Entropic sub-cell shock capturing schemes via Jin-Xin relaxation and Glimm front sampling

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Outline

Joint work with Shi Jin, Jian-Guo Liu and Li Wang.

- Motivation : sharp resolution of entropy satisfying shock solutions
  - *Why sharp resolution while smeared discrete profiles are usually not considered to be a flaw ?*
- The Jin-Xin's relaxation setting and corresponding defect measures
  - *a sub-cell shock capturing technique* but with entropy consistency
- A theoretical framework in the scalar setting
  - *convergence to the Kruvkov solution* for general non-linear flux functions.
  - Consistency with infinitely many entropy pairs must be addressed.

Motivation

Reliable computation of the discontinuous solutions
of first order non-linear PDE models for compressible media

- Classical numerical methods *do perform well* on standard issues
- *Numerical dissipation* : two distinct and opposite issues
  - *cannot be avoided* for consistency with the entropy condition : stability requirement
  - but *usually responsible* for the *smearing* of discrete shock profiles : low resolution (*generally not considered as a flaw*)

Increasing demand for calculations in *non-standard issues* reveals that
numerical dissipation may be responsible for various pitfalls
in the approximation of discontinuous solutions

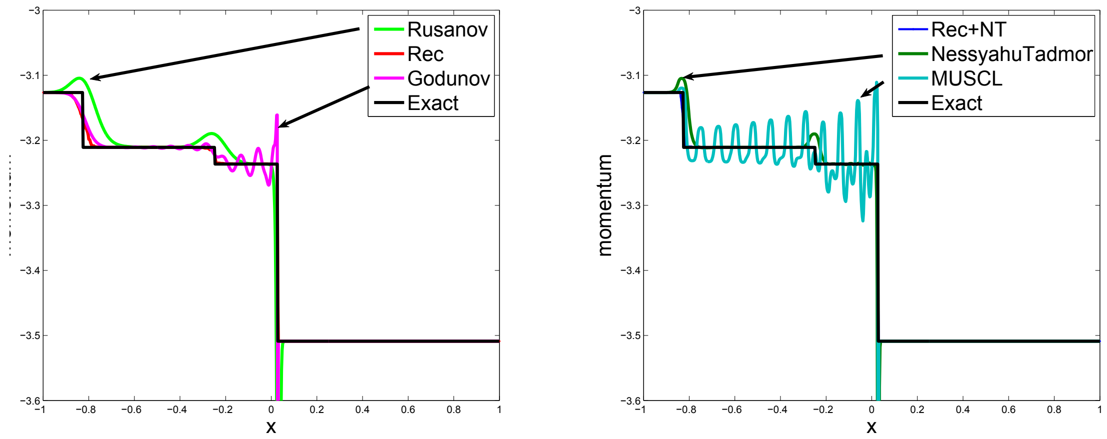
Motivation

Pitfalls may be observed already within the frame of *standard PDE models*

- Euler system for polytropic gases
  - Post shock persistent oscillations in slowly moving shock solutions (JG Liu - S Jin)
  - Theoretical studies show numerical instabilities of *smeared discrete shock profiles* (blow up of the BV bound) (B. Baiti - A. Bressan)
- Scalar conservation laws with stiff source terms exhibiting multiple equilibria
  - numerical shock speed is driven by the CFL number and not by the physics.
- Naturally extends to combustion problems, reacting flows...

A numerical illustration : *slowly moving shock solutions*

Numerical experiments



Motivation

Much severe pitfalls within the frame of *non-classical shock solutions*

- ▷ Exact shock solutions are sensitive with respect to underlying regularizing mechanisms

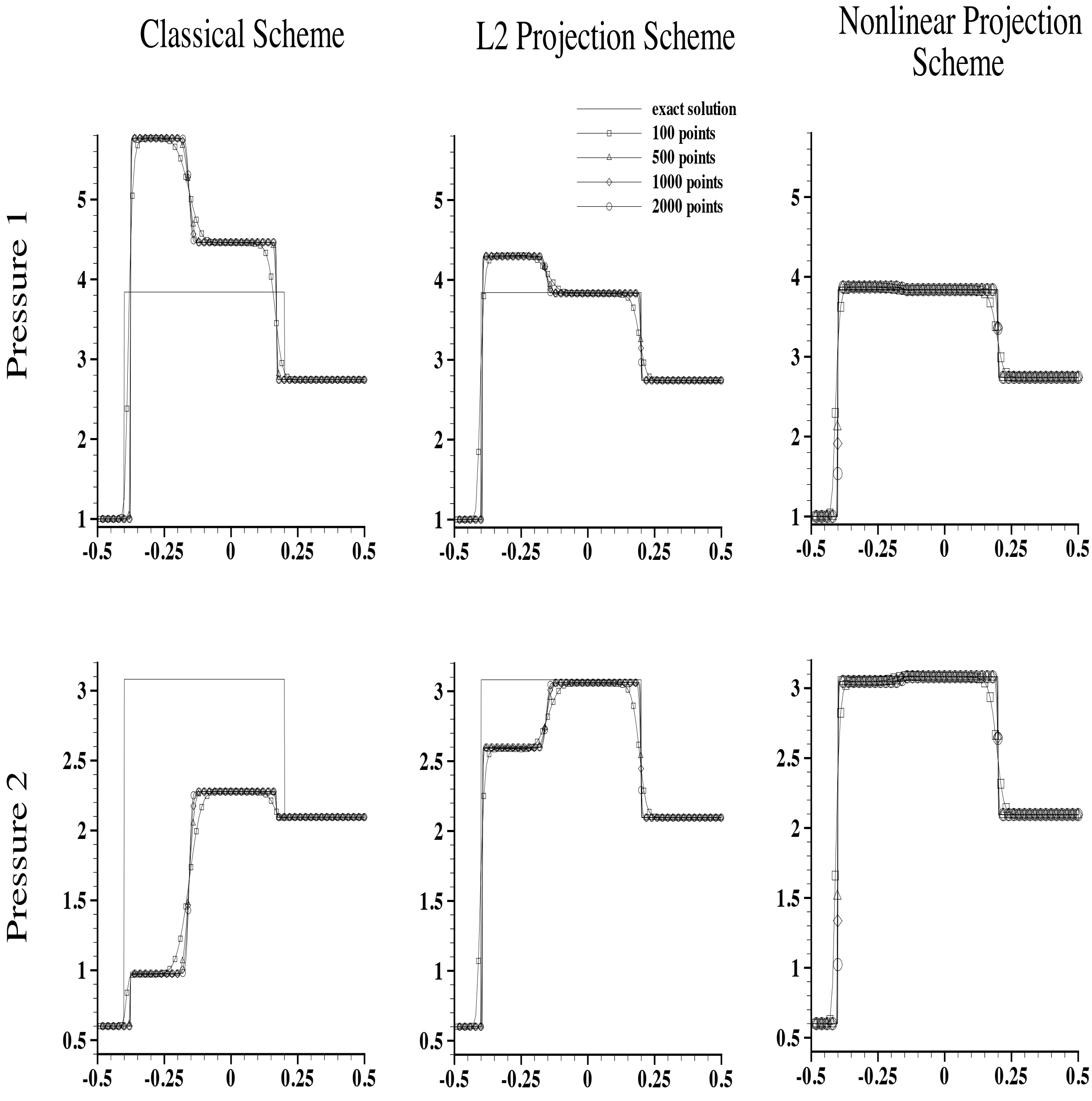
  e.g. viscous and/or dispersive effect

- ▷ Their numerical capture may be grossly corrupted by the artificial *numerical dissipation and/or dispersion*

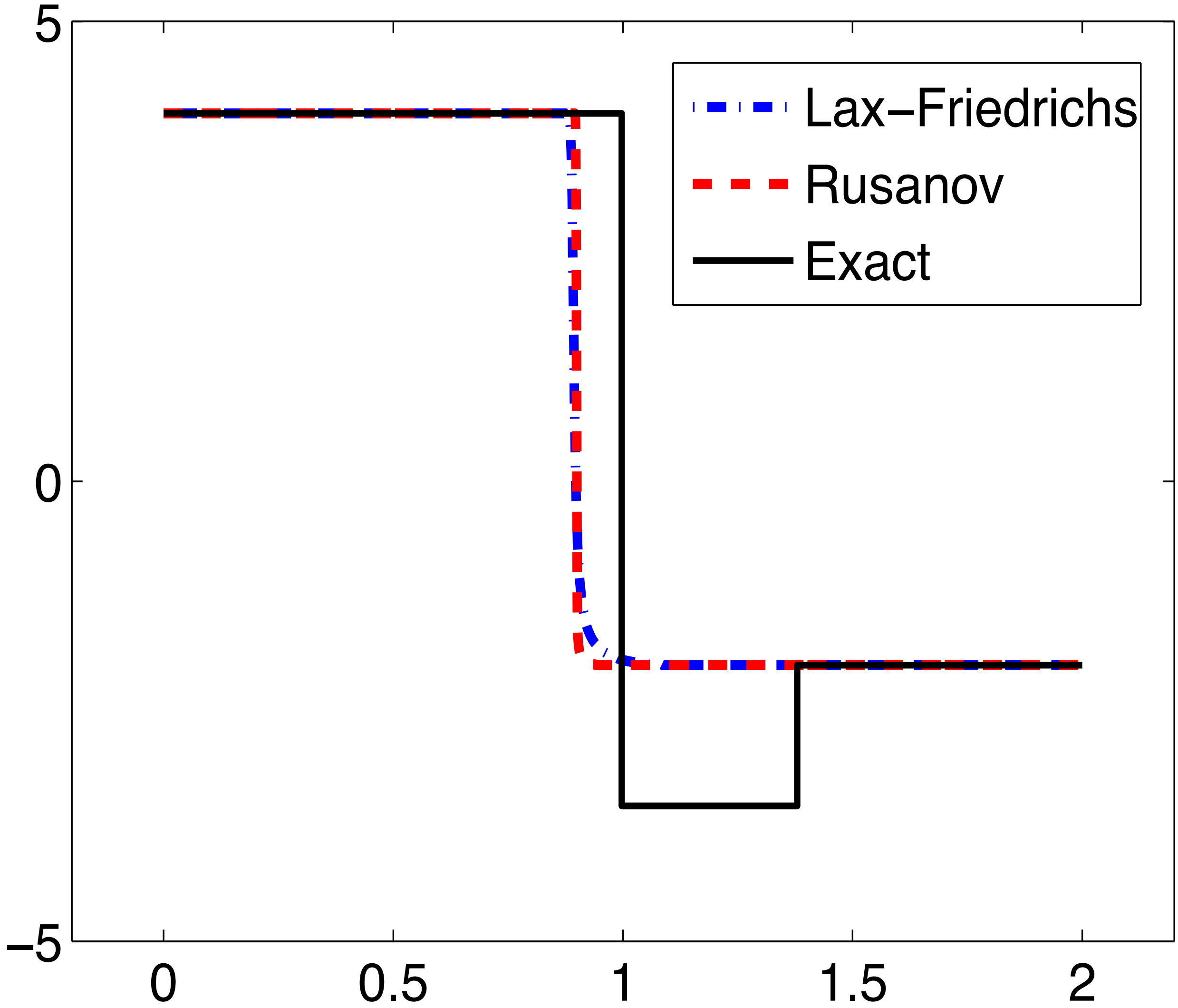
- ▷ Shock solutions of convex hyperbolic PDEs in *non-conservation form*
- ▷ Transition waves in *non-convex hyperbolic* PDEs (phase transition problems, MHD,...)
- ▷ Transition waves in *mixte elliptic-hyperbolic* PDEs (phase transition problems)

Numerical illustrations :

- ▷ Shock solutions in a non-conservative setting : multi-pressure Euler equations (C. Berthon, FC)
- ▷ Transition waves for a non-convex scalar conservation law (P. LeFloch)
- ▷ Transition waves for a elliptic-hyperbolic Euler model (C. Chalons, FC, P. Engel, C. Rohde)

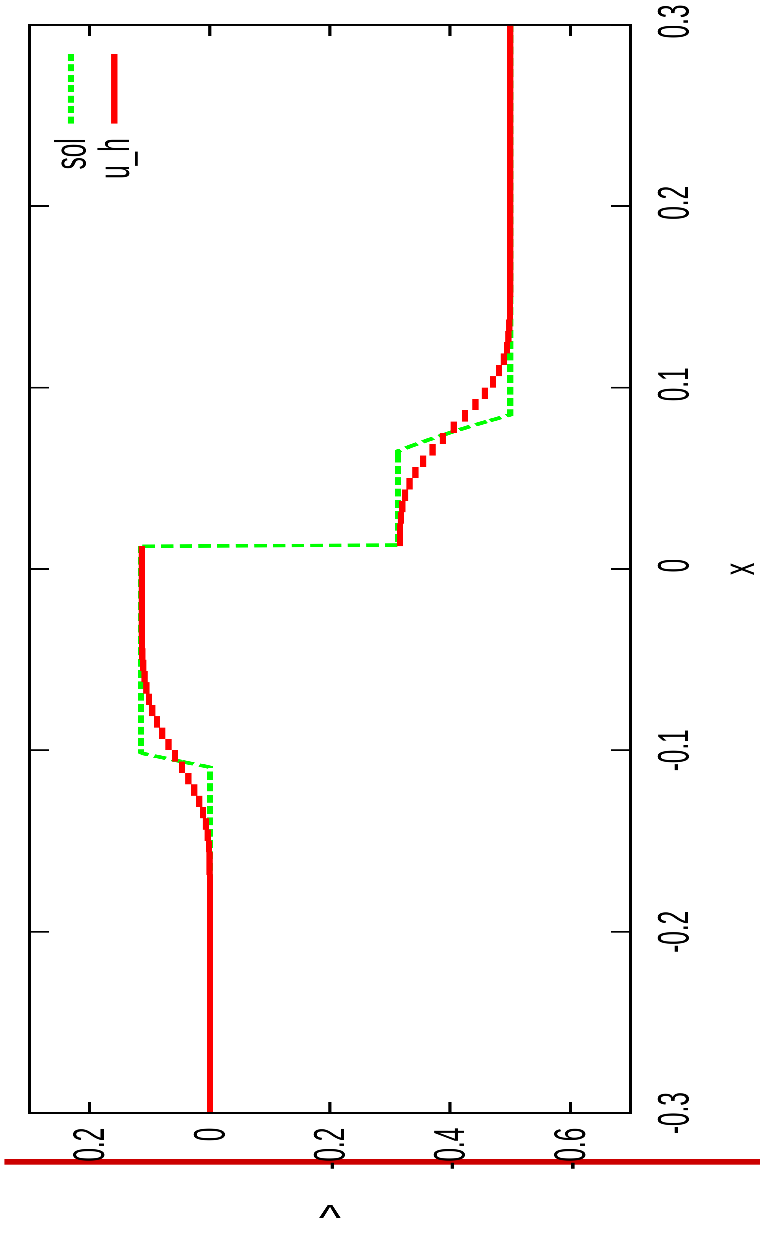


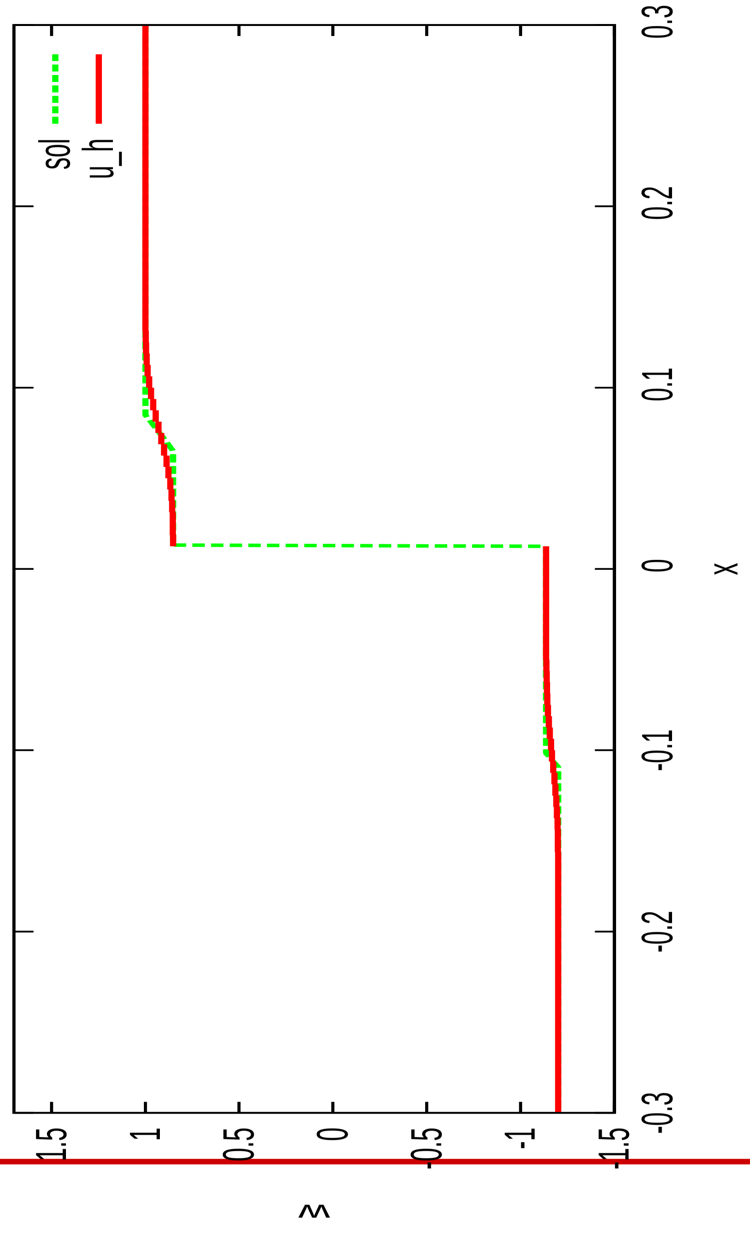
Classical Scheme
L2 Projection Scheme
Nonlinear Projection Scheme
Pressure 1
Pressure 2
exact solution
100 points
500 points
1000 points
2000 points
-0.5
-0.25
0
0.25
0.5



5
0
−5
0
0.5
1
1.5
2
Lax−Friedrichs
Rusanov
Exact

Phase transition in mixed hyperbolic-elliptic systems

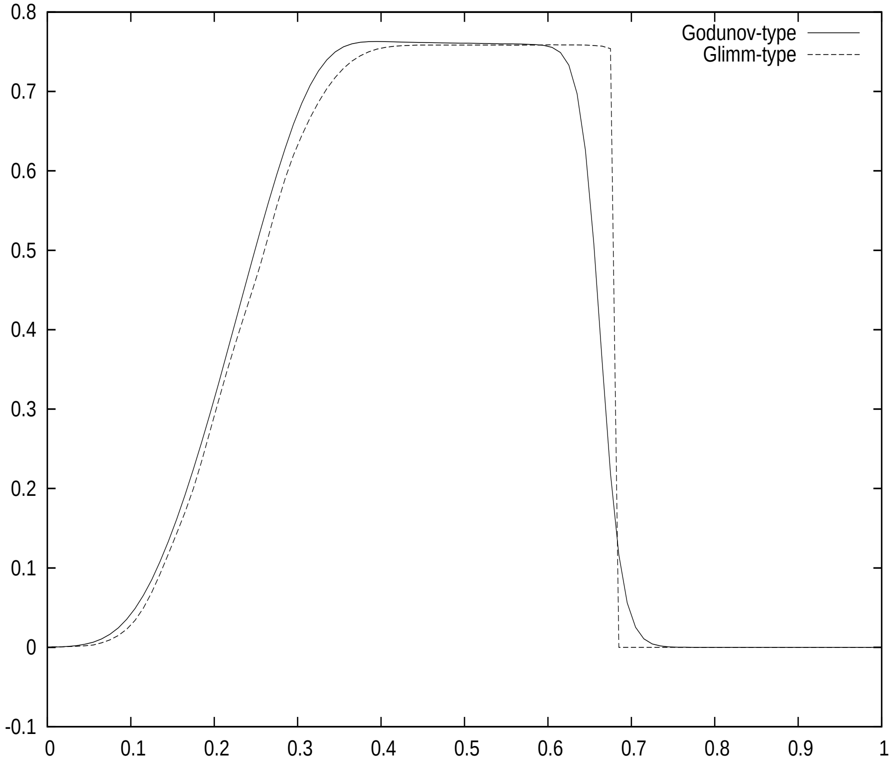




Motivation

Pitfalls are inherently induced by the *smearing of discrete shock profiles*

Prevent discrete shock profiles from smearing



Prevent discrete shock profiles from smearing

A wide variety of approaches for tracking the discontinuities

- Popular approaches in 1D in the frame of non-classical shock solutions
  - VOF or Level set methods
  - Glimm's scheme
- *But in both cases*, difficulty is :

  knowledge of the exact solution of the Riemann problem :
  costly and frequently unknown in the non-classical setting

- Other approaches based on approximate Riemann solvers
  - Sub-cell shock capturing method : Harten
  - Glimm's sampling with *approximate Riemann solvers* : Harten-Hyman, Harten-Lax
- *But in both cases*, difficulty is :

  Satisfying the entropy condition :
  Entropy violation is triggered in the absence of smearing

Prevent discrete shock profiles from smearing

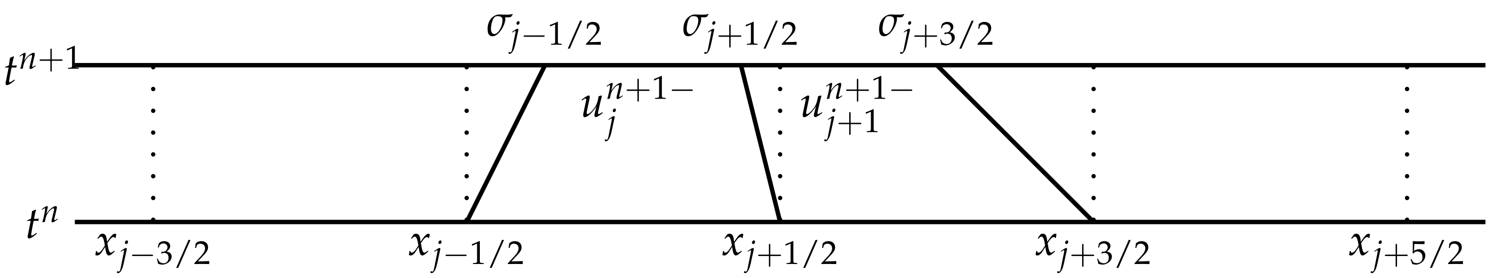
Entropic sub-cell shock capturing schemes
via Jin-Xin relaxation and Glimm front sampling

Combine

- Jin-Xin relaxation framework
  - Fairly easy algebra
  - Positivity preserving properties
  - Built in entropy condition
- Glimm's front sampling
  - Facilitate the analysis of convergence (*scalar setting*)
- Propose a theoretical framework for entropy consistency for scalar conservation laws with *general non-linear flux functions*

Infinitely many entropy pairs must be addressed

Glimm's sampling with approximate Riemann solvers

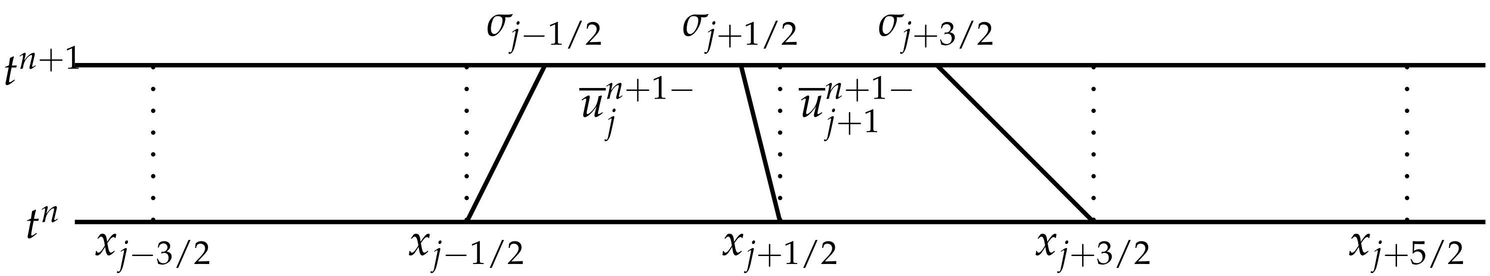


- At t^n, solve (exactly or approximately) a sequence of non-interacting Riemann problems at the interfaces $x_{j+1/2}$. Locate a shock with speed $\sigma_{j+1/2}^n$, if none set $\sigma_{j+1/2}^n = 0$.
- At $t^{n+1-} = t^n + \Delta t^-$, average the resulting solution over shifted cells $[\overline{x}_{j-1/2}^n, \overline{x}_{j+1/2}^n]$,

$$u_j^{n+1-} = \frac{1}{\overline{\Delta x_j}} \int_{\overline{x}_{j-1/2}^n}^{\overline{x}_{j+1/2}^n} u(x, \Delta t) dt, \quad \overline{x}_{j+1/2}^n = x_{j+1/2} + \sigma_{j+1/2}^n \Delta t, \quad \overline{\Delta x}_j = \overline{x}_{j+1/2}^n - \overline{x}_{j-1/2}^n.$$

- To avoid remeshing, sample the discrete constant values in each original cell to define a new constant state u_j^{n+1} at time t^{n+1}.

Glimm's sampling with approximate Riemann solvers



Let be given $(a_n)_n$ a well-distributed sequence in $(0,1)$ ($e.g.$ van der Corput sequence)

- the sampling procedure reads

$$u_j^{n+1} = \begin{cases} u_{j-1}^{n+1-} & \text{if} \quad a_n \in (0, \frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{n,+}), \\ u_j^{n+1-} & \text{if} \quad a_n \in (\frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{n,+}, 1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{n,-}), \\ u_{j+1}^{n+1-} & \text{if} \quad a_n \in (1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{n,-}, 1), \end{cases}$$

with $\sigma_{j+1/2}^{n,+} = \max(\sigma_{j+1/2}^n, 0), \quad \sigma_{j+1/2}^{n,-} = \min(\sigma_{j+1/2}^n, 0).$

The Jin and Xin's relaxation framework

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + a^2 \partial_x u^\epsilon = \dfrac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon), \end{cases}$$

with well-prepared initial data $u^\epsilon(0,x) = u_0(x),\ v^\epsilon(0,x) = f(u_0(x))$.

- Natalini : Let $u_0 \in BV \cap L^\infty(\mathbb{R})$. Under the sub-characteristic condition $a > \sup_{|u| \leqslant ||u_0||_{L^\infty}} |f'(u)|$, u^ϵ converges as ϵ goes to zero in a relevant topology to the Kruzkov's solution of the scalar conservation law with initial data u_0.
  - $u_0(x) = u_L + (u_R - u_L)H(x)$ where u_L and u_R satisfy

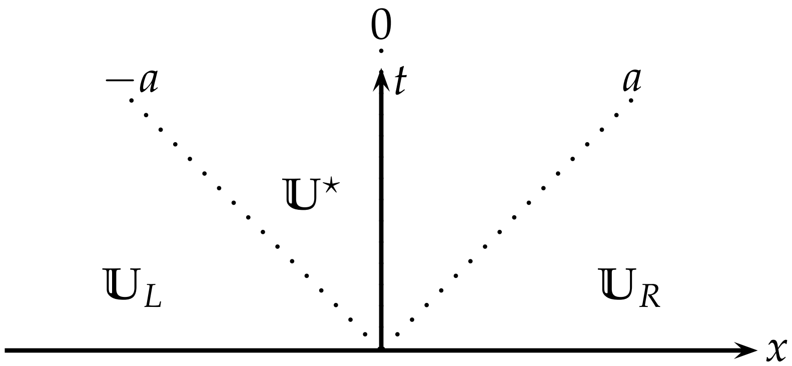
    $$-\sigma(u_L, u_R)(u_R - u_L) + f(u_R) - f(u_L) = 0,$$
    $$-\sigma(u_L, u_R)\left(\mathcal{U}(u_R) - \mathcal{U}(u_L)\right) + \mathcal{F}(u_R) - \mathcal{F}(u_L) \leqslant 0, \forall (\mathcal{U}, \mathcal{F})$$

    converges to the entropy shock solution $u(t,x) = u_L + (u_R - u_L)H(x - \sigma(u_L, u_R)t)$
- What about the discrete approach with fixed $\Delta x > 0$ and $\epsilon \to 0^+$?
  - Difficulty : handle the regime $\epsilon \to 0^+$ in the absence of self-similar solutions

The usual splitting strategy and the sub-characteristic condition

- First step : Solve a sequence of non-interacting Riemann problem

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + a^2 \partial_x u = 0, \end{cases}$$



- Second step : Solve $\begin{cases} \partial_t u^\epsilon = 0, \\ \partial_t v^\epsilon = \dfrac{1}{\epsilon}\left(f(u^\epsilon) - v^\epsilon\right), \end{cases}$ in the limit $\epsilon \to 0$

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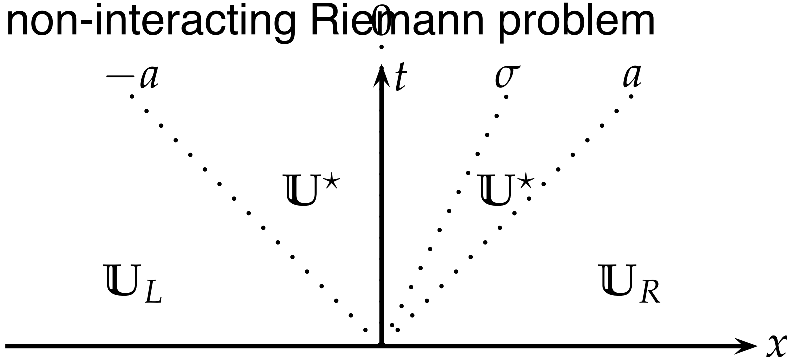


- Second step : Solve $\begin{cases} \partial_t u^\epsilon = 0, \\ \partial_t v^\epsilon = \dfrac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon), \end{cases}$ in the limit $\epsilon \to 0$

> Due to the sub-characteristic condition $a > |\sigma(u_L, u_R)|$, in the first step : an isolated shock-solution is averaged within the intermediate state $\mathbb{U}^\star$.
> Too little from the relaxation mechanisms in the limit $\epsilon \to 0$ have been retained in the first step

The limit $\epsilon \to 0$ and defect measures at shocks

Back to the original relaxation framework

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + a^2 \partial_x u^\epsilon = \dfrac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon), \end{cases}$$

- Evaluating the singular relaxation source term in the limit $\epsilon \to 0$ for an isolated entropy shock
$$\begin{aligned} \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon) &= \Big\{ -\sigma(f(u_R) - f(u_L)) + a^2(u_R - u_L) \Big\} \delta_{x-\sigma t} \\ &= (a^2 - \sigma^2)(u_R - u_L)\delta_{x-\sigma t}, \quad \mathcal{D}'. \end{aligned}$$
- Such a singular limit is referred to as a (relaxation) defect measure
- Due to Natalini's theorem, the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + a^2 \partial_x u = (a^2 - \sigma^2)(u_R - u_L)\delta_{x-\sigma t} \end{cases}$$

with $u_0(x) = u_L + (u_R - u_L)H(x), \quad v_0(x) = f(u_0(x))$ has a unique self-similar solution which coincides with the entropy shock solution in its u-component.
Claim : Because of self-similarity : easily handled for fixed $\Delta x > 0$

The splitting strategy with defect measure

For general initial data u_0, split the relaxation source term in the limit $\epsilon \to 0$ into two contributions

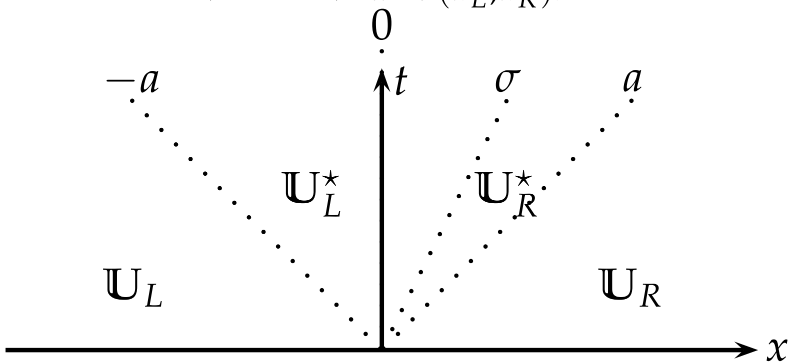
$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon) = \sum_{shocks} (a^2 - \sigma^2)(u_+ - u_-)\delta_{x-\sigma t} + \left\{ \partial_t f(u) + a^2 \partial_x u \right\}$$

- First singular contribution due to entropy satisfying shocks in the limit solution u
  - defect measure to be involved in the first step
- Second smooth contribution coming from the smooth part of the limit solution
  - to be involved in the second step

The splitting strategy with defect measure

- First step : Solve a sequence of non-interacting Riemann problem with defect measure correction

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + a^2 \partial_x u = m(u_L, u_R)\delta_{x-\sigma(u_L,u_R)t}, \end{cases}$$



  - predict $\sigma(u_L, u_R)$ and $m(u_L, u_R)$ so as to achieve stability and accuracy (*exact capture of isolated entropy shocks*).

- Second step : Solve $\begin{cases} \partial_t u^\epsilon = 0, \\ \partial_t v^\epsilon = \dfrac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon), \end{cases}$ in the limit $\epsilon \to 0$
- Third Step : Local averagings avoiding propagating shocks and sampling procedure

Design principle of $\sigma(u_L, u_R)$ and $m(u_L, u_R)$

The u-component of $\mathbb{U}(., \mathbb{U}_L, \mathbb{U}_R)$ must mimic central properties of the Riemann solution of

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \end{cases} \tag{1}$$

supplemented with the entropy differential inequalities

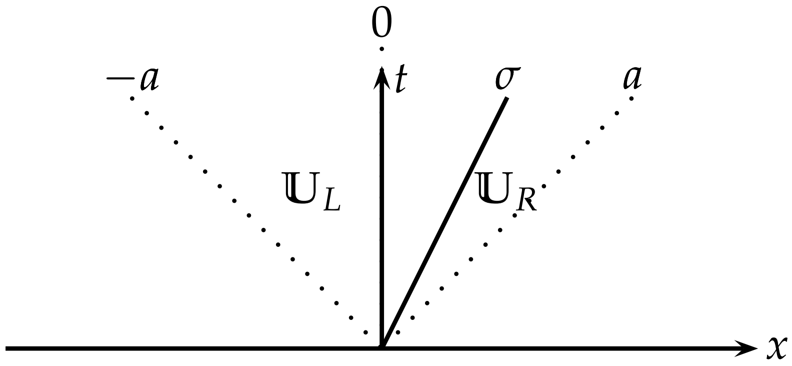
$$\partial_t \mathcal{U}(u) + \partial_x \mathcal{F}(u) \leqslant 0, \quad \mathcal{F}'(u) = f'(u)\mathcal{U}'(u) \text{ for all } u,\ \mathcal{U}(u) \text{convex.} \tag{2}$$

- Stability :
  - Preserve the monotonicity property :
    $||u||_{L^\infty} \leqslant \max(|u_L|, |u_R|),\ TV(u) \leqslant |u_R - u_L|$
  - Respect in a sense to be specified the entropy inequalities (2)
- Accuracy : restore exactly isolated entropy shock solutions of (1)

$$\sigma(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L}, \quad m(u_L, u_R) = (a^2 - \sigma^2(u_L, u_R))(u_R - u_L). \tag{3}$$

Exact capture of isolated entropy shock solutions

$$\begin{aligned} &\partial_t u + \partial_x v = 0 \\ &\partial_t v + a^2 \partial_x u = m(u_L, u_R)\, \delta_{x=\sigma(u_L,u_R)t} \end{aligned} \tag{4}$$



$$-\sigma(u_R - u_L) + (v_R - v_L) = 0, \quad -\sigma(v_R - v_L) + a^2(u_R - u_L) = m(u_L, u_R). \tag{5}$$

$$\sigma(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L}, \quad m(u_L, u_R) = (a^2 - \sigma^2(u_L, u_R))(u_R - u_L). \tag{6}$$

The entropy condition plays no role here !

About general pairs of states cont.

Whatever are $\sigma(u_L, u_R), m(\theta, u_L, u_R)$, The Riemann problem with defect measure correction

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2 \partial_x u = \theta(u_L, u_R)\big(a^2 - \sigma^2(u_L, u_R)\big)(u_R - u_L)\,\delta_{x=\sigma(u_L,u_R)t} \end{cases}$$

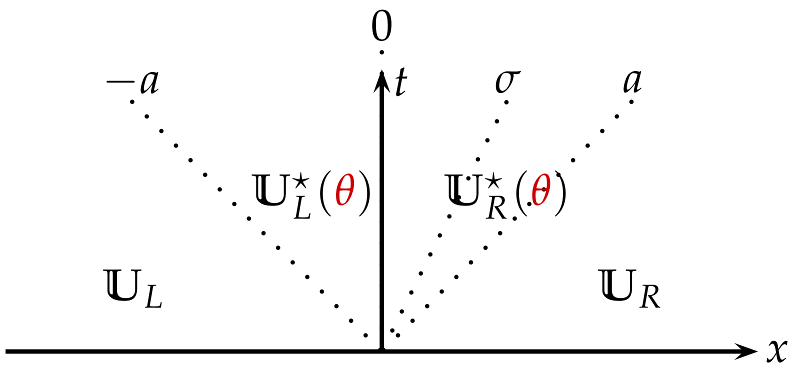
admits a unique solution iff $|\sigma(u_L, u_R)| < a$

Define $\sigma(u_L, u_R), m(\theta, u_L, u_R)$ so as to achieve stability and accuracy

- Exact capture of isolated entropy satisfying discontinuity : $\theta(u_L, u_R) = 1$
- Caution: choosing systematically $\theta(u_L, u_R) = 1$ with $\sigma(u_L, u_R)$ such that $-\sigma(u_L, u_R)(u_R - u_L) + (f(u_R) - f(u_L)) = 0$ always restores a single propagating discontinuity, entropy satisfying or not ! *Such a strategy yields a Roe solver, known to be entropy violating*.
- **Besides monotonicity preserving, entropy consistency is mandatory**

About general pairs of states

$\rightarrow$ Define $\sigma(u_L, u_R), m(u_L, u_R)$



so as to achieve stability conditions

- ▷ Monotonicity preserving properties
- ▷ some entropy consistency condition with respect to the original entropy pairs $(\mathcal{U}, \mathcal{F})$

Keep unchanged $\sigma(u_L, u_R)$ but properly tune the mass of the defect measure correction :

$$\sigma(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L}, \quad m(\theta, u_L, u_R) = \theta(u_L, u_R)(a^2 - \sigma^2(u_L, u_R))(u_R - u_L). \tag{7}$$

Define the tuning parameter $\theta(u_L, u_R)$ so as to meet the above requirements plus...

The monotonicity preserving condition

Under the sub-characteristic condition

$$\sup_{u\in[\min(u_L,u_R),\max(u_L,u_R)]} |f'(u)| < a, \tag{8}$$

the u-component of the solution $\mathbb{U}(.;u_L,u_R)$ of the Riemann problem (**??**)–(**??**) verifies the following monotonicity preserving properties

$$\mathrm{TV}\left(u(\cdot;u_L,u_R)\right) < |u_R - u_L|, \quad \min(u_L,u_R) \leqslant u(\cdot;u_L,u_R) \leqslant \max(u_L,u_R), \tag{9}$$

if and only if

$$0 \leqslant \theta(u_L,u_R) \leqslant 1. \tag{10}$$

- ▷ The sub-characteristic condition is preserved for all $\theta \in (0,1)$
- ▷ The accuracy property $\theta(u_L,u_R) = 1$ is permitted...
- ▷ but to be achieved only under some entropy consistency condition !

Towards the entropy consistency condition : an invariant domain

Define the characteristic variables at equilibrium

$$h_{\pm}(u) = f(u) \pm au, \quad u \in \mathcal{K} = \{u\ \mathbb{R};\ a > |f'(u)|\}, \tag{11}$$

- Consider the compact intervals $\mathcal{K}_- = h_-(\mathcal{K})$ and $\mathcal{K}_+ = h_+(\mathcal{K})$.
- The following compact domain of R^2 built from the interval $\mathcal{K}$ is invariant for the exact Jin-Xin PDEs

$$\mathcal{D}_{\mathcal{K}} \equiv \{\mathbb{U} = (u,v) \in R^2;\ r_-(\mathbb{U}) = v - au \in \mathcal{K}_- \text{ and } r_+(\mathbb{U}) = v + au \in \mathcal{K}_+\}. \tag{12}$$

  - if $\mathbb{U}_0(x) \in \mathcal{D}_{\mathcal{K}}$, then $\mathbb{U}^{\epsilon}(t,x) \in \mathcal{D}_{\mathcal{K}}$ for all $\epsilon > 0$.
  - **Invariance property essential for entropy consistency**
- Is it true for $\mathbb{U}(.,\theta,u_L,u_R)$? in which $m(u_L,u_R)$ is an *approximation of the exact mass attached to exact defect measures*.

Assume the sub-characteristic condition, then the Riemann solution $\mathbb{U}(.,\theta,u_L,u_R)$ with defect measure correction keeps value in $\mathcal{D}_{\mathcal{K}(u_L,u_R)}$ if and only if the monotonicity preserving condition holds true : $0 \leqslant \theta(u_L,u_R) \leqslant 1$

The invariant domain and the relaxation entropy pairs

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + a^2 \partial_x u^\epsilon = \dfrac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon), \end{cases}$$

(Φ, Ψ) is said to be a relaxation entropy pair consistent with the equilibrium pair $(\mathcal{U}, \mathcal{F})$ if

$$\partial_t \Phi(u^\epsilon, v^\epsilon) + \partial_x \Psi(u^\epsilon, v^\epsilon) = \frac{1}{\epsilon} \partial_v \Phi(u^\epsilon, v^\epsilon)(f(u^\epsilon) - v^\epsilon)$$

- $(u, v) \in \mathcal{D}_{\mathcal{K}(u_L, u_R)} \to \Phi(u, v) \in \mathbb{R}$ strictly convex.
  - *For any given fixed u, $\Phi(u, v)$ admits a unique minimum in v*
- $\partial_v \Phi(u, v)(f(u) - v) \leqslant 0$, for any given $(u, v) \in \mathcal{D}_{\mathcal{K}(u_L, u_R)}$.
  - *Convex entropy Φ dissipated with respect to relaxation mechanisms*
  - *For vanishing ϵ, given u^ϵ, $\Phi(u^\epsilon, v^\epsilon)$ reaches its global minimum in v^ϵ*
- $\Phi(u, f(u)) = \mathcal{U}(u)$, $\Psi(u, f(u)) = \mathcal{F}(u)$, for all $u \in \mathcal{K}(u_L, u_R)$.
  - *For vanishing ϵ, v^ϵ reaches the stable equilibrium $f(u^\epsilon)$*

Theses consistency requirements are valid iff (u^ϵ, v^ϵ) belongs to the invariant domain $\mathcal{D}_{\mathcal{K}(u_L, u_R)}$

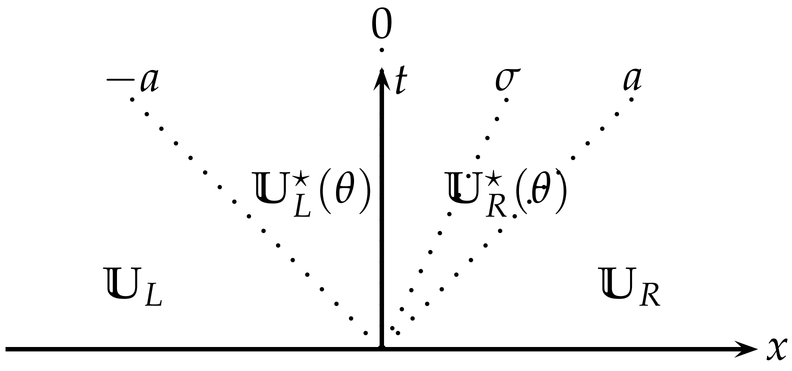
$$0 \leqslant \theta(u_L, u_R) \leqslant 1$$

The entropy consistency requirement

$$\partial_t \Phi(\mathbb{U}(\theta)) + \partial_x \Psi(\mathbb{U}(\theta)) \leqslant 0.$$

$$\begin{aligned} +a\big(\Phi(\mathbb{U}_L^\star(\theta;u_L,u_R)) - \Phi(\mathbb{U}_L)\big) + \Psi(\mathbb{U}_L^\star(\theta;u_L,u_R)) - \Psi(\mathbb{U}_L) &= 0, \\ -a\big(\Phi(\mathbb{U}_R) - \Phi(\mathbb{U}_R^\star(\theta;u_L,u_R))\big) + \Psi(\mathbb{U}_R) - \Psi(\mathbb{U}_R^\star(\theta;u_L,u_R)) &= 0, \end{aligned}$$

Entropy is preserved at the extreme waves, but not across the intermediate one

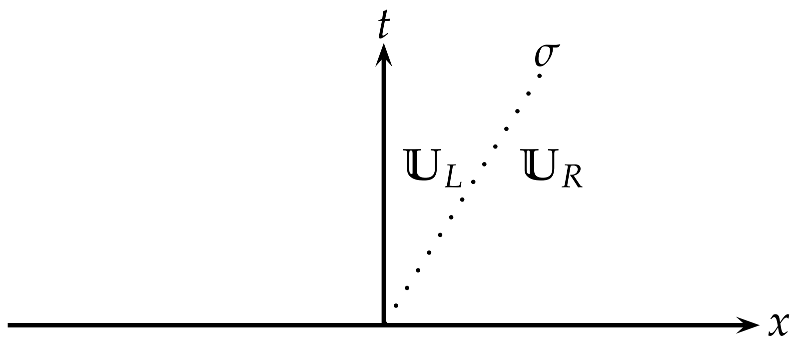


The defect measure correction $m(\theta, u_L, u_R)$ must be consistent with the dissipative property $\frac{1}{\epsilon}\partial_v \Phi(u^\epsilon, v^\epsilon)(f(u^\epsilon) - v^\epsilon) \leqslant 0$. Choose θ so that :

$$\begin{aligned} &\mathcal{E}\{\mathcal{U}\}(\theta;u_L,u_R) \equiv \\ &-\sigma\big(\Phi(\mathbb{U}_R^\star(\theta;u_L,u_R)) - \Phi(\mathbb{U}_L^\star(\theta;u_L,u_R))\big) + \Psi(\mathbb{U}_R^\star(\theta;u_L,u_R)) - \Psi(\mathbb{U}_L^\star(\theta;u_L,u_R)) \leqslant 0. \end{aligned}$$

The entropy consistency requirement for an isolated entropy shock

Is $\theta(u_L, u_R) = 1$ permitted ?



$$
\begin{aligned}
&\mathcal{E}\{\mathcal{U}\}(1; u_L, u_R) \\
&= -\sigma(u_L, u_R)\big(\Phi(\mathbb{U}_R^\star(1; u_L, u_R)) - \Phi(\mathbb{U}_L^\star(1; u_L, u_R))\big) + \Psi(\mathbb{U}_R^\star(1; u_L, u_R)) - \Psi(\mathbb{U}_L^\star(1; u_L, u_R)) \\
&= -\sigma(u_L, u_R)\big(\Phi(\mathbb{U}_R) - \Phi(\mathbb{U}_L)\big) + \Psi(\mathbb{U}_R) - \Psi(\mathbb{U}_L) \\
&= -\sigma(u_L, u_R)\big(\mathcal{U}(u_R) - \mathcal{U}(u_L)\big) + \mathcal{F}(u_R) - \mathcal{F}(u_L) \\
&\leqslant 0.
\end{aligned}
$$

Yes !

The entropy consistency requirement

To select the unique Kruzkov's solution

- For a genuinely non-linear flux $f(u)$ (either strictly convex or concave) : a single strictly convex entropy pair suffices (Panov)

$$\mathcal{U}(u) = \frac{u^2}{2}, \qquad \mathcal{F}(u) = \int_0^u v f'(v) dv.$$

- For a general non-linear flux : infinitely many entropy pairs are in order (the Kruzkov's entropy pairs)

$$\mathcal{U}_k = |u - k|, \quad \mathcal{F}_k(u) = sign(u - k)\big(f(u) - f(k)\big), \quad k \in R.$$

The genuinely non-linear flux framework

Let us consider the entropy pair $(\mathcal{U}(u), \mathcal{F}(u))$ with $\mathcal{U}(u) = u^2/2$ and the associated relaxation entropy pair (Φ, Ψ). Assume the sub-characteristic condition. Then the monotonicity preserving condition and the entropy requirement $\mathcal{E}\{\mathcal{U}\}(\theta; u_L, u_R) \leqslant 0$ are satisfied provided that $\theta(u_L, u_R)$ is chosen so as to verify :

$$0 \leqslant \theta(u_L, u_R) \leqslant \Theta(u_L, u_R) \equiv \max(0, \min(1, 1 + \Gamma(u_L, u_R)), \tag{13}$$

$$\Gamma(u_L, u_R) = \begin{cases} -2\,\gamma(u_L, u_R) \dfrac{\left(-\sigma(\mathcal{U}(u_R) - \mathcal{U}(u_L)) + (\mathcal{F}(u_R) - \mathcal{F}(u_L))\right)}{|u_R - u_L|^2}, & u_L \neq u_R, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

$$\gamma(u_L, u_R) = \begin{cases} \dfrac{a - \max(|f'(u_L)|, |f'(u_R)|)}{\left(a^2 - \sigma^2(u_L, u_R)\right)}, & u_L \neq u_R, \\ 1/\left(a + |f'(u_L)|\right), & \text{otherwise.} \end{cases} \geqslant 0! \tag{15}$$

$\Theta(u_L, u_R) \in (0, 1)$ (*Monotonicity*), $\Theta(u_L, u_R) = 1$ *for entropy satisfying shocks*,
$\Theta(u_L, u_R) \simeq 1$ (*zone of smoothness*)

The general non-linear flux setting

Consider the Oleinik entropy conditions

$$\mathcal{K}(k;u_L,u_R) = -\sigma(u_L,u_R)\left(\frac{u_L+u_R}{2}-k\right) + \left(\frac{f(u_R)+f(u_L)}{2}-f(k)\right) \leqslant 0, \quad k \in \lfloor u_L,u_R \rceil.$$

$\mathcal{E}\{\mathcal{U}_k\}(\theta;u_L,u_R) \leqslant 0$ for all $k \in \lfloor u_L,u_R \rceil$, provided that $\theta(u_L,u_R)$ verifies :

$$0 \leqslant \theta(u_L,u_R) \leqslant \Theta(u_L,u_R) = \min_{k\in\lfloor u_L,u_R\rceil} \Big(1+\Gamma_{\mathcal{K}}(k;u_L,u_R)\Big), \tag{16}$$

$$\Gamma_{\mathcal{K}}(k;u_L,u_R) = -2\gamma(u_L,u_R)\begin{cases} \dfrac{-\sigma(u_L,u_R)\left(\frac{u_L+u_R}{2}-k\right)+\left(\frac{f(u_L)+f(u_R)}{2}-f(k)\right)}{u_R-u_L}, & \text{if } u_L \neq u_R, \\ 0, \quad \text{otherwise}, \end{cases}$$

$$\gamma(u_L,u_R) = 2a/\left(a^2-\sigma^2(u_L,u_R)\right) \geqslant 0$$

$\Theta(u_L,u_R) = 1$ if $\mathcal{K}(k;u_L,u_R) \leqslant 0$ for all $k \in \lfloor u_L,u_R \rceil$, $\quad 0 < \Theta(u_L,u_R) < 1$ otherwise.

A convergence result

Let be given $u_0 \in L^\infty(R) \cap BV(R)$. Assume the sub-characteristic condition and the CFL condition $CFL \leqslant 0.5$. Assume that the mapping $\theta(u_L, u_R)$ is monotonicity preserving and consistent with the entropy consistency requirement, namely with the quadratic entropy pair in the case of a genuinely non-linear flux and with the whole Kruzkov's family in the case of a general non-linear flux function. Then for almost any given sampling sequence $\alpha = (\alpha_1, \alpha_2, ...) \in (0,1)^N := \mathcal{A}$, the family of approximate solutions $\left\{u^\alpha_{\Delta x}\right\}_{\Delta x > 0}$ given by the Jin-Xin scheme with defect measure correction converges in $L^\infty\big((0,T), L^1_{loc}(R)\big)$ for all $T > 0$ and a.e. as $\Delta x \to 0$ with $\frac{\Delta t}{\Delta x}$ kept fixed to the Kruzkov's solution of the corresponding equilibrium Cauchy problem.

- BV framework for a Glimm type of analysis
- The sampling sequence has to be well-distributed (*e.g.* Van der Corput)

Numerical experiments

$$\partial_t u + \partial_x \left(\frac{u^3}{3}\right) = 0, \quad t > 0, x \in (0,1),$$

$$u(0,x) = u_0(x) = \begin{cases} u_L = -1, & x < 0.5, \\ u_R = +1, & x > 0.5, \end{cases}$$

Exact solution made of a *shock attached to a rarefaction wave*.

- Initial data such that

$$-\sigma(u_L, u_R)\left(\frac{u_R^2}{2} - \frac{u_L^2}{2}\right) + \left(\frac{u_R^4}{4} - \frac{u_L^4}{4}\right) = 0, \quad \sigma(u_L, u_R) = \frac{1}{3}.$$

  - In the genuinely nonlinear framework, $\Theta(u_L, u_R) = \max(0, \min(1, 1 + 0)) = 1$
  - Capture an entropy violating shock solution !
- $0 < \Theta_{\text{Kruzkov}}(u_L, u_R) < 1$
  - In the nonlinear framework without genuine nonlinearity, $\Theta(u_L, u_R)$ has to be designed according to infinitely many entropy pairs

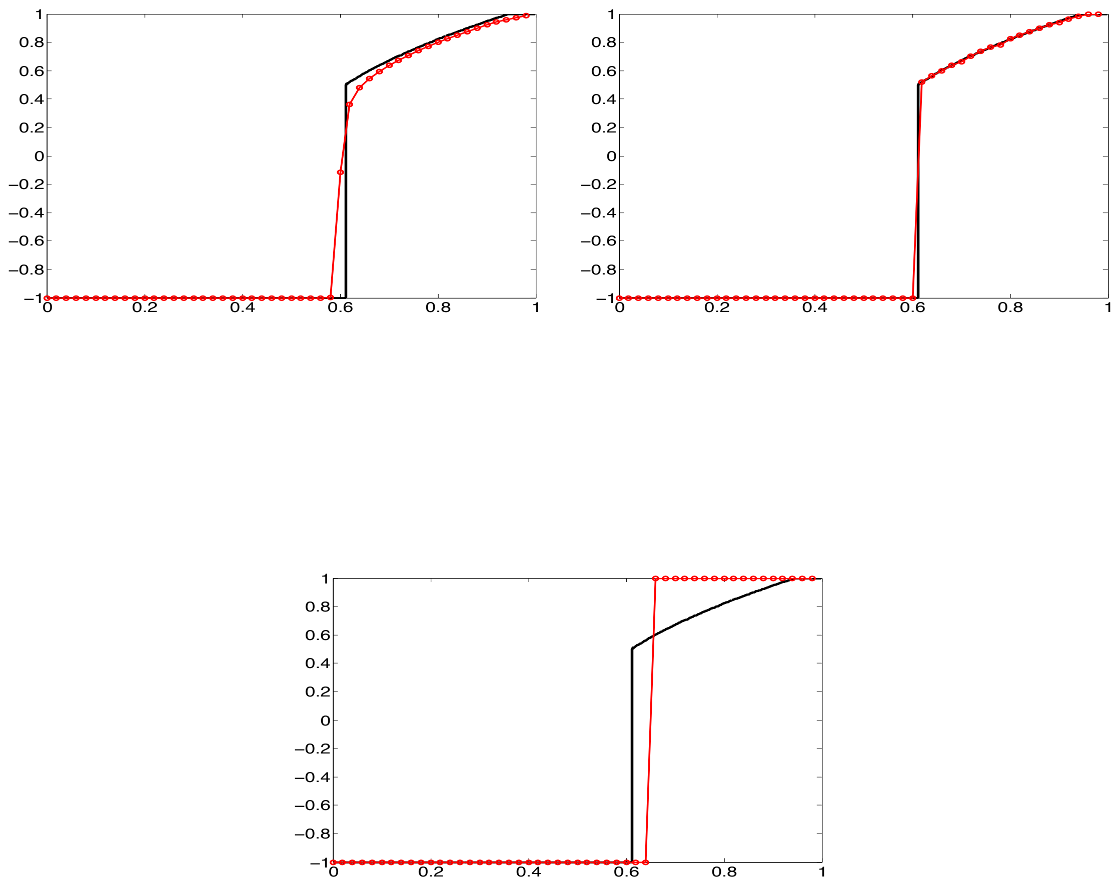


Figure 1:

□