Mean field limits for interacting Bose gases and the Cauchy problem for Gross-Pitaevskii hierarchies

Thomas Chen
University of Texas at Austin

Joint works with: N. Pavlović, K. Taliaferro (UT), N. Tzirakis (UIUC)

KI-Net Conference at University of Maryland, May 2013

Bose gas: System of N bosons in GP scaling

N-body Schrödinger equation, $\Psi_N(x_i, \dots, x_N) \in L^2_{sym}(\mathbb{R}^{dN})$,

$$i\partial_t \Psi_N \;=\; H_N \, \Psi_N \quad , \quad \Psi_N(0) \;=\; \Psi_{N,0}$$

$$H_N \;:=\; \sum_{j=1}^{N}(-\Delta_{x_j}) \;+\; \sum_{\ell=1}^{N} V_{ext}(x_\ell) \;+\; \frac{1}{N} \sum_{1\le i<j\le N} V_N(x_i - x_j)\,,$$

$$V_N(x) = N^{d\beta} V(N^\beta x)$$

V sufficiently regular, $0 < \beta \le 1$.

Marginal density matrices

- Define the **N-particle density matrix**

$$\gamma_N(t,\underline{x}_N;\underline{x}'_N) = \Psi_N(t,\underline{x}_N)\overline{\Psi_N(t,\underline{x}'_N)}$$

- and **k-particle marginals for $k = 1,\dots,N$,**

$$\gamma_N^{(k)}(t,\underline{x}_k;\underline{x}'_k) = \int d\underline{x}_{N-k}\gamma_N(t,\underline{x}_k,\underline{x}_{N-k};\underline{x}'_k,\underline{x}_{N-k}),$$

  where $\underline{x}_k := (x_1,\dots,x_k)$, $\underline{x}_{N-k} := (x_{k+1},\dots,x_N)$.

Key properties: *Positive definite* and *admissible*:

$$\gamma_N^{(k)} = \mathrm{Tr}_{k+1}\gamma_N^{(k+1)} \geq 0$$

Key question: Mean field properties for $N \to \infty$.

1. **Proof of Bose-Einstein condensation, ground states**
   [Lieb-Seiringer-Yngvason; Aizenman-L-S-Solovej-Y]

$$\Phi_N \text{ ground state of } H_N \;\Rightarrow\; \gamma^{(1)}_{\Phi_N} \to |\phi\rangle\langle\phi|$$

   where ϕ minimizes the GP functional.

2. **Derivation of nonlinear Schrödinger or Hartree equation**
   - Via Fock space: *Hepp, Ginibre-Velo, Rodnianski-Schlein, Grillakis-Machedon-Margetis, Grillakis-Machedon*
   - Via BBGKY: *Spohn, Erdös-Schlein-Yau, Elgart-E-S-Y, Adami-Bardos-Golse-Teta*
   - Via BBGKY & PDE-type approach: *Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani, C-Pavlović, X.Chen, X.C.-Holmer*
   - Other approaches: *Fröhlich-Graffi-Schwarz, F-Knowles-Pizzo, Anapolitanos-Sigal, Pickl*

Convergence rate, approach via Fock space

Fock space $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n\geq 1}(L^2(\mathbb{R}^d))^{\otimes_s n}$.

Bosonic creation-, annihilation operators a_x^*, a_x, satisfying CCR

$$[a_x, a_y^*] = \delta(x-y) \quad , \quad [a_x^{(*)}, a_y^{(*)}] = 0 \quad , \quad a_x\Omega = 0 \; \forall x \, .$$

with Fock vacuum $\Omega = (1, 0, 0, \dots)$. Second quantized Hamiltonian

$$\mathcal{H}_N = \int \nabla a_x^* \nabla a_x dx + \frac{1}{N}\int a_x^* a_y^* V_N(x-y) a_y a_x dx dy$$

Coherent initial data,

$$\Phi_{\phi_0} = \left(\frac{1}{k!}(-\sqrt{N}\phi_0)^{\otimes k} \right)_{k=0}^{\infty} = e^{A(\phi_0)}\Omega$$

Convergence

$$e^{-it\mathcal{H}_N}\Phi_{\phi_0} - \Phi_{\phi_t} \longrightarrow 0 \qquad (N \to \infty)$$

ϕ_t solves Hartree ($\beta = 0$) [Hepp] or NLS ($0 < \beta < 1$):

$$i\partial_t \phi_t = -\Delta\phi + (V * |\phi|^2)\phi \qquad \text{or} \qquad i\partial_t \phi_t = -\Delta\phi + |\phi|^2\phi$$

Hartree: $(\beta = 0)$

- *[Rodnianski-Schlein]:* Convergence rate

$$\mathrm{Tr}\Big(\Big|\gamma^{(1)}_{e^{-it\mathcal{H}_N}\Phi_{\phi_0}} - |\phi(t)\rangle\langle\phi(t)|\,\Big|\Big) \leq C\,\frac{e^{Kt}}{N}$$

See also [L.Chen-Lee-Schlein].

- *[Grillakis-Machedon-Margetis], [Grillakis-Machedon]:*

$$\big\| e^{-it\mathcal{H}_N}\Phi_{\phi_0} - e^{A(\phi_t)}e^{B(t)}\Omega\big\|_{\mathcal{F}} \leq C\,\frac{\sqrt{1+t}}{\sqrt{N}}$$

Second order terms via Bogoliubov rotation $e^{B(t)}$.

NLS:

- *[Grillakis-Machedon]:* $(0 < \beta < \frac{1}{3})$

$$\big\| e^{-it\mathcal{H}_N}\Phi_{\phi_0} - e^{A(\phi_t)}e^{B(t)}\Omega\big\|_{\mathcal{F}} \leq C\,\frac{(1+t)\log^4(1+t)}{N^{(1-3\beta)/2}}$$

Approach via BBGKY hierarchy

Steps of [Erdös-Schlein-Yau] approach:

$$
\begin{aligned}
i\partial_t \gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) &= -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) \\
&+ \frac{1}{N} \sum_{1 \le i < j \le k} [V_N(x_i - x_j) - V_N(x'_i - x'_j)]\, \gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) \\
&+ \frac{N-k}{N} \sum_{i=1}^{k} \Big(\mathrm{Tr}_{k+1}[V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1})]\, \gamma_N^{(k+1)} \Big)(t, \underline{x}_k; \underline{x}'_k)
\end{aligned}
$$

Mean field limit $N \to \infty$:

- *Error term:* $\frac{k^2}{N} \to 0$ for any fixed k.
- *Main term:* $\frac{N-k}{N} \to 1$, for any fixed k. For $0 < \beta < 1$,

$$
V_N(x_i - x_j) \rightharpoonup \Big(\int dx\, V(x) \Big)\, \delta(x_i - x_j)
$$

[ESY] Weak-* convergence $\gamma_N^{(k)} \rightharpoonup \gamma_\infty^{(k)}$ along a subsequence, for fixed k.

BBGKY $\to$ GP hierarchy

$$i\partial_t \gamma_\infty^{(k)} \quad = \quad -\sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x_j'})\, \gamma_\infty^{(k)} \;+\; \mu \sum_{j=1}^{k} B_{j;k+1} \gamma_\infty^{(k+1)}$$

Interaction term via **"contraction operator"**

$$\begin{aligned} &\left(B_{j;k+1} \gamma_\infty^{(k+1)} \right) (t, x_1, \ldots, x_k; x_1', \ldots, x_k') \\ &\quad := \int dx_{k+1} dx_{k+1}' \\ &\qquad \left[\, \delta(x_j - x_{k+1}) \delta(x_{k+1} - x_{k+1}') - \delta(x_j' - x_{k+1}) \delta(x_{k+1} - x_{k+1}') \,\right] \\ &\qquad\qquad \gamma_\infty^{(k+1)} (t, x_1, \ldots, x_j, \ldots, x_k, x_{k+1}; x_1', \ldots, x_k', x_{k+1}') \,. \end{aligned} \tag{0.1}$$

NLS

The GP hierarchy preserves factorization of solutions: If

$$\gamma_\infty^{(k)}(0) \;=\; \prod_{j=1}^{k} \phi_0(x_j)\,\overline{\phi_0(x_j')}$$

then

$$\gamma_\infty^{(k)}(t) \;=\; \prod_{j=1}^{k} \phi(t,x_j)\,\overline{\phi(t,x_j')}$$

$$i\partial_t\phi \;=\; -\Delta_x\phi \,+\, \mu\,|\phi|^2\,\phi \;=\; 0$$

Cubic NLS with $\phi_0 \in L^2(\mathbb{R}^d)$.

Uniqueness of solutions to GP hierarchy

[ESY] Weak subsequential limit: Uniqueness of limit requires separate proof. Via **Feynman graph expansions**.

Most difficult part of the program !

Solution spaces of [ESY]

$$\|\gamma^{(k)}\|_{\mathfrak{h}^1} \;:=\; \mathrm{Tr}\big(\,|\,S^{(k,1)}\gamma^{(k)}\,|\,\big) \;<\; C^k$$

$$S^{(k,\alpha)} \;:=\; \prod_{j=1}^{k}\langle\nabla_{x_j}\rangle^{\alpha}\langle\nabla_{x'_j}\rangle^{\alpha}$$

"L^1-type trace Sobolev norm", and $\langle x\rangle := \sqrt{1+x^2}$.

Klainerman-Machedon approach to uniqueness of GP

[Klainerman and Machedon, CMP'08] Different approach to uniqueness, inspired by methods in **dispersive nonlinear PDE's**.

Instead of L^1-type, consider Hilbert-Schmidt L^2-type Sobolev norm

$$\|\gamma^{(k)}\|_{H^1} := \left(\mathrm{Tr}\left(|S^{(k,1)}\gamma^{(k)}|^2 \right) \right)^{\frac{1}{2}}. \qquad (0.2)$$

The method is based on Duhamel expansion, combined with:

- control of combinatorics via **"board game argument"**
- use of **space-time norms** and **Strichartz estimates**.

Thm: *[K-M, CMP'08] Solutions to the cubic GP hierarchy in 3D are unique* **conditional** *under assumption that the a priori space-time bound*

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L^1_{t\in[0,T]}H^1} < C^k, \qquad \text{(KM condition)}$$

holds for all $k \in \mathbb{N}$, with C independent of k.

Subsequently:

- *[Kirkpatrick-Schlein-Staffilani, AJM'11]* proved the KM condition for the derivation of the cubic GP in $d = 2$.
- *[C-Pavlović, JFA'11]:* Proof of the KM condition for the quintic GP in $d = 1, 2$, for the derivation of quintic defocusing NLS.

Key problem: Derivation of 3D cubic GP and proof of KM condition.

The Cauchy problem for Gross-Pitaevskii hierarchies

Joint with N. Pavlović: Study the Cauchy problem for GP hierarchies.

<u>*Compact notation for the GP hierarchy*</u>

$$\Gamma \, = \, (\, \gamma^{(k)}(\, t, x_1, \dots, x_k; x_1', \dots, x_k') \,)_{k \in \mathbb{N}} \;,$$

$$\mathbf{i}\partial_{\mathbf{t}}\mathbf{\Gamma} \, = \, -\widehat{\mathbf{\Delta}}_{\pm}\mathbf{\Gamma} \, + \, \mu \widehat{\mathbf{B}}\mathbf{\Gamma} \, . \tag{0.3}$$

with $\mu = \pm 1$ (de)focusing.

$$\Delta_{\pm}^{(k)} \, = \, \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k} \, , \quad \text{with} \quad \Delta_{\underline{x}_k} \, = \, \sum_{j=1}^{k} \Delta_{x_j} \, .$$

$$\widehat{\Delta}_{\pm}\Gamma \, := \, (\, \Delta_{\pm}^{(k)} \gamma^{(k)} \,)_{k \in \mathbb{N}} \;,$$

$$\widehat{B}\Gamma \, := \, (\, B_{k+1} \gamma^{(k+1)} \,)_{k \in \mathbb{N}} \, .$$

Banach spaces of sequences of density matrices

Problem: The equations for $\gamma^{(k)}$ don't close & no fixed point argument.

Solution: [C-P] Endow *the space of sequences* Γ *with a suitable topology.* Let

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$$

be the space of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k\in\mathbb{N}}.$$

Introduce generalized Sobolev spaces $\mathcal{H}^\alpha_\xi$ based on Hilbert-Schmidt type Sobolev norms

$$\|\Gamma\|_{\mathcal{H}^\alpha_\xi} := \sum_{k\in\mathbb{N}} \xi^k \|\gamma^{(k)}\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} , \qquad 0 < \xi < 1 .$$

Properties:

- **Finiteness:** $\|\Gamma\|_{\mathcal{H}^\alpha_\xi} < C$ implies that $\|\gamma^{(k)}\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} < C\xi^{-k}$.
- **Interpretation:** ξ^{-1} upper bound on typical H^α-energy per particle.

Joint results with N. Pavlovic, N. Tzirakis, K. Taliaferro:

1. **Local in time existence and uniqueness** of solutions to focusing and defocusing GP hierarchies [C-P].

2. **Blow-up** [C-P-Tz] for L^2-supercritical focusing GP hierarchies, via conserved 1-particle energy functional & virial identity.

3. **Interaction Morawetz identities** for the GP hierarchy, [C-P-Tz].

4. **Global well-posedness** for GP via higher order conserved energy functionals, assuming *positive semi-definiteness* [C-P].

5. **Existence** of solutions for GP **without KM condition**, [C-P].

6. **Derivation of the 3D cubic GP hierarchy** in [KM] spaces for $\mathcal{H}_\xi^{1+\delta}$ initial data, [C-P].

7. **Global well-posedness** for cubic defocusing GP [C-Ta] including derivation of GP for $\mathcal{H}_\xi^1$ initial data.

Local in time existence and uniqueness

[C-P, DCDS'10] GP can be written as system of integral equations

$$\begin{aligned}\Gamma(t) &= e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)\\ \underline{\widehat{B}\Gamma(t)} &= \widehat{B}\, e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\,\widehat{B}\, e^{i(t-s)\widehat{\Delta}_\pm}\underline{\widehat{B}\Gamma(s)}\,,\end{aligned}$$

Prove local well-posedness via **fixed point argument** in the space

$$\mathfrak{W}^\alpha_\xi(I) := \{\ (\Gamma,\Theta) \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi \times L^2_{t\in I}\mathcal{H}^\alpha_\xi\ \}, \tag{0.4}$$

$$\|(\Gamma\,,\,\Theta)\|_{\mathfrak{W}^\alpha_\xi(I)} := \|\Gamma\|_{L^\infty_{t\in I}\mathcal{H}^\alpha_\xi} + \|\Theta\|_{L^2_{t\in I}\mathcal{H}^\alpha_\xi}$$

where $I = [0,T]$ and $\Theta = \widehat{B}\Gamma$.

Existence of solutions to GP without KM condition

[C-P, PAMS'13] Question: Is the Klainerman-Machedon condition

$$\widehat{B}\Gamma \in L^1_{t\in[0,T]}\mathcal{H}^1_\xi$$

necessary for both existence and uniqueness ?

- In fact, for the existence part, it is not required.
- However, the solution obtained in new approach satisfies the KM condition as an *a posteriori* result.

Flavor of the proof: Fix $K \in \mathbb{N}$. We consider solutions $\Gamma^K(t)$ of the GP hierarchy,

$$i\partial_t \Gamma^K = \widehat{\Delta}_\pm \Gamma^K + \mu \widehat{B} \Gamma^K ,$$

for the *truncated initial data* $\Gamma^K(0) = (\gamma_0^{(1)}, \dots, \gamma_0^{(K)}, 0, 0, \dots)$, where the m-th component of $\Gamma^K(t) = 0$ for all $m > K$, and establish:

Step 1 Existence of solutions to the truncated GP

Step 2 Existence of the strong limit:

$$\Theta = \lim_{K\to\infty} \widehat{B}\Gamma^K \quad \in L^1_{t\in I}\mathcal{H}^\alpha_{\xi''} .$$

Step 3 Existence of the strong limit:

$$\Gamma = \lim_{K\to\infty} \Gamma^K \quad \in L^\infty_{t\in I}\mathcal{H}^\alpha_{\xi} ,$$

that satisfies the GP hierarchy, given the initial data $\Gamma_0 \in \mathcal{H}^\alpha_{\xi'}$.

Step 4 Comparing the equations satisfied by Θ and Γ, we prove that

$$\widehat{B}\Gamma = \Theta.$$

Derivation of cubic GP hierarchy in 3D

Thm *[C-P, AHP'13] Let* $\delta > 0$ *arbitrary. Let* $0 < \beta < \frac{1}{4+2\delta}$.

Let Φ_N *denote a solution of the* N*-body Schrödinger equation, for which*

$$\Gamma^{\Phi_N}(0) = (\gamma^{(1)}_{\Phi_N}(0), \dots, \gamma^{(N)}_{\Phi_N}(0), 0, 0, \dots)$$

has a strong limit

$$\Gamma_0 = \lim_{N\to\infty} \Gamma^{\Phi_N}(0) \in \mathcal{H}^{1+\delta}_{\xi'}. \tag{0.5}$$

Denote by

$$\Gamma^{\Phi_N}(t) := (\gamma^{(1)}_{\Phi_N}(t), \dots, \gamma^{(N)}_{\Phi_N}(t), 0, 0, \dots, 0, \dots) \tag{0.6}$$

the solution to the associated BBGKY hierarchy.

Define the truncation operator $P_{\leq K}$ *by*

$$P_{\leq K}\Gamma = (\gamma^{(1)}, \dots, \gamma^{(K)}, 0, 0, \dots). \tag{0.7}$$

Then, letting

$$K = K(N) = b_0 \log N \,,$$

for a sufficiently large constant $b_0 > 0$*, we have*

$$\lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N} = \Gamma \tag{0.8}$$

strongly in $L^\infty_{t\in I}\mathcal{H}^1_\xi$*, and*

$$\lim_{N\to\infty} \widehat{B}_N P_{\leq K(N)}\Gamma^{\Phi_N} = \widehat{B}\Gamma \tag{0.9}$$

strongly in $L^2_{t\in[0,T]}\mathcal{H}^1_\xi$*, for* $\xi > 0$ *sufficiently small.*

In particular, Γ *solves the cubic, defocusing GP hierarchy with* $\Gamma(0) = \Gamma_0$*, and* $(\Gamma, \widehat{B}\Gamma)$ *is an element of the space* $\mathfrak{W}^1_\xi([0,T])$ *with*

$$\|(\Gamma\,,\,\Theta)\|_{\mathfrak{W}^\alpha_\xi(I)} := \|\Gamma\|_{L^\infty_{t\in I}\mathcal{H}^\alpha_\xi} + \|\Theta\|_{L^2_{t\in I}\mathcal{H}^\alpha_\xi}$$

Remarks:

- The result implies that the N-BBGKY hierarchy has a limit in the space $\mathfrak{W}^1_\xi([0,T])$ introduced in [CP], which is based on the space considered by [KM]. For factorized solutions, this provides the derivation of the cubic defocusing NLS in those spaces.
- We assume that the i.d. has a limit, $\Gamma^{\phi_N}(0) \to \Gamma_0 \in \mathcal{H}^{1+\delta}_{\xi'}$ as $N \to \infty$, which does not need to be factorized. We note that in [ESY], i.d. is assumed to be asymptotically factorized.
- In [ESY], the limit $\gamma^{(k)}_{\Phi_N} \rightharpoonup \gamma^{(k)}$ of solutions to the BBGKY to solutions to the GP holds in the weak, subsequential sense, for an arbirary but fixed k. In our approach, we prove strong convergence for a sequence of suitably truncated solutions to the BBGKY, in an entirely different space.

Key idea of the proof: Use of auxiliary truncations from our recent proof of existence of solutions to the GP.

The proof contains four main steps:

1. Prove existence of solution Γ_N^K to N-BBGKY hierarchy in $\mathfrak{W}_\xi^{1+\delta}([0,T])$ with truncated initial data $P_{\leq K}\Gamma_N(0)$.

2. Compare to solution Γ^K of GP with truncated initial data $P_{\leq K}\Gamma_0$,

$$\|(\Gamma_N^{K(N)}, \widehat{B}_N\Gamma_N^{K(N)}) - (\Gamma^{K(N)}, \widehat{B}\Gamma^{K(N)})\|_{\mathfrak{W}_\xi^1([0,T])} \to 0 \quad (N \to \infty)$$

3. Also compare to the truncated solution $P_{\leq K(N)}\Gamma^{\Phi_N}$ of N-BBGKY,

$$\|(\Gamma_N^{K(N)}, \widehat{B}_N\Gamma_N^{K(N)}) - (P_{\leq K(N)}\Gamma^{\Phi_N}, \widehat{B}_N P_{\leq K(N)}\Gamma^{\Phi_N})\|_{\mathfrak{W}_\xi^1([0,T])} \to 0$$

4. Prove that $(\Gamma^{K(N)}, \widehat{B}\Gamma^{K(N)}) \to (\Gamma, \widehat{B}\Gamma)$ in $\mathfrak{W}_\xi^1([0,T])$ where Γ solves the cubic defocusing GP hierarchy. Already done in PAMS paper.

Related works

[X.Chen-Holmer '13]: Derivation of cubic GP in $\mathbb{R}^3$ for $\beta \in (0, \frac{2}{3})$.

Via weak-* limit of $\gamma_N^{(k)}$, and proof that KM condition is satisfied, uniformly in N.

Proof uses X_b- and Koch-Tataru spaces.

GWP for cubic defocusing GP

Thm: [C-P] Define **higher order energy functionals**

$$\langle K^{(m)}\rangle_{\Gamma(t)} := \mathrm{Tr}_{1,3,5,\dots,2(m-1)+1}(K^{(m)}\gamma^{(2m)}(t))$$

for $\ell \in \mathbb{N}$, and

$$K_\ell := \frac{1}{2}(1-\Delta_{x_\ell})\mathrm{Tr}_{\ell+1} + \frac{1}{4}B^+_{\ell;\ell+1}$$
$$K^{(m)} := K_1 K_3 \cdots K_{2(m-1)+1}.$$

Let $\Gamma \in \mathfrak{H}^1_\xi$ be symmetric, admissible solution of GP. Then,

$$\langle K^{(m)}\rangle_{\Gamma(t)} = \langle K^{(m)}\rangle_{\Gamma_0}$$

are conserved $\forall m \in \mathbb{N}$.

If $\gamma^{(k)}(t)$ *positive semidefinite*, $\langle K^{(m)}\rangle_{\Gamma(t)}$ is upper bound on $\|\gamma^{(k)}(t)\|_{\mathfrak{h}^1}$

$\Rightarrow$ Global well-posedness

Thm: *[C-Taliaferro] Let* $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$ *be positive semidefinite, admissible,* $\mathrm{Tr}\,\gamma_0^{(1)} = 1$*. Then, for* $0 < \xi' < 1$, $\exists\ \xi = \xi(\xi')$ *so that* $\exists!$ *global solution*

$$\Gamma \ \in \ \mathcal{V}^1_\xi(\mathbb{R}) := \Big\{ \Gamma \in C(\mathbb{R}, \mathcal{H}^1_\xi) \Big| B^+\Gamma, B^-\Gamma \in L^2_{loc}(\mathbb{R}, \mathcal{H}^1_\xi) \Big\}$$

to cubic defocusing GP with initial data Γ_0*.* $\Gamma(t)$ *is positive semidefinite,*

$$\|\Gamma(t)\|_{\mathcal{H}^1_{\xi_1}} \leq \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}} \quad \forall t$$

Proof idea: Given GP, truncate initial data above N-th term, $\Gamma_{0,N}$.

Solve N-BBGKY with initial data $\Gamma_{0,N}$ (not a pure state !) with an *auxiliary* N-body Schrödinger Hamiltonian H_N.

Then, $\Gamma_N(t)$ is *positive semidefinite* for all N.

Prove $\Gamma_N(t) \to \Gamma(t)$ in $\mathcal{V}^1_\xi(\mathbb{R})$, as $N \to \infty$.

With $\mathcal{H}^1_\xi$ instead of $\mathcal{H}^{1+\delta}_\xi$ initial data, can use conservation of higher energy functionals *iteratively* to enhance LWP to GWP.

Thank you !!!