, and R. Tavarone, From empirical data to inter-individual interactions: unveiling the rules of collective animal behavior Math. Models Methods Appl. Sci., 20 (2010), 1491-1510.

• R. Eftimie, G. De Vries, and M.A. Lewis, Complex spatial group patterns result from different animal communication mechanisms, Proc. Nat. Acad. Sci., 104 (2007), 6974-6979.

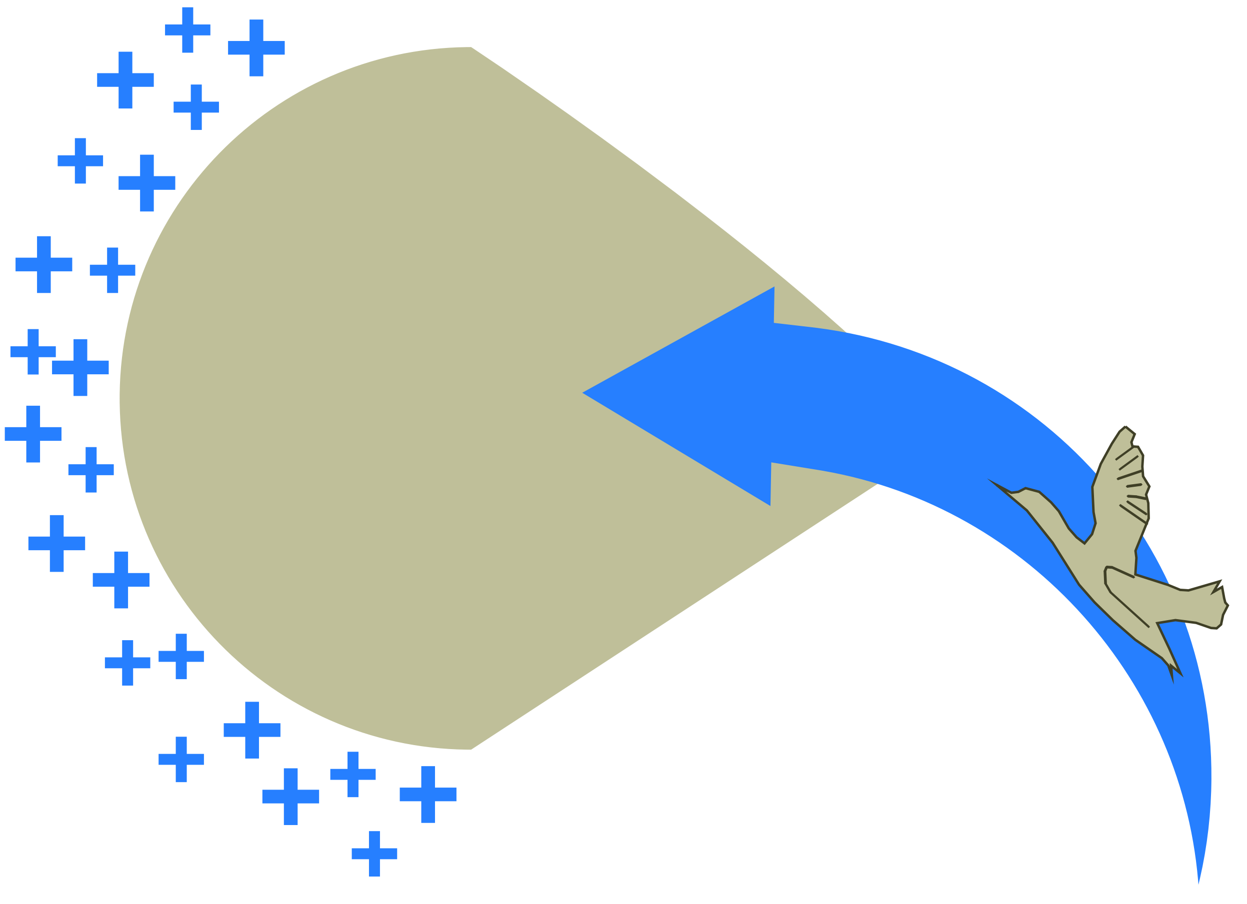
• R. Eftimie, Hyperbolic and kinetic models for self-organized biological aggregations and movement: a brief overview, J. Math. Biology, (2012).

• N. Bellomo and J. Soler, On the mathematical theory of the dynamics of swarms viewed as complex systems, Math. Models Methods Appl. Sci., 22, Suppl. 1: 1140006, (2012).

Dynamics of the swarm



Attack of a predator



Complexity Features of swarms

1. *Ability to express a strategy*
2. *Interactions by topological metrics*
3. *Heterogeneity and Hierarchy*
4. *Learning dynamics*
5. *Large deviations*

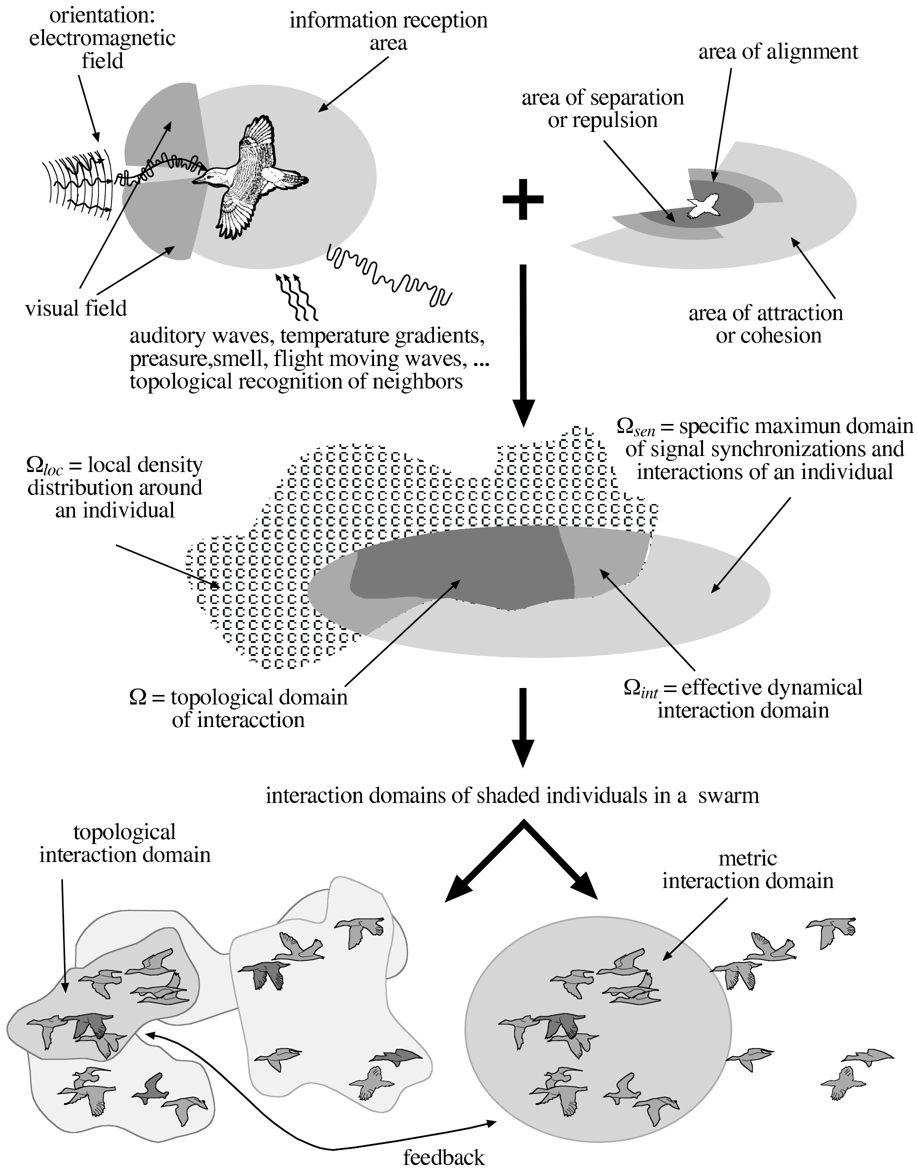
Technical Complexity

– The dynamics of interactions differs in the various zones of the swarm. For instance, from the border to the center of the domain occupied by the swarm. Small stochastic behaviors could be an important characteristics for the dynamics to take into account small fluctuations induced by heterogeneous behaviors.

– Validation of models should be based on their ability to depict observed emerging behaviors, which depends on the type of individuals composing the swarm. Various emerging behaviors can be studied, such as milling formation, flocking phenomena, and many others. Generally, the qualitative behavior is preserved in the case of similar input data; however, small deviations of them may induce large deviations of the quantitative values of the dynamics.

– The number of individuals involved in the swarm is not large enough to justify the approach of continuum mechanics. Moreover, even the continuity assumption of the probability distribution over the microscopic state of the individuals of the swarm needs to be put in discussion and to be treated as an approximation of physical reality.

Lecture 4 - Looking at the Beautiful Shapes of Swarms



Towards a mathematical approach

• The overall state of the system is described by the probability density distribution over the micro-state:

$$f = f(t, \mathbf{x}, v, \theta, u) \ : \quad [0, T] \times \Sigma_t \times [0, 1] \times [0, 2\pi) \times D_u \ \to \ \mathbb{R}_+ \, ,$$

which is positively defined, referred to N, and provides via $f \, d\mathbf{x} \, v dv \, d\theta \, du$ the probability of finding an active particle in the elementary volume of the space of the microscopic states $[\mathbf{x}, \mathbf{x} + d\mathbf{x}] \times [v, v + dv] \times [\theta, \theta + d\theta] \times [u, u + du]$.

• If the particles are subdivided into *functional subsystems* by a hierarchy, the following representation can be used:

$$f = f(t, \mathbf{x}, v, \theta, u) = \sum_{i=1}^{p} f_i(t, \mathbf{x}, v, \theta) \, \delta(u - u_i).$$

Mathematical structures

The strategy to derive these equations follows the guidelines given in the preceding slides, namely, by using a balance equation for net flow of particles in the elementary volume of the space of the microscopic state taking into account transport and interactions.
In this specific case both long range interactions with the whole swarm and local interactions are taken into account referring to the following formal equation:

$$\partial_t f + \mathbf{v} \cdot \partial_{\mathbf{x}} f + \partial_{\mathbf{v}} (\mathcal{F}[f] f) = J[f] \,,$$

where $f = f(t, \mathbf{x}, \mathbf{v})$, while long range interactions within the swarm are modeled by $\mathcal{F}[f](\mathbf{x})$, the left-hand-side models the transport of particles, and where the right-hand side $J[f]$ represents the short range interaction among particles.

Towards a Modeling Approach

The assumptions which lead to a specific model are as follows:
H.1. The *test* particle $\mathbf{x}, \mathbf{v}$ is subject to an action of the whole storm of the type:

$$\mathcal{F}[f](t,\mathbf{x}) = \frac{1}{\rho_M} \int_{\Sigma_t} \varphi(\mathbf{x},\mathbf{x}^*,\mathbf{v},\mathbf{v}^*,\mathbf{U}[f](t))\, f(t,\mathbf{x}^*,\mathbf{v}^*)\, d\mathbf{x}^*\, d\mathbf{v}^*\,,$$

where Σ_t is the domain occupied by the swarm, φ is the individual action of the field particle $(\mathbf{x}^*, \mathbf{v}^*)$ over the test particle. This acceleration term depends on the distance between particles inducing a flocking action, but also induces an attraction towards the mean direction of the swarm:

$$\varphi = \frac{1}{\varepsilon + |\mathbf{x} - \mathbf{x}^*|^a} \psi(\mathbf{v}, \vec{\nu}_S)\,, \quad a > 0, \qquad \vec{\nu}_S = \frac{\mathbf{U}}{|\mathbf{U}|},$$

where ε is a small positive dimensionless quantity. The term ψ models the attraction of individuals toward the main stream of the swarm. It increases with increasing angle between the individual direction and the mean direction of the swarm.

H.2. There is an interaction rate η related to ρ_{Ω_β} as follows:
$\eta[f](t,\mathbf{x}) = \eta_0(t,\mathbf{x})\,\mu[f]$ where, depending on time and position,

$$\mu[f] = \left(\frac{\rho_{\Omega_\beta}}{\rho_{\Omega_{int}}}\right)^b, \quad b > 0\,,$$

and where $\rho_{\Omega_{int}}(t)$ represents the mean density on Ω_{int} and η_0 represents the interaction rate when the two domains coincide.

H.3. The *candidate* particle with microscopic state $(\mathbf{x}_*, \mathbf{v}_*)$ at the time t interacts with *field* particles $(\mathbf{x}^*, \mathbf{v}^*)$ with rate η and acquires the state of the *test* particles. The candidate particle modifies its state according to the probability density $\mathcal{A}$ which depends on the state of the interacting particles and on the mean velocity $\mathbf{V}$:

$$\mathcal{A}[f](\mathbf{v}_* \to \mathbf{v}|\mathbf{V}[f])\,, \qquad \int_{D_\mathbf{v}} \mathcal{A}[f](\mathbf{v}_* \to \mathbf{v}|\mathbf{V}[f])\,d\mathbf{v} = 1\,,$$

for all conditioning inputs, while *test* particles interact with field particles and lose their state. More precisely, Candidate particles have a trend to adjust their velocity both to the mean velocity of the cluster and to that of the field particles.

$$\Big(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} + \partial_{\mathbf{v}} \mathcal{F}[f]\Big) f(t, \mathbf{x}, \mathbf{v}) = J[f](t, \mathbf{x}, \mathbf{v})$$

$$= \int_{\Omega_\beta[f]} \int_{(D_{\mathbf{v}})^2} \mu[f]\, \mathcal{A}(\mathbf{v}_* \to \mathbf{v} | \mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f]) f(t, \mathbf{x}, \mathbf{v}_*)\, f(t, \mathbf{x}^*, \mathbf{v}^*)\, d\mathbf{x}^*\, d\mathbf{v}_*\, d\mathbf{v}^*$$

$$- f(t, \mathbf{x}, \mathbf{v}) \int_{\Omega_\beta[f]} \int_{D_{\mathbf{v}}} \mu[f]\, f(t, \mathbf{x}^*, \mathbf{v}^*)\, d\mathbf{x}^*\, d\mathbf{v}^*.$$

This structure can be rapidly generalized to the presence of a hierarchy, as follows:

$$\Big(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} + \sum_{j=1}^{p} \partial_{\mathbf{v}} \mathcal{F}_{ij}[f]\Big) f_i(t, \mathbf{x}, \mathbf{v}) = J_i[f](t, \mathbf{x}, \mathbf{v}) = \sum_{j=1}^{p} J_{ij}[f](t, \mathbf{x}, \mathbf{v}),$$

$$J_{ij} = \int_{\Omega_\beta[f]} \int_{(D_{\mathbf{v}})^2} \mu[\mathbf{f}]\, \mathcal{A}_{ij}[f](\mathbf{v}_* \to \mathbf{v}) f_i(t, \mathbf{x}, \mathbf{v}_*)\, f_j(t, \mathbf{x}^*, \mathbf{v}^*)\, d\mathbf{x}^*\, d\mathbf{v}_*\, d\mathbf{v}^*$$

$$- f_i(t, \mathbf{x}, \mathbf{v}) \int_{\Omega_\beta[f]} \int_{D_{\mathbf{v}}} \mu[\mathbf{f}]\, f_j(t, \mathbf{x}^*, \mathbf{v}^*)\, d\mathbf{x}^*\, d\mathbf{v}^*.$$

The density within the interaction domain Ω_β is computed as follows:

$$\rho_{\Omega_\beta}(t) = \rho_{\Omega_\beta}[f](t) = \int_{D_v} \int_{\Omega_\beta} f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}.$$

Moreover, the mean velocity of the particles included in Ω is computed by

$$\mathbf{V}_{\Omega_\beta}(t) = \mathbf{V}_{\Omega_\beta}[f](t) = \frac{1}{\rho_{\Omega_\beta}(t)} \int_{D_v} \int_{\Omega_\beta} \mathbf{v}\, f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}.$$

This is the mean velocity within the cluster and we define the direction

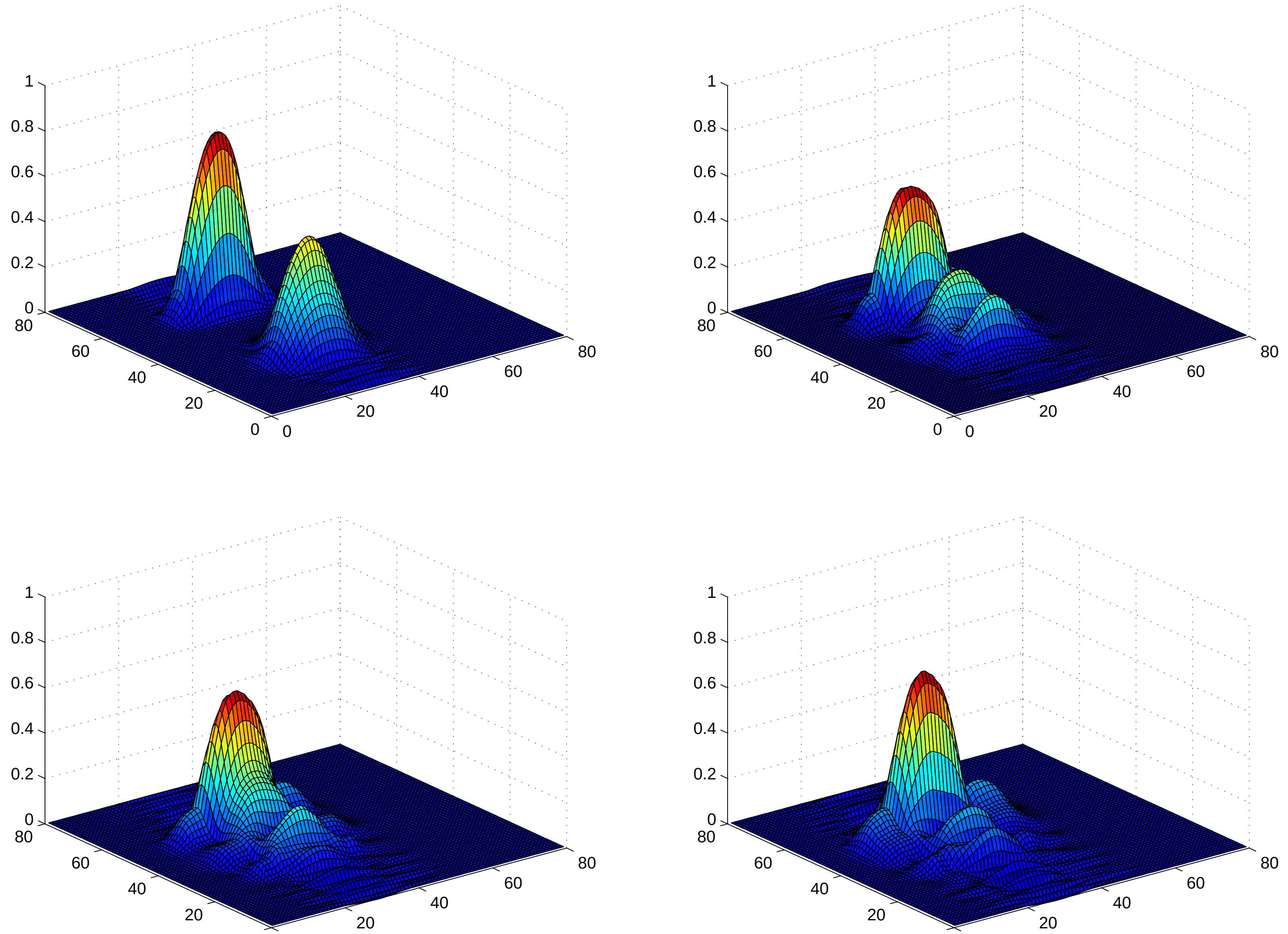
$$\vec{\nu}_{\Omega_\beta}(t) = \frac{\mathbf{V}_{\Omega_\beta}(t)}{|\mathbf{V}_{\Omega_\beta}(t)|}.$$

An additional quantity of interest is the mean velocity of the whole storm:

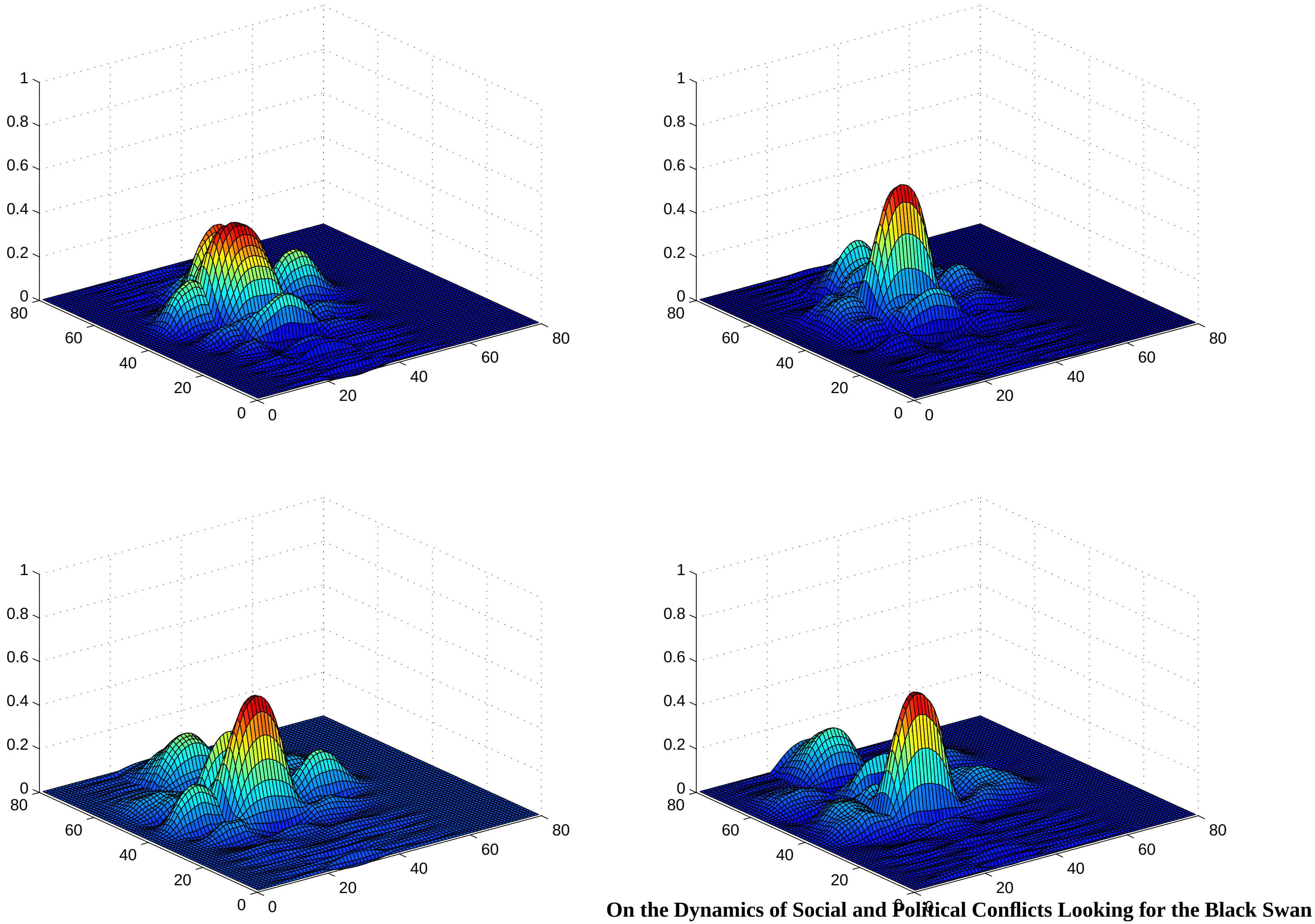
$$\mathbf{U}(t) = \mathbf{U}[f](t) = \frac{1}{\rho(t)} \int_{D_v} \int_{\Sigma_t} \mathbf{v}\, f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}\,,$$

$$\rho(t) = \rho[f](t) = \int_{D_v} \int_{\Sigma_t} f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}.$$

Case Study 1

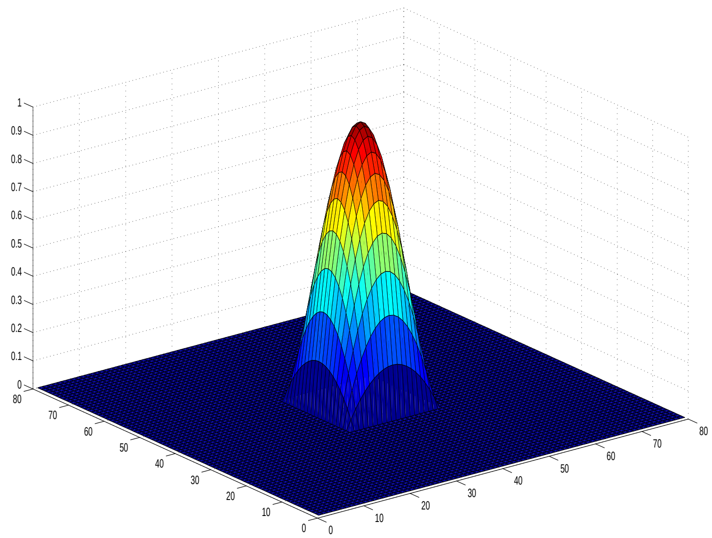


Case Study 1

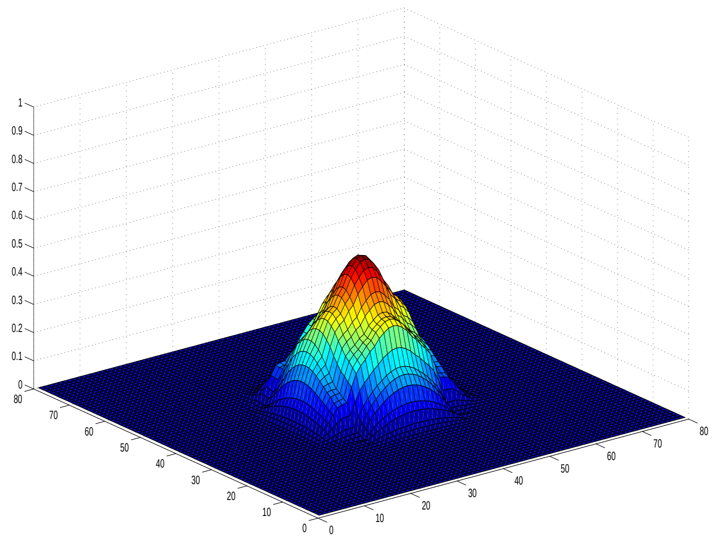


Lecture 4 - Looking at the Beautiful Shapes of Swarms

Case study 2: *A group is initially concentrated in a certain zone, and that suddenly some panic situation occurs within this zone. So, they start to move out from this (danger) point, taking different directions of movement.*

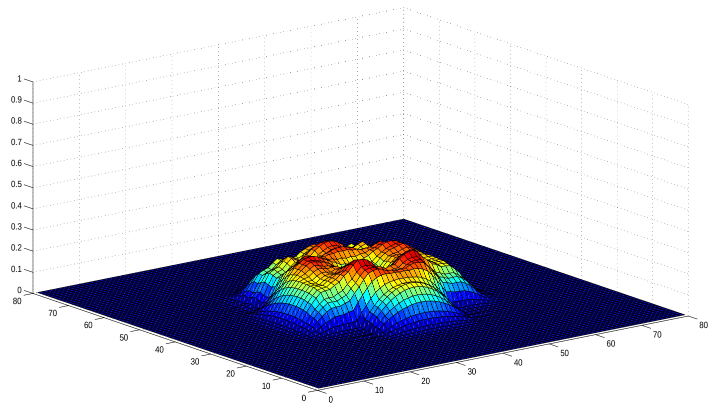


(a)



(b)

Lecture 4 - Looking at the Beautiful Shapes of Swarms



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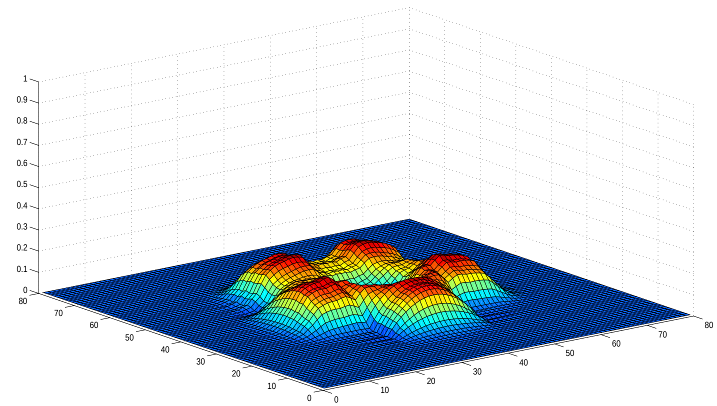

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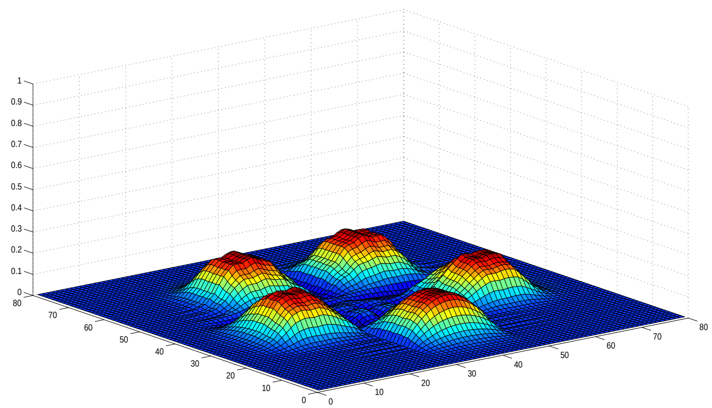
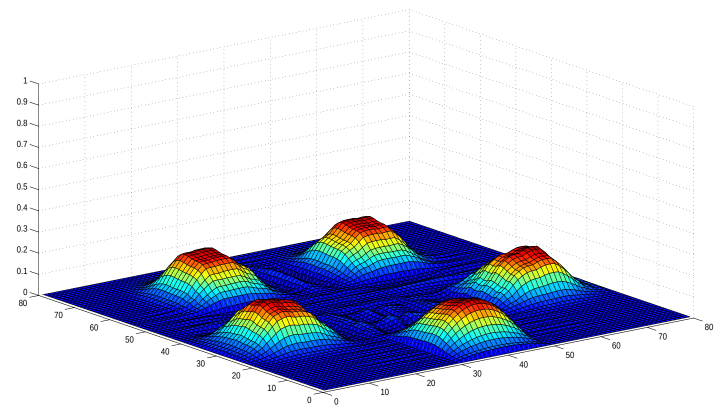


(e)



(f)