Emergence of collective dynamics from a purely stochastic origin

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i) some simple rank-based dynamics in one space dimension,
ii) some models of inviscid chemotaxis generalizing the inviscid Burgers equation in higher dimensions,
ii) the classical Newtonian model of gravitation (*)

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AN EXAMPLE OF RANK-BASED DYNAMICS IN 1D

Consider $N$ taxpayers labelled by $\alpha \in \{1, \cdots, N\}$.

$Z_n(\alpha) \geq 0$ is the taxable income of year $n$.

$\sigma_n(\alpha) \in \{1, \cdots, N\}$ is the rank of $Z_n(\alpha)$ in $\{Z_n(1), \cdots, Z_n(N)\}$.
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Model: $Z_{n+1}(\alpha) = Z_n(\alpha) \exp(r \tau) \exp(-G(\sigma_n) \tau)$ with a uniform growth rate $r$ for all incomes and a tax rate $G$ that depends only on the rank.
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This can be related to hyperbolic scalar conservation laws, the formation of shock waves corresponding to the emergence of classes.
Example: formation of 2 classes

Evolution of the income distribution, starting from a linear profile, with formation of two classes (i.e. two "shocks" in terms of conservation laws).
(Data: $N = 100$, $\tau = 0.01$, $F(u) = u + \frac{\sin(4\pi u)}{4}$, $u \in [0, 1]$, $t \in [0, 1]$, $\tau = 0.01$.)
RANK BASED DYNAMICS IN 1D: (old) RESULTS

For the slightly more general model (with pseudo-noise in option)

\[ X_{n+1}(\alpha) = X_n(\alpha) + \tau F(w) + (-1)^{(N-1)w} \sqrt{2\eta\tau} R(w), \quad w = \frac{\sigma_n(\alpha) - 1}{N - 1} \]
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1) Asymptotic behavior \( \tau \ll 1, N \gg 1 \), for \( u_n(x) = \frac{1}{N} \sum_{\alpha=1}^{N} 1\{x > X_n(\alpha)\} \)

\[ \partial_t u + \partial_x(f(u)) = \eta \partial_{xx}(r(u)), \quad F(u) = f'(u), \quad R(u) = r'(u) \geq 0 \]
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1) Asymptotic behavior \( \tau \ll 1, N \gg 1 \), for \( u_n(x) = \frac{1}{N} \sum_{\alpha=1}^{N} 1\{x > x_n(\alpha)\} \)

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2) A unique "class" emerges whenever \( \forall u \in ]0, 1[ \), \( f(u) > f(0) = f(1) \).

We consider $N$ particles in $\mathbb{R}^d$ subject to independent Brownian motions and issued from a cubic lattice $\{A(\alpha) \in \mathbb{R}^d, \alpha = 1, \cdots, N\}$

$$Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon}B_t(\alpha), \quad \alpha = 1, \cdots, N$$

We call "point cloud at time $t$" the collection of positions $\{Y_t(\alpha)\}$ reached by these particles, disregarding their label $\alpha \in \{1, \cdots, N\}$. 
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We call "point cloud at time $t$" the collection of positions $\{Y_t(\alpha)\}$ reached by these particles, disregarding their label $\alpha \in \{1, \ldots, N\}$. In other words, the cloud lives in $(\mathbb{R}^d)^N / S_N$, where $S_N$ is the symmetric group (of all permutations of the $N$ first integers).
WHERE IS THE BROWNIAN CLOUD AT TIME $T$?

At a fixed time $T > 0$, the probability for the moving cloud

$$Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon}B_t(\alpha), \quad \alpha = 1, \ldots, N$$

to be observed at $X_T = (X_T(\alpha), \alpha = 1, \ldots, N) \in \mathbb{R}^{dN}$ has density

$$\frac{1}{Z} \sum_{\sigma \in S_N} \prod_{\alpha=1}^{N} \exp\left(-\frac{|X_T(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T}\right)$$

Here, we crucially used that the particles are indistinguishable!!!
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$$Z = N! \sqrt{2\pi \epsilon T}^{Nd}, \quad |.| \text{ and } ||.||| = \text{euclidean norms in } \mathbb{R}^d \text{ and } \mathbb{R}^{Nd}.$$ 

Here, we crucially used that the particles are indistinguishable!!!
As a simple consequence of the “large deviation principle”, we note that, as $\epsilon \to 0$, the observer at time $T$ feels that the particles have travelled along straight lines by “optimal transport” $X_t = (1 - tT)A_{\sigma_{\text{opt}}} + tT X_T$, $\sigma_{\text{opt}} = \text{Argsup}_{\sigma \in S_N} \langle (X_T, A_\sigma) \rangle$, $t \in [0, T]$.

$$\lim_{\epsilon \to 0} \epsilon \log \frac{1}{Z} \sum_{\sigma \in S_N} \exp\left(-\frac{\|X_T - A_\sigma\|^2}{2\epsilon T}\right) = \frac{1}{2T} \inf_{\sigma \in S_N} \|X_T - A_\sigma\|^2$$
As a simple consequence of the "large deviation principle", we note that, as $\epsilon \to 0$, the observer at time $T$ feels that the particles have travelled along straight lines by "optimal transport"

$$X_t = (1 - \frac{t}{T}) A_{\sigma_{opt}} + \frac{t}{T} X_T, \quad \sigma_{opt} = \text{Argsup}_{\sigma \in S_N} \left( (X_T, A_{\sigma}) \right), \quad t \in [0, T]$$
LAW AND DISORDER!

From the apparent motion of the cloud up to time $T$

\[ X_t = (1 - \frac{t}{T}) A_{\sigma_{opt}} + \frac{t}{T} X_T, \quad \sigma_{opt} = \text{Arginf}_{\sigma \in S_N} \|X_T - A_{\sigma}\|^2 \]

we easily deduce \[ \sigma_{opt} = \text{Arginf}_{\sigma \in S_N} \|X_t - A_{\sigma}\|^2, \quad \forall t \in ]0, T[ \]
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we easily deduce

$$\sigma_{opt} = \text{Arginf}_{\sigma \in S_N} ||X_t - A_{\sigma}||^2, \quad \forall t \in ]0, T]$$

This leads to the apparent "law"

$$\frac{dX_t}{dt} = \frac{X_t - A_{\sigma_{opt}}}{t}, \quad \sigma_{opt} = \text{Arginf}_{\sigma \in S_N} ||X_t - A_{\sigma}||^2, \quad t \in ]0, T]$$

just resulting of the observation of a purely random motion!
\( t = e^\theta \) leads to

\[
\frac{dX_\theta}{d\theta} = X_\theta - A_{\sigma_{opt}}, \quad \sigma_{opt} = \text{Arginf}_{\sigma \in S_N} \| X_\theta - A_\sigma \|^2
\]
ZELDOVICH MODEL AND INVISCID CHEMOTAXIS

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Using optimal transport tools, we find, as formal continuous limit,

\[ \partial_\theta \rho - \nabla \cdot (\rho \nabla x \varphi) = 0, \quad \det(I + D_x^2 \varphi) = \rho; \quad \rho \geq 0, \quad \varphi \in \mathbb{R}, \quad (\theta, x) \in \mathbb{R}^{1+d} \]
ZELDOVICH MODEL AND INVISCID CHEMOTAXIS

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\]

This is a multidimensional generalization of the rank based dynamics discussed at the beginning of this talk. It is equivalent to the Zeldovich model (1970) in Cosmology. It can also be seen as a fully nonlinear version of the (inviscid) chemotaxis model:

\[
\partial_\theta \rho - \nabla \cdot (\rho \nabla x \varphi) = 0, \quad \Delta \varphi = \rho - \bar{\rho}, \quad \bar{\rho} = \int \rho(t, x) dx = 1
\]
Monge-Ampère gravitation: a simulation of the Zeldovich model
We first observe that the probability density we found for the Brownian point cloud to be found at \( X \in \mathbb{R}^{Nd} \) at time \( t > 0 \)

\[
\frac{1}{N! \sqrt{2\pi \epsilon t}^{Nd}} \sum_{\sigma \in S_N} \exp\left(-\frac{\|X - A_\sigma\|^2}{2\epsilon t}\right), \quad X \in \mathbb{R}^{Nd}
\]

is just the solution \( \rho(t, X) \) of the heat equation in \( \mathbb{R}^{Nd}/S_N \)

\[
\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \Delta \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in S_N} \delta(X - A_\sigma), \quad X \in \mathbb{R}^{Nd}
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"SURFING THE HEAT WAVE"

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\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in S_N} \delta(X - A_\sigma), \quad X \in \mathbb{R}^{Nd}
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\]

For arbitrarily chosen position \( X_{t_0} \in \mathbb{R}^{Nd} \) at \( t_0 > 0 \), let us "surf" the "heat wave" by solving the ODE

\[
\frac{dX_t}{dt} = \nu(t, X_t), \quad \nu(t, X) = -\frac{\epsilon}{2} \nabla X \log \rho(t, X), \quad t \geq t_0
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For arbitrarily chosen position $X_{t_0} \in \mathbb{R}^{Nd}$ at $t_0 > 0$, let us "surf" the "heat wave" by solving the ODE

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This is an adaptation of de Broglie's "onde pilote" concept. As a matter of fact, a similar calculation also works for the free Schrödinger equation:

$$(i \partial_t + \triangle)\psi = 0, \quad \psi(0, X) = \sum_{\sigma} \exp(-||X - A_\sigma||^2/a^2), \quad \nu = \nabla \text{Im} \log \psi$$
Using $t = e^{2\theta}$, the "heat wave" ODE explicitly reads

$$\frac{dX_\theta}{d\theta} = v_\epsilon(\theta, X_\theta), \quad v_\epsilon(\theta, X) = X - \sum_{\sigma \in S_N} A_\sigma \exp\left(\frac{-\|X - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right) \sum_{\sigma \in S_N} \exp\left(\frac{-\|X - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)$$
SURFING THE "HEAT WAVE" SYSTEM
... WITH ADDITIONAL NOISE!

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$$

To get Newton’s gravitation, our key idea is now to consider large deviations of this ODE subject to additional noise:

$$
\frac{dX_\theta}{d\theta} = v_\epsilon(\theta, X_\theta) + \sqrt{\eta} \frac{dB_\theta}{d\theta}
$$
THROUGH LARGE DEVIATION AND LEAST ACTION PRINCIPLES

we end up, as $\epsilon, \eta \to 0$, with the following dynamical system

$$\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha)), \quad X_\theta(\alpha) \in \mathbb{R}^d, \quad \alpha = 1, \ldots, N$$

$$\sigma_{opt} = \text{Arginf} \sigma \in S_N \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2$$
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Let

$$\sigma_{opt} = \text{Arginf} \, \sigma \in S_N \, \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2$$

involving, at each time $t$, a discrete optimal transport problem which leads, in the limit $N \to \infty$, to a Monge-Ampère equation.
WRITTEN IN KINETIC TERMS:

\[
\partial_\theta f(\theta, x, \xi) + \nabla_x \cdot (\xi f(\theta, x, \xi)) - \nabla_\xi \cdot (\nabla_x \varphi(\theta, x)f(\theta, x, \xi)) = 0
\]

\[
\det(\mathbb{I} + D_x^2 \varphi(\theta, x)) = \int_{\mathbb{R}^d} f(\theta, x, d\xi), \quad (\theta, x, \xi) \in \mathbb{R}^{1+d+d}
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\[ \partial_{\theta} f(\theta, x, \xi) + \nabla_x \cdot (\xi \, f(\theta, x, \xi)) - \nabla_{\xi} \cdot (\nabla_x \varphi(\theta, x)f(\theta, x, \xi)) = 0 \]

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(with possible large scale computations thanks to recent efficient Monge-Ampère solvers by Quentin Mérigot and Bruno Lévy.)

For weak fields \( \varphi \), we asymptotically recover the Poisson equation \( \Box \varphi = \int_{\mathbb{R}^d} f(\theta, x, d\xi) - 1 \) which describes Newtonian gravitation.

SPECIAL THANKS TO ANDREA, ANIL, EITAN, GIANLUCA, SIDDHARTHA and TRISTAN!!!!

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We first pass to the limit $\eta \to 0$, while $\epsilon > 0$ is kept fixed. The large deviation theory tells us that the probability to join point $X_{\theta_0}$ at $\theta = \theta_0$ and point $X_{\theta_1}$ at later time $\theta = \theta_1$ behaves as

$$\exp\left(-\frac{\mathcal{A}}{\eta}\right), \quad \eta \to 0,$$

$$\mathcal{A} = \frac{1}{2} \int_{\theta_0}^{\theta_1} \| \frac{dX_{\theta}}{d\theta} - v_\epsilon(\theta, X_{\theta}) \|^2 d\theta$$

where we call $\mathcal{A}$ the Freidlin-Vencel action.
Γ—LIMIT OF THE VENCEL-FREIDLIN ACTION

We now pass to the $\Gamma-$limit $\epsilon \downarrow 0$ (*) in the Vencel-Freidlin action

$$A = \frac{1}{2} \int_{\theta_0}^{\theta_1} \left\| \frac{dX_\theta}{d\theta} - v_\epsilon(\theta, X_\theta) \right\|^2 d\theta,$$

$$v_\epsilon(\theta, X) = -\nabla_X \Phi_\epsilon(\theta, X), \quad \Phi_\epsilon(\theta, X) = \epsilon e^{2\theta} \log \sum_{\sigma \in S_N} \exp \left( -\frac{\|X - A_\sigma\|^2}{2\epsilon e^{2\theta}} \right)$$

noticing that

$$\lim_{\epsilon \downarrow 0} \Phi_\epsilon(\theta, X) = -\frac{1}{2} \inf_{\sigma \in S_N} \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2$$

(*) thanks to L. Ambrosio, private communication.