The N-player War of Attrition

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Introduction

John Maynard Smith (1974)

2 Players:

\[
\text{Time cost} = -t
\]

\[
\text{Prize} = V > 0
\]
Payoff-function

\[ J_x(\tau_x, \tau_y) := \begin{cases} 
V - \tau_y, & \text{if } \tau_x > \tau_y \\
-\tau_x, & \text{if } \tau_x < \tau_y.
\end{cases} \]

How to play?

Pure Strategy

Mixed Strategy

\[ t \in \mathbb{R}_+ \]

\[ \mu(dt) \in \mathcal{M}_1(\mathbb{R}_+) \]
The 2-player case

In 1976 Bishop and Cannings showed that the classical 2-player War of Attrition admits a unique ESS, namely:

\[ \mu(dt) = \frac{1}{V} e^{-x/V} \]

(having a very long tail!?)

**Recall:** the mixed strategy \( \mu \) is an ESS (Evolutionary Stable Strategy) if and only if either

* \( \mathcal{J}(\mu, \mu) > \mathcal{J}(\mu, \pi) \)

for any other mixed strategy \( \pi \) or, if “=” for some \( \hat{\pi} \), then

* \( \mathcal{J}(\mu, \hat{\pi}) > \mathcal{J}(\hat{\pi}, \hat{\pi}) \)
N-player generalizations

"The n-person War of Attrition"
(1989)
The Dynamic N-player model

N available prizes: \( \{V_k\}_{k=1}^{N} \)

1st round

(i) Each of the N players choose a waiting time.  
(ii) The player having the least time receives the prize \( V_1 \), pays the waiting time cost and leaves the game.  
(iii) The remaining players pay the same time cost and proceeds to the next round.

2nd round \( \rightarrow \) ... ... \( \rightarrow \) (N-1)th round
The Static N-player model

N available prizes: $\{V_k\}_{k=1}^N$

One-Shot game

(i) Each of the N players choose a waiting time.

(ii) The prizes are handed out according to the order of the chosen waiting times, i.e. the player with the least waiting time receives $V_1$ and so forth.

(iii) All players pay their individual waiting time.

How to play in these models?
Evolutionary Stable Strategy (ESS)

A mixed strategy $\mu^*$ is an $N$-player ESS if either

(i) $\mathcal{I}_N(\mu^* | \mu^*, \ldots, \mu^*) > \mathcal{I}_N(\mu | \mu^*, \ldots, \mu^*)$

or, if "=" in (i) for some $\bar{\mu}$, then

(ii) $\mathcal{I}_N(\mu^* | \mu^*, \ldots, \bar{\mu}) > \mathcal{I}_N(\bar{\mu} | \mu^*, \ldots, \bar{\mu})$

Note: An ESS is also a Nash-equilibrium, but the opposite is false!
ESS in the N-player War of Attrition?

The dynamic model always has a unique ESS!

\[ \mu(d\tau) = \frac{1}{(N - k)(V_{k+1} - V_k)} \exp \left\{ - \frac{\tau}{(N - k)(V_{k+1} - V_k)} \right\} d\tau \]

in round \((k + 1)\).

The static model ... ... has a ... ... ESS?

\[ \text{Ex:} \]

(i) If \( \{V_k\}_{k=1}^N \) linj. increasing there is a unique ESS.

(ii) If \( V_1 = 1, V_2 = 4, V_3 = 6 \) there is a candidate ESS, but it is not! (it is a Nash-equilibrium)

(iii) If \( V_1 = 1, V_2 = 2, V_3 = 1 \) there is not even a Nash-equilibrium.
Consider the limit when \( N \) tends to infinity!

**The Dynamic Model:**

The "game evolution" can be seen as a C.T.M.C

\[
X(t) = \sum_{k=1}^{N-1} \frac{1}{N} \mathbb{1}\{T_1 + \ldots + T_k \leq t\}, \quad T_k \sim \exp \left( \frac{N - k + 1}{(N - k) (V_{k+1} - V_k)} \right)
\]

\[
X(t): 0 \quad \frac{1}{N} \quad \frac{(N-2)}{N} \quad \frac{(N-1)}{N}
\]

and after some calculations one finds that

\[
\mathbb{E}[X(t)] = \sum_{i=1}^{N-1} \frac{i}{N} \sum_{l=1}^{i+1} \prod_{k=1, k \neq l}^{i} \lambda_k \prod_{k=1, k \neq l}^{i+1} \lambda_k \cdot e^{-\lambda_l t}
\]
A useful lemma: Let \( \{\lambda_i\}_{i=1}^{n} \) be a sequence of positive and distinct real numbers. Then, if \( f_i(t) = \lambda_i e^{-\lambda_i t} \chi_{[0,\infty)} \), it holds that

\[
f_1 \ast f_2 \ast \ldots \ast f_n(t) = \sum_{l=1}^{n} \frac{\prod_{k=1}^{n} \lambda_k}{\prod_{k=1, k \neq l}^{n} (\lambda_k - \lambda_l)} \cdot e^{-\lambda_l t}
\]

Consider \( \mathcal{L}(\mathbb{E}[X(t)]) \) and pass to the limit!
If $V(x) \in C^1[0, 1]$ is increasing, $V(0) = 0$, and $V_k = V(k/N)$, then one can prove

"Theorem 1": $\lim_{N \to \infty} \mathbb{E}[X(t)] = V^{-1}(t)$

"Theorem 2": In the limit the dynamic model is (in a sense) static, and the limiting strategy is $d/dt(V^{-1})(t)dt$
The Static Model:

Consider a $N$-player situation in which:

(N-1) players play $g_N(t) \in \mathcal{M}_1(\mathbb{R}^+)$

1 player play $\delta_x \in \mathcal{M}_1(\mathbb{R}^+)$ (quit at $t=x$)

Then $g_N(t)$ is a Nash-equilibrium (and ESS candidate) iff. the expected payoff of playing $\delta_x$ is constant w.r.t $x$.

\begin{align*}
\Rightarrow \quad & \left\{ \frac{dG_N}{dx} = \frac{1-G_N^{N-1}}{(N-1) \sum_{r=0}^{N-2} \binom{N-2}{r} G_N^r (1-G_N)^{N-2-r}} =: \Xi(G_N) \\
& G_N(0) = 0,
\right.
\end{align*}

where $G_N(t)$ is the c.d.f of $g_N(t)$.

Note: $G_N(t) \leftrightarrow \mathbb{E}[X(t)]$
**Theorem:** Let $V(x)$ be an increasing $C^1$-function on $[0,1]$ such that $V(0) = 0$ and define $\{V_k\}_{k=1}^N$ by $V_k := V(k/N)$. Then, if $G_N$ is the unique solution to the ode-problem, it holds that

$$G_N(t) \rightarrow \begin{cases} V^{-1}(t), & 0 \leq t \leq V(1) \\ 1, & t > V(1) \end{cases}$$

uniformly as $N \rightarrow \infty$.

**Intuition:** The dynamic and the static models "coincides" in the limit!
Sketch of proof:

\[ \Xi_N(x) = \frac{1 - x^{N-1}}{(N - 1) \sum_{r=0}^{N-2} \binom{N-2}{r} x^r (1-x)^{N-2-r}} \xrightarrow{} \frac{1}{V'(x)} \]

uniformly in \( x \).

* Thus, if \( y_N(x) \) solves the N-player eq. and \( y(x) = V^{-1}(x) \), we get the pointwise estimate:

\[ |y_N(x) - y(x)| \leq \varepsilon_N xe^{x \xi_N}, \quad x \in [0, V(1)] \]

* Pointwise convergence \( \Rightarrow \) Uniform convergence

since \( \{y_N\}_{N=2}^{\infty} \) is a sequence of monotone functions.
Games Having a Continuum of Players

\[ \mathcal{P} = (P, \mathcal{P}, \mu) \] - space of players

\[ \mathcal{A} = (A, \mathcal{B}(A)) \] - space of possible actions

A measure valued mapping \( \Delta : P \rightarrow \mathcal{M}_1(A) \) (mixed action profile) keeping track of what strategies the players use \( (\Delta(p)(A) = 1) \).

\[ J : \mathcal{R} \times P \rightarrow [-\infty, \infty) \] - payoff function \( (\Delta \in \mathcal{R}) \)

A GAME is a triple

\[ \mathcal{G} = (\mathcal{P}, \mathcal{A}, J) \]

In this framework we can define what an ESS should be!
Assume a Continuum of Players playing the War of Attrition...

The payoff function is then given by:

$$J(\Delta, p) := \int_0^\infty \left[ V \left( \int_0^t \Delta(dx) \right) - t \right] \Delta(p)(dt)$$

where $V(x)$ is an increasing $C^2$-function on $[0,1]$. (prize function)
A calculation shows that the limit strategy $q(t) := d/dt(V^{-1}(t))$ is an ESS in the continuum limit of the static war of attrition if $V(x)$ is a \textbf{convex} function. Moreover, for a \textbf{concave} prize function, the limit strategy $q(t)$ does worse than any other strategy.

\textbf{IS THIS REFLECTED IN THE FINITE N-PLAYER GAME?}
A sufficient condition for the N-player candidate strategy $G_N(t)$ to be an ESS is to have strict positivity in the function

$$Q [G_N] = 2G_N^{N-2} + \frac{d}{dt} \left \{ \sum_{r=0}^{N-2} c_r \binom{N-2}{r} G_N^r (1 - G_N)^{N-2-r} \right \}$$

Positive if $N$ large enough and prize sequence is convex?

**Theorem:** If the prize sequence $\{V_k\}_{k=1}^N \subset \mathbb{R}_+$ is convex, then $G_N(t)$ is an ESS (unique) for **ALL** $N \geq 2$. 
The Concave Case

The sufficient condition cannot be used in this case...

**IDEA:** pick a strategy and prove that \( g_N(t) \) does not fulfill the (N-player) ESS conditions against this strategy.

**EX:** let \( \delta_0 \) play against a population of \( g_N(t) \)-players. If

\[
\Delta^\delta_0_N := \mathcal{I}_N(g_N | g_N^{(N-2)}, \delta_0) - \mathcal{I}_N(\delta_0 | g_N^{(N-2)}, \delta_0) < 0,
\]

then \( g_N(t) \) is not an ESS.

Hard to investigate \( \Delta^\delta_0_N \) for a general prize sequence, but if we consider the case \( V_k := (k/N)^\alpha \) so that the sequence is concave if \( 0 < \alpha < 1 \), then \( \Delta^\delta_0_N \) is negative for \( N \) large enough!