Developmental Partial Differential Equations

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Kinet Young Researchers' Workshop
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Outline

1. Motivation: A description of oogenesis
2. The heat equation on time-varying manifolds
3. A "Lie bracket" between transport and heat
4. Control of growth via a signal
5. Future Directions

Drosophila oogenesis

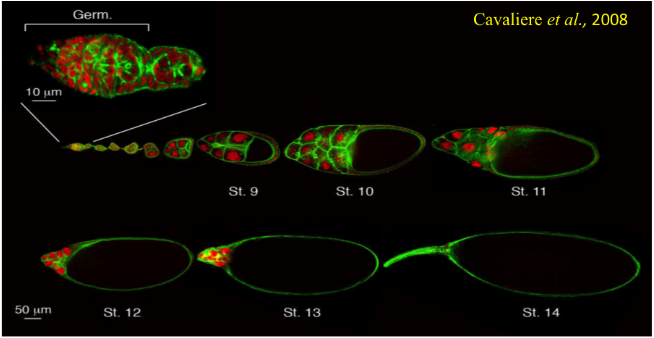
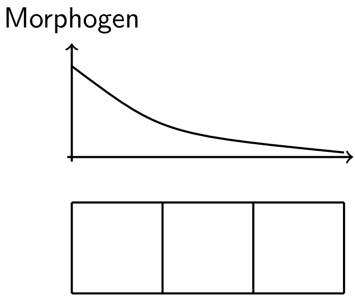


Figure: *Drosophila melanogaster* oogenesis

Morphogens



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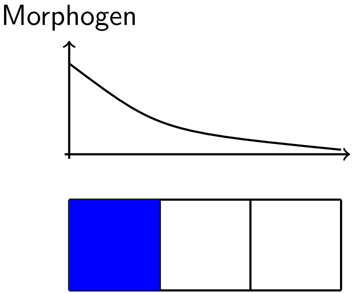

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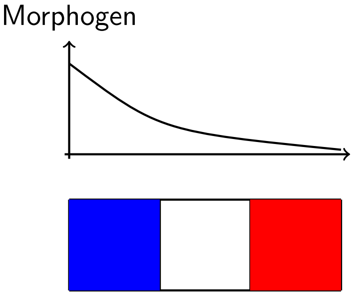

Figure: "French flag model"

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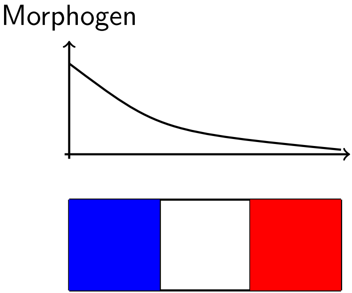

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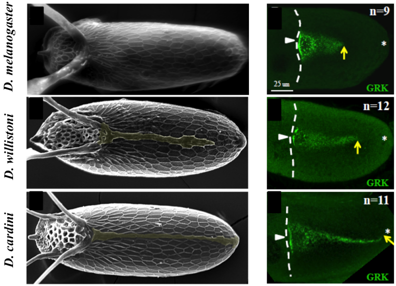

Figure: Morphlogies of *Drosophila* eggshells and Gurken patterning

Mechanism of Gurken diffusion and internalization

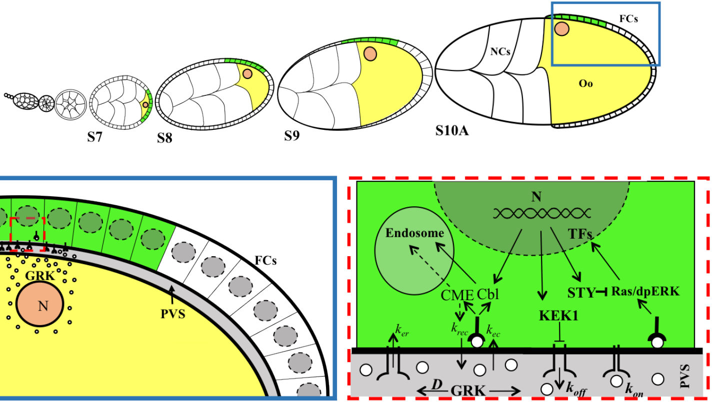

Figure: Gurken diffusion from oocyte nucleus in the perivitelline space and internalization into the follicle cells

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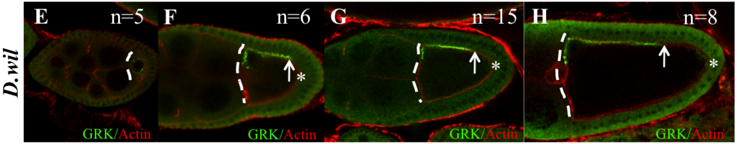
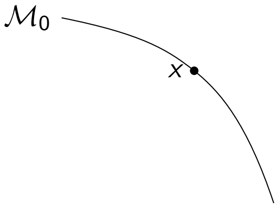


Figure: Gurken in *Drosophila willistoni*

General model

- $\mathcal{M}_t$: time-varying compact manifold of dimension n embedded in $\mathbb{R}^d = \mathbb{R}^{n+1}$
  Organism's membrane



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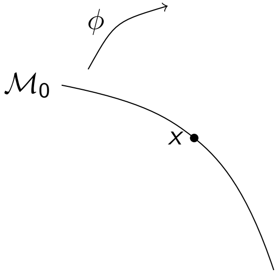
  Growth vector field



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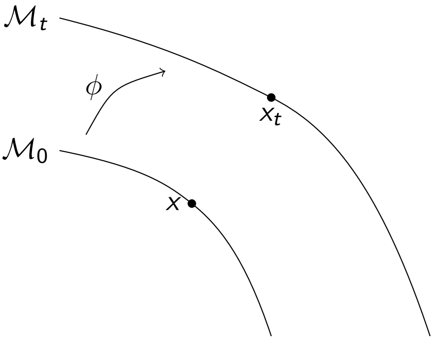

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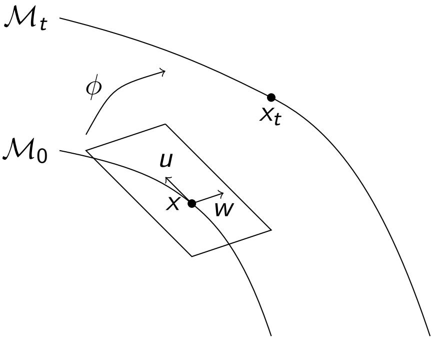

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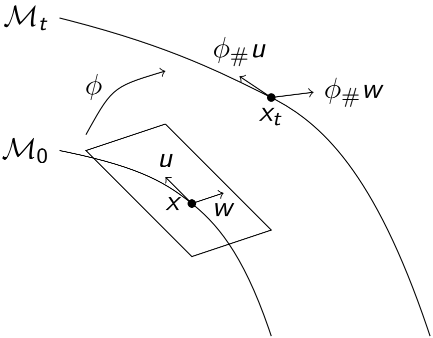

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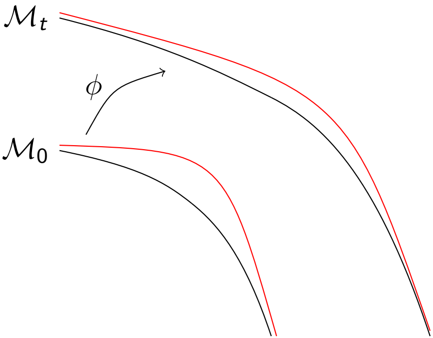
- $\mu_t \in \mathcal{P}(\mathcal{M}_t)$: probability measure on $\mathcal{M}_t$ (also, $\mu_t \in \mathcal{P}_c(\mathbb{R}^d)$)
  Morphogen diffusing in intercellular space



Coupling of diffusion and manifold evolution

Evolution of μ_t by the combined transport and diffusion:

Transport-diffusion PDE

$$\partial_t \mu_t + \nabla \cdot (v[\mu_t]\mu_t) = \Delta_t \mu_t \tag{1}$$

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- v: a Lipschitz function w.r.t. the Wasserstein distance W_2

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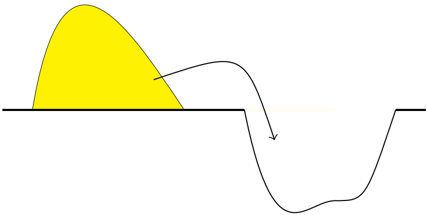
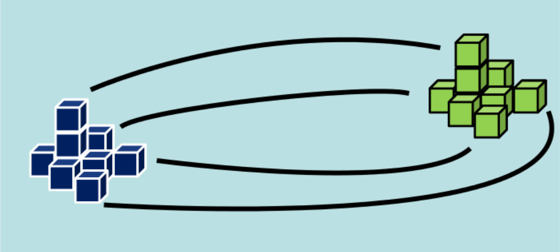
Wasserstein distance: Monge transportation problem



How do you best move a pile of sand to fill up a given hole of the same total volume?

Wasserstein distance: Monge transportation problem



Monge's problem (1781)

Given $\mu, \nu \in \mathcal{P}(X)$ and $c : X \times X \to \mathbb{R}^+$ a Borel-measurable function,

$$\text{Minimize } \int_X c(x, T(x)) d\mu(x)$$

among all transport maps $T : X \to X$ s.t. $T\#\mu = \nu$.

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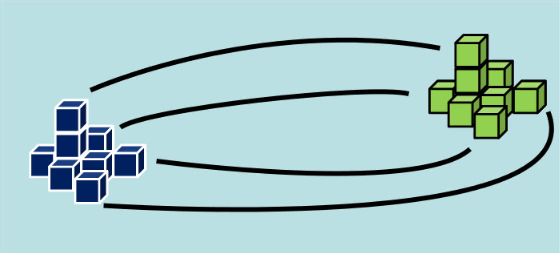

Kantorovich's formulation (1940's)

Given $\mu, \nu \in \mathcal{P}(X)$ and $c : X \times X \to \mathbb{R}^+$ a Borel-measurable function,

$$\text{Minimize } \int_{X \times X} c(x,y) d\gamma(x,y)$$

where $\gamma \in \Pi(\mu, \nu) := \{\rho \in \mathcal{P}(X \times X) \mid \pi_1 \# \rho = \mu,\ \pi_2 \# \rho = \nu\}$.

p-Wasserstein distance

$$\mathcal{W}_p(\mu,\nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

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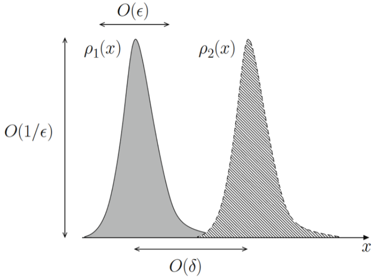

Figure: Two measures with different L^1 and $\mathcal{W}_1$ distances (respectively $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$).

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Laplace-Beltrami operator

Laplace-Beltrami operator: generalization of the Laplacian on Riemannian manifolds.

$$\Delta f := \nabla \cdot \nabla f$$

Let $(x_i)_{i\in\{1,\ldots,n\}}$ be a coordinate system on $\mathcal{M}_t$ and g_t be the metric tensor of $\mathcal{M}_t$.
Let $f \in \mathcal{C}^\infty(\mathcal{M}_t)$.

$$\Delta_t f = \frac{1}{\sqrt{|g_t|}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\sqrt{|g_t|} \sum_{j=1}^{n} g_t^{ij} \frac{\partial}{\partial x_j} f\Big)$$

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Theorem (Piccoli, Pouradier Duteil, Rossi)

There exists a unique solution to Equation (1).

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Weak formulation

For all $f \in C^\infty(\mathbb{R}, \mathbb{R}^d)$

$$\partial_t \int_{\mathbb{R}^d} f \, d\mu_t - \int_{\mathbb{R}^d} (\nabla f \cdot v[\mu_t]) d\mu_t = \int_{\mathcal{M}_t} \Delta_t f \, d\mu_t. \tag{2}$$

Proof of existence

Sketch of proof (Existence).

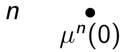
- Introduce a discrete scheme that alternates time steps of transport and diffusion, and prove that it admits a convergent subsequence
- Prove that the limit is a solution to the PDE (1)

□

Proof of existence and uniqueness: Discrete scheme

Scheme $\mathbb{S}$

Define $\tau_n = t_n := 2^{-n}T$. Let $\mu^n(0) := \mu_0$.



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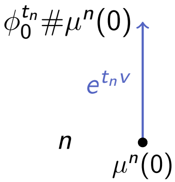

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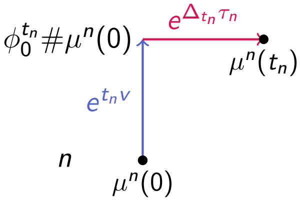

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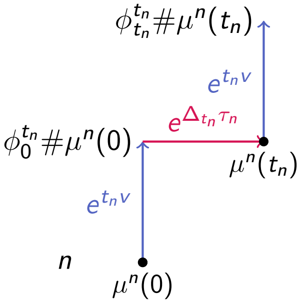

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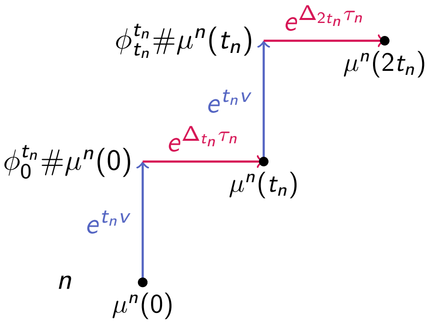

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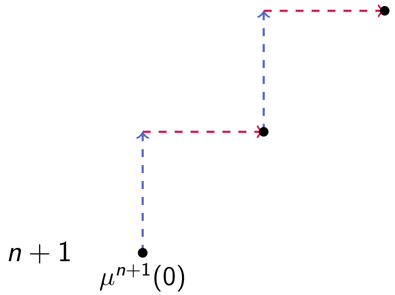

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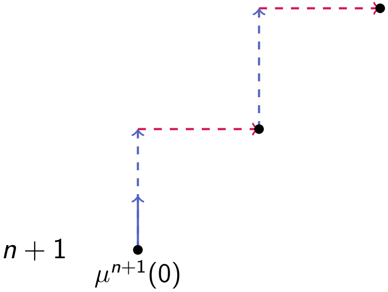

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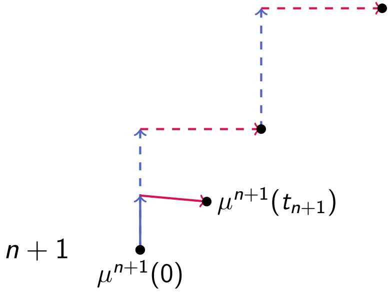

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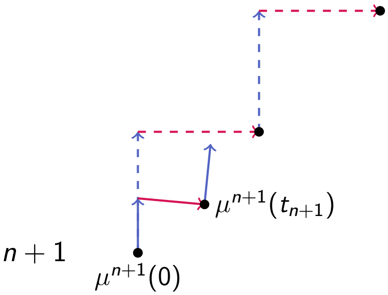

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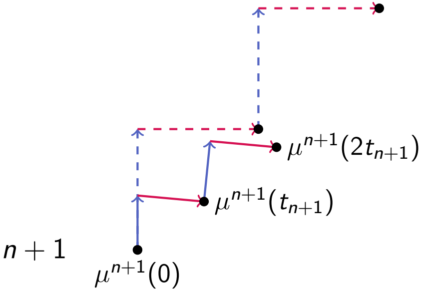

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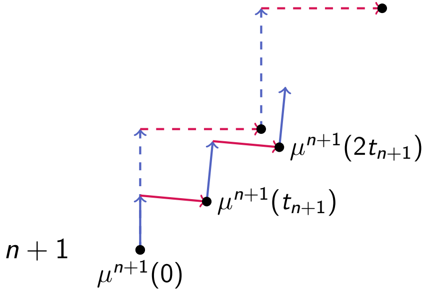

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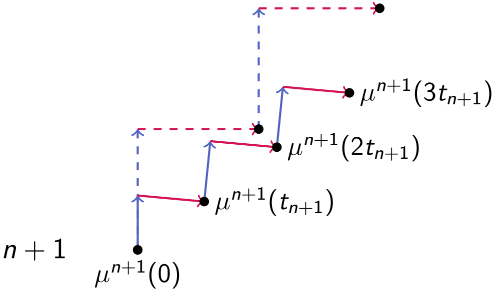

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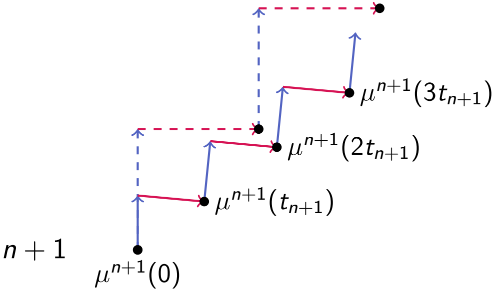

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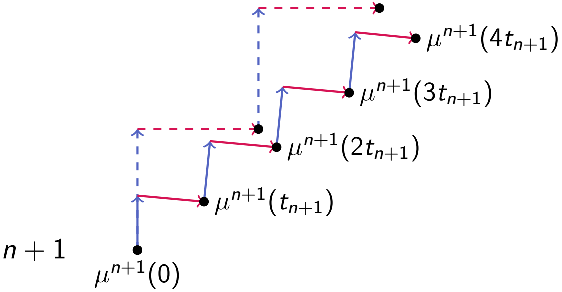
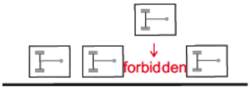
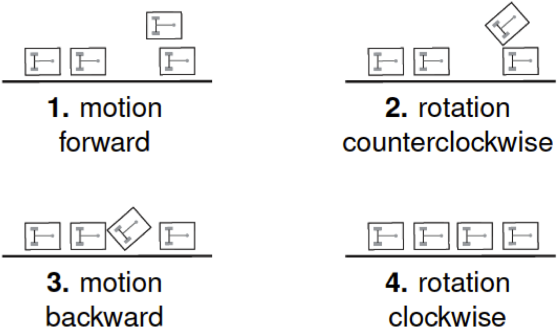


Lie bracket: intuitive example



Four motions with the same amplitude perform forbidden motion:



Non commutativity of transport and heat

Definition: "Lie bracket" between transport and heat

$$[\Delta, v]\mu := \lim_{t=\tau\to 0} \frac{\Phi_{-t}\#\left(e^{\tau\Delta_t}(\Phi_t\#\mu)\right) - e^{\tau\Delta_0}\mu}{t\tau}$$

with $\Phi_t\#$: push-forward via the flow generated by v

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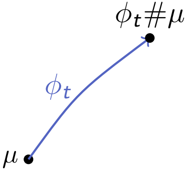

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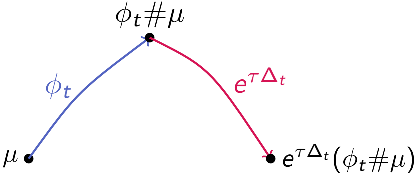

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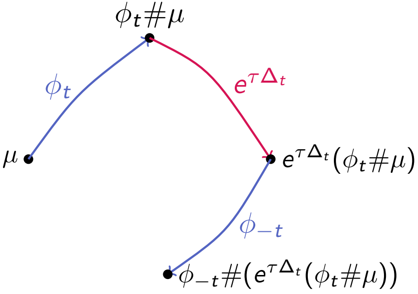

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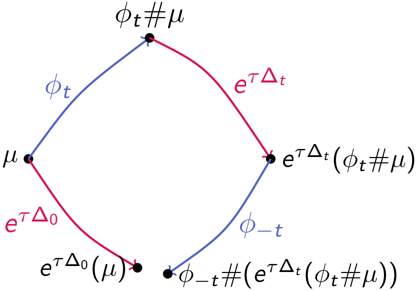


Simple example: Transport of S^1

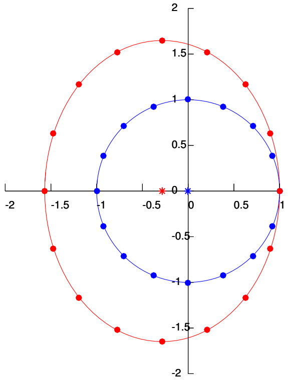
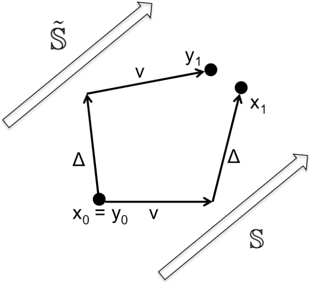


Figure: Transport of S^1 by $v(x,y) := (x-1, 2y)$. At $t = 0.25$, the resulting ellipse is centered at $(1-e^{0.25}, 0)$.

Simple example: Discrete scheme



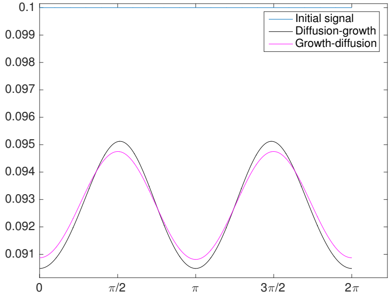


Figure: Iterative diffusion and transport

Simple example: Convergence of the bracket

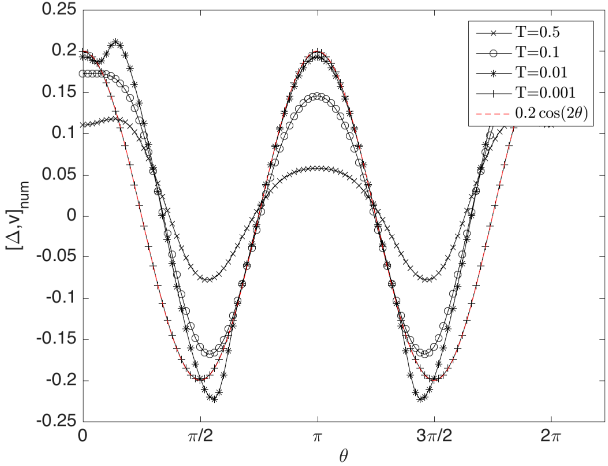


Figure: Convergence of the numerical approximations of the bracket to the theoretical expression for the initial signals $\mu_0(\theta) = 0.1 d\theta$.

Simple example: Convergence of the bracket

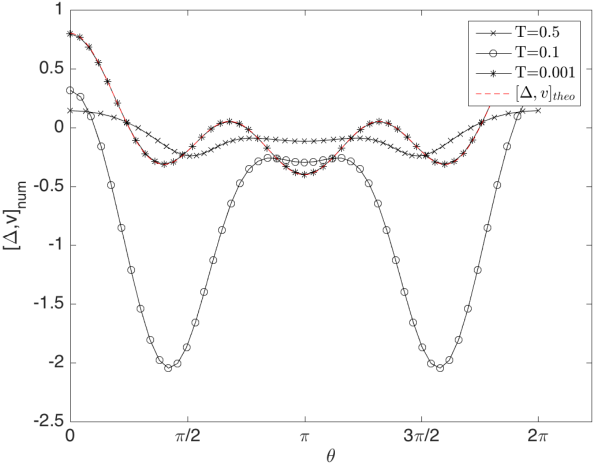


Figure: Convergence of the numerical approximations of the bracket to the theoretical expression for the initial signal $\mu_0(\theta) = 0.1(\cos(\theta) + 1)d\theta$.

Control of manifold evolution

Complete coupling of signal s and manifold r with control of s at a point.

$$\begin{cases} \partial_t r = s, \\ \partial_t s = \Delta_r s, \\ s(t, \theta = 0) = u(t). \end{cases} \tag{2}$$

where

- $r(t, \theta)$: radius of the cell;
- $s(t, \theta)$: growing signal (solving the heat equation);
- Δ_r: Laplace-Beltrami operator (depending on r);
- $u(t)$: control (value of s at a given point).

Example: S^1

$$\begin{cases} \partial_t r = s, \\ \partial_t s = \Delta_r s, \\ s(t, \theta = 0) = u(t), \\ \partial_\theta s(t, \theta = \pi) = 0. \end{cases} \tag{3}$$

with $r(0, \theta) = 1$ (constant radius) and $s(0, \theta) = 0$ (zero signal).
The Laplace-Beltrami operator is:

$$\Delta_r s = \frac{1}{r^2 + r_\theta^2} \partial_\theta^2 s - \frac{r r_\theta + r_\theta \partial_\theta^2 r}{(r^2 + r_\theta^2)^2} \partial_\theta s \tag{4}$$

Simulations: constant control

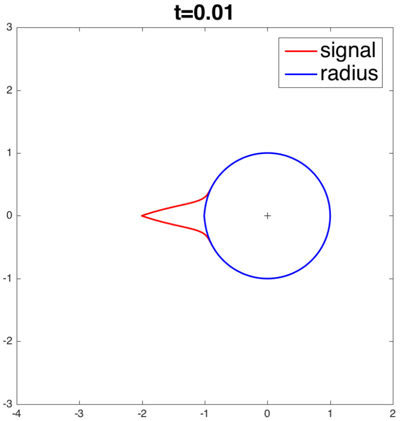


Figure: Simulations with a constant control $u \equiv 1$.

Simulations: sine control

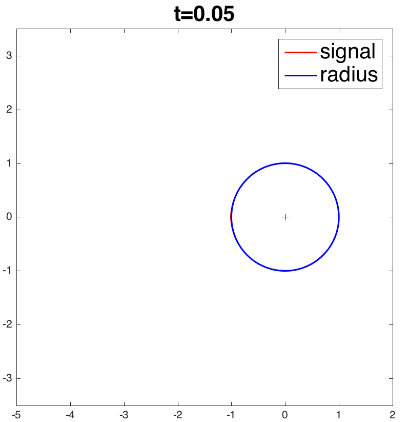


Figure: Simulations with a sinusoidal control.

Simulations: growth of circle

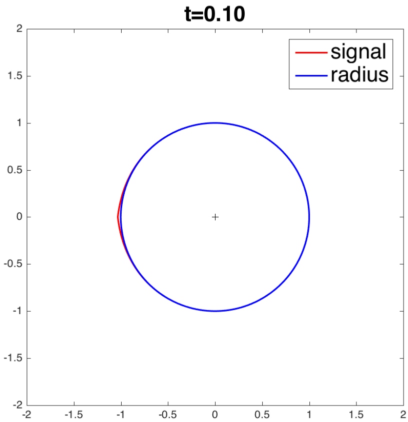


Figure: Simulations with a control $u(t) = 0.25\sin(\frac{2\pi}{5}t)$ for $t \in [0, 2.5]$ and $u(t) = 0$ for $t \in [2.5, 10]$.

Controllability

Exact controllability

Find a control $u : [0; T] \to \mathbb{R}$ such that the unique solution of (3) with $\forall\theta \in [0, 2\pi],\ r(t = 0, \theta) = r_0$ and $s(t = 0, \theta) = 0$ satisfies $\forall\theta \in [0, 2\pi],\ r(t = T, \theta) = r_1(\theta)$ and $s(t = T, \theta) = 0$.

Exact controllability cannot be obtained in general (e.g. for non-smooth configurations). Hence we relax our goal:

Approximate controllability

Find a control $u : [0; T] \to \mathbb{R}$ such that the unique solution of (3) with $\forall\theta \in [0, 2\pi],\ r(t = 0, \theta) = r_0(\theta)$ and $s(t = 0, \theta) = 0$ satisfies $\|r(t = T, \cdot) - r_1(\cdot)\|_{L^2} < \epsilon$ and $\|s(t = T)\|_{L^2} < \epsilon$.

Approximate controllability

Theorem

The system

$$\begin{cases} \partial_t r = s, \\ \partial_t s = \Delta_r s, \\ s(t, \theta = 0) = u(t), \\ \partial_\theta s(t, \theta = \pi) = 0 \end{cases}$$

is approximately controllable for r on $[0, T]$.

Future Directions

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- Reaction-diffusion equations - generalized Wasserstein distance

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- Reaction-diffusion equations - generalized Wasserstein distance
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- Cost of control - optimal control given a number of harmonics in Fourier series

Thank you for your attention!
Any questions?