An asymptotic preserving unified gas kinetic scheme for the grey radiative transfer equations

Song Jiang

Institute of Applied Physics and Computational Mathematics, Beijing

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(joint work with Wenjun Sun, Kun Xu)

Introduction

- An asymptotic limit associated with a PDE is a limit in which certain terms in the PDE are made "small" relative to other terms (ordering in size via a scaling parameter, say ϵ).
- In many instances, the scale lengths associated with solutions of the limiting equation are much larger than the smallest scale lengths associated with solutions of the original PDE.

When this is the case, an asymptotic-preserving (AP) discretization is necessary for near-asymptotic problems to avoid impractical mesh resolution requirement ($\Delta x \sim O(\epsilon)$).

Aim of this talk: To present an AP scheme for the grey radiative transfer system (and for the frequency-dependent radiative transfer system)

• Outline:

1. Governing equations
2. An AP scheme for the system
3. Asymptotic analysis, AP property
4. Numerical experiments
5. Frequency-dependent radiative transfer system
6. conclusions
7. Future studies

1. Governing equations

Grey radiative transfer equations:

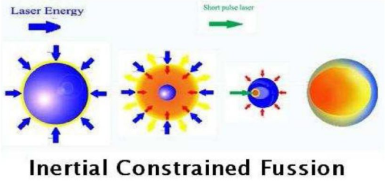
$$\frac{\epsilon^2}{c}\frac{\partial I}{\partial t} + \epsilon\vec{\Omega}\cdot\nabla I = \sigma\left(\frac{1}{4\pi}acT^4 - I\right), \qquad (1)$$

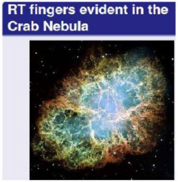
$$\epsilon^2 C_v \frac{\partial T}{\partial t} = \sigma\left(\int I d\vec{\Omega} - acT^4\right). \qquad (2)$$

$I(\vec{r}, \vec{\Omega}, t)$: Radiation intensity, $T(\vec{r}, t)$: Material temperature,
$\sigma(\vec{r}, T)$: Opacity, a: Radiation constant, c: Speed of light,
$\epsilon > 0$: Knudsen number, $C_v(\vec{r}, t)$: Specific heat,
$\vec{r}$: Spatial variable, $\vec{\Omega}$: Angular variable, t: Time variable.

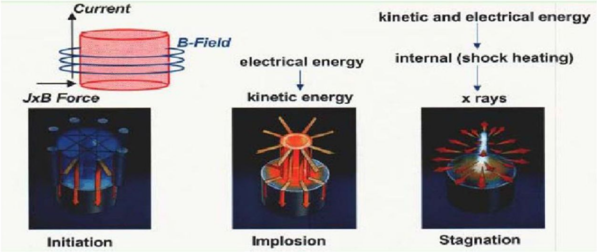
σI: absorption term; $\frac{\sigma}{4\pi}acT^4$: emission term.

Backgrounds





Z-pinch



Grey radiative transfer system finds applications in the study of ICF (inertia constrained fusion), Z-pinch, astrophysics, etc.

Numerical challenges

- **High dimensional problem**

  time: t, spatial: $\vec{r}$, direction : $\vec{\Omega}$.

- **For multi-material problems (multi-scale)**

  optically thick material: small ϵ/σ

  optically thin material: large ϵ/σ

- **To resolve physical scales, normally $\Delta x \sim O(\epsilon/\sigma)$**

  huge computing costs for small ϵ

Formally, as $\epsilon \to 0$ (optically thick), (1)-(2) behaves like a diffusion equation.

Expand I and T in ϵ and insert the expansions into (1)-(2) $\Rightarrow$ the leading-order term satisfies (formally):
$I^{(0)} = \frac{1}{4\pi} ac(T^{(0)})^4$, and a diffusion equation

$$C_v \frac{\partial}{\partial t} T^{(0)} + a \frac{\partial}{\partial t} (T^{(0)})^4 = \nabla \cdot \frac{ac}{3\sigma} \nabla (T^{(0)})^4. \tag{3}$$

An asymptotic preserving (AP) scheme for (1)-(2) is a numerical scheme that discretizes (1)-(2) in such a way that it leads to a correct discretization of the diffusion limit (3) when ϵ/σ small.

In the last decades, there have been a number of articles on AP schemes for radiative (and neutron) transfer problems (e.g., by Larsen, Morel, Miller, Klar, Jin, Levermore, Pareschi, Toscani, $\cdots\cdots$, '87–'13)

Recently, a different approach by Xu & Huang'10 for rarefied gas dynamics based on a unified gas kinetic scheme (UGKS), further development by Mieussens'13 for a linear transport model.

(scalar equations are dealt with, not coupled with other eqs.)

We want to use the idea for UGKS to construct an AP scheme for the system (1)–(2) (also for the frequency-dependent radiative transfer system).

2.1. Angular discretization

In 2D Cartesian coordinates, we can rewrite (1), (2) as

$$\begin{aligned} &\frac{\epsilon}{c}\partial_t I + \mu\partial_x I + \xi\partial_y I = \frac{\sigma}{\epsilon}\Big(\frac{1}{2\pi}\phi - I\Big),\\ &\epsilon^2 C_v\frac{\partial T}{\partial t} = \sigma\Big(\int I d\vec{\Omega} - \phi\Big), \qquad \phi = acT^4. \end{aligned} \tag{4}$$

where $\mu = \sqrt{1-\zeta^2}\cos\theta$, $\xi = \sqrt{1-\zeta^2}\sin\theta$,
$\zeta \in [0,1]$: cosine value of the angle between the propagation direction $\vec{\Omega}$ and z-axis,
$\theta \in [0,2\pi)$: projection vector of $\vec{\Omega}$ onto the xy-plane and the x-axis.

2.1. Angular discretization

We use the usual discrete ordinate method to discretize (4) in directions.

Write the propagation direction (μ, ξ) as some discrete directions (μ_m, ξ_m), $m = 1, \cdots, M \;\; (= N(N+2)/2)$, together with the corresponding integration weights $\omega_m \Longrightarrow$ angular discretized equations:

$$\begin{cases} \frac{\epsilon}{c}\partial_t I_m + \mu_m \partial_x I_m + \xi_m \partial_y I_m = \frac{\sigma}{\epsilon}\Big(\frac{1}{2\pi}\phi - I_m\Big), \\ \epsilon^2 C_v \frac{\partial T}{\partial t} = \sigma\Big(\sum_{m=1}^{M} I_m \omega_m - \phi\Big), \qquad \phi = acT^4. \end{cases} \tag{5}$$

Denote $I^n_{i,j,m}$, $T^n_{i,j}$: cell average values of I_m, T at t^n in cell

$$(i,j) := \{(x,y);\; x_{i-1/2} < x < x_{i+1/2}, y_{j-1/2} < y < y_{j+1/2}\},$$

and integrate (5) $\Rightarrow$

FV discretization of (5) reads as

$$I^{n+1}_{i,j,m} = I^n_{i,j,m} + \frac{\Delta t}{\Delta x}(F_{i-1/2,j,m} - F_{i+1/2,j,m}) + \frac{\Delta t}{\Delta y}(H_{i,j-1/2,m} - H_{i,j+1/2,m}) + c\Delta t \frac{\sigma}{\epsilon^2}\left(\frac{1}{2\pi}\tilde{\phi}_{i,j} - \tilde{I}_{i,j,m}\right), \tag{6}$$

$$C_v T^{n+1}_{i,j} = C_v T^n_{i,j} + \Delta t \frac{\sigma}{\epsilon^2}\left(\sum_{m=1}^{M} \tilde{I}_{i,j,m}\omega_m - \tilde{\phi}_{i,j}\right),$$

where $F_{i-1/2,j,m}$, $H_{i,j-1/2,m}$: numerical fluxes in the x-, y-directions, $\tilde{\phi}_{i,j}, \tilde{I}_{i,j,m}$: average of ϕ, I_m in $(t^n, t^{n+1}) \times cell(i,j)$; and

$$F_{i\pm 1/2,j,m} = \frac{c\mu_m}{\epsilon\Delta t}\int_{t^n}^{t^{n+1}} I_m(t, x_{i\pm 1/2}, y_j, \mu_m, \xi_m)dt,$$
$$H_{i,j\pm 1/2,m} = \frac{c\xi_m}{\epsilon\Delta t}\int_{t^n}^{t^{n+1}} I_m(t, x_i, y_{j\pm 1/2}, \mu_m, \xi_m)dt, \qquad (7)$$

and we take here approximation (implicitly in time):

$$\tilde{I}_{i,j,m} \approx I_{i,j,m}^{n+1}, \quad \tilde{\phi}_{i,j} \approx \phi_{i,j}^{n+1}.$$

We have to construct the numerical fluxes in (7).

2.3. Construction of the numerical fluxes

To evaluate $F_{i-1/2,j,m}$, we use the idea of gas kinetic schemes, i.e., solve $(5)_1$ in the x-direction at the cell boundary $x = x_{i-1/2}, y = y_j$:

$$\begin{aligned} &\frac{\epsilon}{c}\partial_t I_m + \mu_m \partial_x I_m = \frac{\sigma}{\epsilon}(\frac{1}{2\pi}\phi - I_m), \\ &I_m(x, y_j, t)|_{t=t^n} = I_{m,0}(x, y_j, t^n) \end{aligned} \tag{8}$$

to have the explicit solution

$$\begin{aligned} I_m(t, x_{i-1/2}, y_j, \mu_m, \xi_m) &= e^{-\nu_{i-1/2,j}(t-t^n)} I_{m,0}(x_{i-1/2} - \frac{c\mu_m}{\epsilon}(t - t^n)) \\ &+ \int_{t^n}^{t} e^{-\nu_{i-1/2,j}(t-s)} \frac{c\sigma_{i-1/2,j}}{2\pi\epsilon^2} \phi(s, x_{i-1/2} - \frac{c\mu_m}{\epsilon}(t - s))ds, \end{aligned} \tag{9}$$

where $\nu = c\sigma/\epsilon^2$, $\nu_{i-1/2,j}$: value of ν at the corresponding cell boundary. ($I_{m,0}$, ϕ to be determined)

Put (9) into (7) $\Rightarrow$

numerical flux $F_{i-1/2,j,m}$ **in** x,

provided the initial data $I_{m,0}(x, y_j, t^n)$ **in (8),** $\phi(x, y, t)$ **for** $t \in (t^n, t^{n+1})$ **and** (x, y) **around** $(x_{i-1/2}, y_j)$ **in (9) are known.**

(Flux $H_{i,j-1/2,m}$ **in** y **can be constructed in the same manner)**

- **Approximate** $I_{m,0}(x, y_j, t^n)$ **explicitly by piecewise linear polynomials and MUSCL limiter for slopes, using known** $I^n_{i,j,m}$ $\surd$

- Evaluate $\phi(x, y, t)$ implicitly by piecewise polynomials:

$$\phi(x, y_j, t) = \phi^{n+1}_{i-1/2,j} + \delta_t \phi^{n+1}_{i-1/2,j}(t - t^{n+1}) + \begin{cases} \delta_x \phi^{n+1,L}_{i-1/2,j}(x - x_{i-1/2}), & \text{if } x < x_{i-1/2}, \\ \delta_x \phi^{n+1,R}_{i-1/2,j}(x - x_{i-1/2}), & \text{if } x > x_{i-1/2}. \end{cases} \tag{10}$$

Here $\delta_t \phi^{n+1}_{i-1/2,j} = (\phi^{n+1}_{i-1/2,j} - \phi^{n}_{i-1/2,j})/\Delta t$, and

$$\delta_x \phi^{n+1,L}_{i-1/2,j} = \frac{\phi^{n+1}_{i-1/2,j} - \phi^{n+1}_{i-1,j}}{\Delta x/2}, \quad \delta_x \phi^{n+1,R}_{i-1/2,j} = \frac{\phi^{n+1}_{i,j} - \phi^{n+1}_{i-1/2,j}}{\Delta x/2}.$$

The numerical flux $F_{i-1/2,j,m}$ is obtained, provided the implicit terms $\phi^{n+1}_{i-1/2,j}, \phi^{n+1}_{i-1,j}, \phi^{n+1}_{i,j}$ are evaluated.

Remark. By (9)–(10), flux $F_{i-1/2,j,m}$ can be decomposed into

$$\begin{aligned} F_{i-1/2,j,m} &= A_{i-1/2,j}\mu_m(I^-_{i-1/2,j,m}\mathbf{1}_{\mu_m>0} + I^+_{i-1/2,j,m}\mathbf{1}_{\mu_m<0}) \\ &+C_{i-1/2,j}\mu_m\phi^{n+1}_{i-1/2,j} \\ &+D_{i-1/2,j}(\mu_m^2\delta_x\phi^{n+1,L}_{i-1/2,j}\mathbf{1}_{\mu_m>0} + \mu_m^2\delta_x\phi^{n+1,R}_{i-1/2,j}\mathbf{1}_{\mu_m<0}) \\ &+B_{i-1/2,j}(\mu_m^2\delta_x I^n_{i-1,j,m}\mathbf{1}_{\mu_m>0} + \mu_m^2\delta_x I^n_{i,j,m}\mathbf{1}_{\mu_m<0}) \\ &+E_{i-1/2,j}\mu_m\delta_t\phi^{n+1}_{i-1/2,j}, \qquad \text{(involved the values of } I \text{ at } t^n \text{ only)} \end{aligned} \tag{11}$$

where $I^-_{i-1/2,j,m}$, $I^+_{i-1/2,j,m}$ denote the boundary values:

$$I^-_{i-1/2,j,m} = I^n_{i-1,j,m} + \frac{\Delta x}{2}\delta_x I^n_{i-1,j,m}, \quad I^+_{i-1/2,j,m} = I^n_{i,j,m} - \frac{\Delta x}{2}\delta_x I^n_{i,j,m},$$

and $\nu = c\sigma/\epsilon^2$; the coefficients $A_{i-1/2,j} = \cdots$ are explicit functions of $\Delta t, \epsilon, \sigma, \nu$.

2.4. Evaluation of $\phi_{\cdots}^{n+1}$ in the flux $F_{i-1/2,j,m}$

To evaluate $\phi_{\cdots}^{n+1}$, we take the moment of (1).

Denote $\phi = acT^4$, $\rho = \int Id\vec{\Omega}$ as before, take angular moment of (1)

$\Rightarrow$ (1), (2) can be rewritten in the macro-form:

$$\begin{cases} \frac{\epsilon^2}{c}\frac{\partial \rho}{\partial t} + \epsilon \nabla \cdot < \vec{\Omega} I > = \sigma(\phi - \rho), \qquad < \vec{\Omega} I > := \int \vec{\Omega} I d\vec{\Omega}, \\ \epsilon^2 \frac{\partial \phi}{\partial t} = \beta\sigma(\rho - \phi), \qquad \beta = \frac{4acT^3}{C_v}. \end{cases} \tag{12}$$

We discretize (12) implicitly as follows.

$$\begin{aligned} \rho_{i,j}^{n+1} &= \rho_{i,j}^{n} + \frac{\Delta t}{\Delta x}(\Phi_{i-1/2,j}^{n+1} - \Phi_{i+1/2,j}^{n+1}) \\ &+ \frac{\Delta t}{\Delta y}(\Psi_{i,j-1/2}^{n+1} - \Psi_{i,j+1/2}^{n+1}) + \frac{c\sigma_{i,j}^{n+1}\Delta t}{\epsilon^2}(\phi_{i,j}^{n+1} - \rho_{i,j}^{n+1}), \end{aligned} \tag{13}$$

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \frac{(\beta\sigma)_{i,j}^{n+1}\Delta t}{\epsilon^2}(\rho_{i,j}^{n+1} - \phi_{i,j}^{n+1}),$$

where

$$\Phi^{n+1}_{i\pm 1/2,j} = \frac{c}{\epsilon \Delta t}\int_{t^n}^{t^n+\Delta t} < \Omega_x I > (x_{i\pm 1/2}, y_j, t)dt,$$

$$\Psi^{n+1}_{i,j\pm 1/2} = \frac{c}{\epsilon \Delta t}\int_{t^n}^{t^n+\Delta t} < \Omega_y I > (x_i, y_{j\pm 1/2}, t)dt,$$

which can be explicitly evaluated using (9) and (11), e.g.,

$$\Phi^{n+1}_{i-1/2,j} = \sum_{m=1}^{M} \omega_m F_{i-1/2,j,m} = A_{i-1/2,j}\sum_{m=1}^{M}\omega_m \mu_m \Big(I^n_{i-1,j,m} 1_{\mu_m>0}$$

$$+ I^n_{i,j,m} 1_{\mu_m<0}\Big) + \frac{2\pi D_{i-1/2,j}}{3}\Big(\frac{\phi^{n+1}_{i,j} - \phi^{n+1}_{i-1,j}}{\Delta x}\Big)$$

$$+ B_{i-1/2,j}\sum_{m=1}^{M}\omega_m \mu_m^2 (\delta_x I^n_{i-1,j,m} 1_{\mu_m>0} + \delta_x I^n_{i,j,m} 1_{\mu_m<0}),$$

where $A_{i-1/2,j}, \cdots$ are same as before $\left(\sigma_{i-1/2,j} = \frac{2\sigma_{i,j}\sigma_{i-1,j}}{\sigma_{i,j}+\sigma_{i-1,j}}\right)$

($\star$ only involved the values of I at t^n)

(13) is a nonlinear system for $\phi_{i,j}^{n+1}$, $\rho_{i,j}^{n+1}$ with parameters σ, β depending on T, and is solved by a two-level iteration method,

outer iteration: a nonlinear iteration with fixed σ, β $\sim$ a linear algebraic system;

inner iteration: Gauss-Sidel iteration to solve the linear system.

After obtaining $\phi_{i,j}^{n+1}$, we take the cell boundary value $\phi_{i-1/2,j}^{n+1}$ in (10) to be

$$\phi_{i-1/2,j}^{n+1} = \frac{1}{2}(\phi_{i-1,j}^{n+1} + \phi_{i,j}^{n+1}). \tag{14}$$

With (14), the numerical fluxes (10) are determined.

The construction of our AP UGKS scheme is complete □

LOOP of the AP-UGKS: Given $I^n_{i,j,m}$ and $T^n_{i,j}$, we have $\rho^n_{i,j}$ and $\phi^n_{i,j}$. Find I^{n+1} and T^{n+1}.

1) Solve the system (13) to obtain $\phi^{n+1}_{i,j}$, $\rho^{n+1}_{i,j}$;

2) Using $\phi^{n+1}_{i,j}$ to solve the system (6) for I to get $I^{n+1}_{i,j,m}$, $T^{n+1}_{i,j}$;

3) (Corrector step) Using $I^{n+1}_{i,j,m}$ to solve $(12)_2$ (i.e. $\epsilon^2\phi_t = \beta\sigma(\rho - \phi)$) to get a new $\phi^{n+1}_{i,j}$ (explicitly given by

$$\bar{\phi}^{n+1}_{i,j} = \frac{\phi^n_{i,j} + \Delta t(\beta\sigma)^{n+1}_{i,j} \sum_{m=1}^{M} \omega_m I^{n+1}_{i,j,m}}{1 + \Delta t(\beta\sigma)^{n+1}_{i,j}} \quad).$$

And then, to give the corrected temperature by $T^{n+1}_{i,j} = (\bar{\phi}^{n+1}_{i,j}/(ac))^{1/4}$.

4) Goto 1) for the next computational step.

3. Asymptotic analysis, AP property

- **The above constructed scheme is AP**

Recalling (11), $F_{i-1/2,j,m}$ can be decomposed into

$$F_{i-1/2,j,m} = A_{i-1/2,j} \cdots + \cdots + E_{i-1/2,j} \cdots ,$$

where the coefficients satisfy $A, B \to 0$, $D \to -c/(2\pi\sigma)$ as $\epsilon \to 0$.

Thus, the corresponding macroscopic diffusion flux $(\text{Diff})^{n+1}_{i-1/2,j}$:

$$(\text{Diff})^{n+1}_{i-1/2,j} := \int \frac{c\mu}{2\pi\epsilon\Delta t} \int_{t^n}^{t^{n+1}} I(t, x_{i-1/2}, y_j, \mu, \xi) dt d\mu d\xi$$

$$= \frac{1}{2\pi} \sum_{m=1}^{M} \omega_m \mu_m I^m(t, x_{i-\frac{1}{2}}, y_j, \mu_m, \xi_m) \to -\frac{c}{3\sigma^{n+1}_{i-\frac{1}{2},j}} \frac{\phi^{n+1}_{i,j} - \phi^{n+1}_{i-1,j}}{\Delta x}$$

as $\epsilon \to 0$, which gives a numerical flux of the limiting diffusion equation for ϕ $\rightsquigarrow$ 5 points scheme in 2D $\Rightarrow$ an AP scheme.

4. Numerical experiments

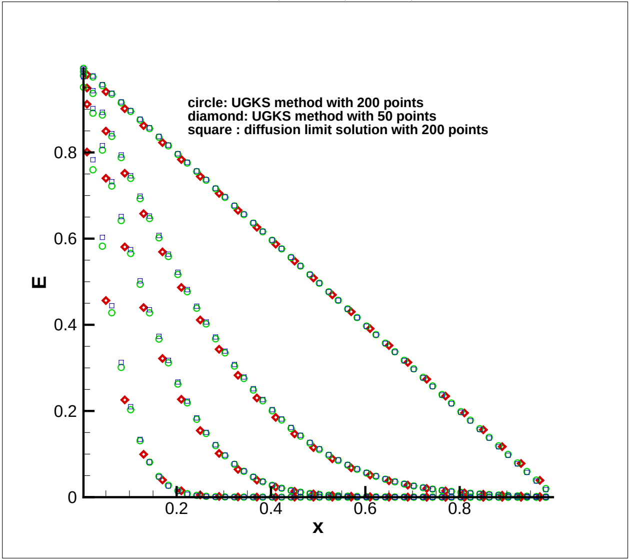
Example 1. A test for the linear kinetic equation:

$$\frac{\epsilon}{c}\partial_t I + \mu\partial_x I = \frac{\sigma}{\epsilon}\Big(\frac{1}{2\pi}\phi - I\Big),$$

where we take $\phi = \int I d\mu$, Domain: $x \in [0,1]$, $\sigma = 1$, $\epsilon = 10^{-8}$; boundary conditions: $I_L(0,\mu) = 1$ ($\mu > 0$ and $I_R(1,\mu) = 0$ ($\mu < 0$); $\Delta x = 5 \times 10^{-3}, 2.5 \times 10^{-3} >> \epsilon$.

This problem corresponds to the equilibrium diffusion approximation.

Computed solution at $t = 0.01, 0.05, 0.15, 2.0$:



Computed solution agrees well with that of the diffusion limiting eq. $\sim$ AP property

Example 2. **(Marshak wave-2B)**

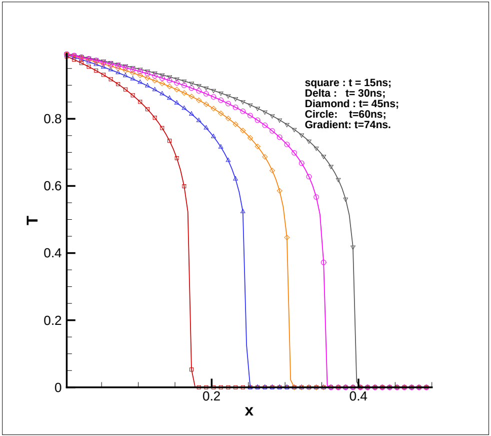
$\sigma = \frac{100}{T^3} cm^2/g$, $T = 10^{-6}$Kev, $\epsilon = 1$, specific heat=0.1$GJ/g/$Kev. This corresponds to the equilibrium diffusion approximation.

$\Delta x = 0.005$cm (200 cells), $\Delta y = 0.01$cm (1 cell), $\Delta x, \Delta y >> \epsilon/\sigma$.

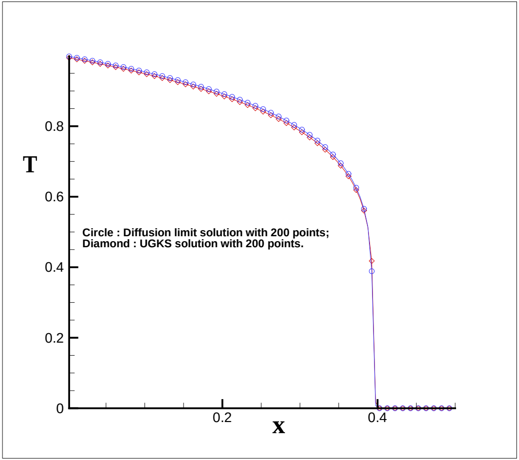
Left boundary: constant incident radiation intensity with a Planckian distribution at 1Kev;

Right boundary: outflow

Computed material temperature at $t = 15, 30, 45, 60, 74$ns



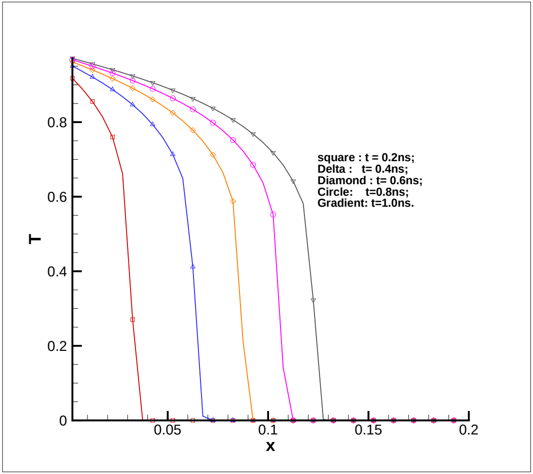
Computed material temperature for both grey radiation transfer system and diffusion limiting equation at 74ns



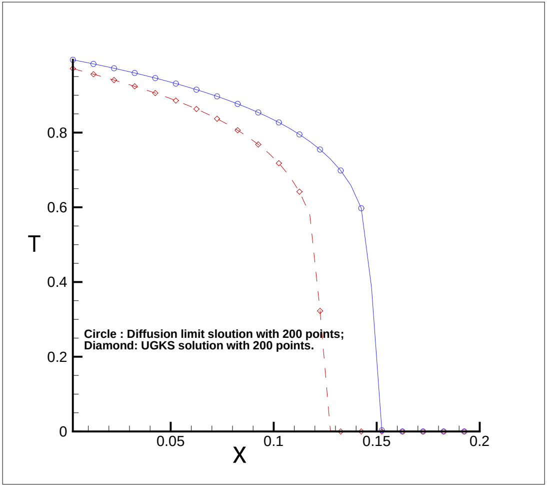
Results computed by AP UGKS agree very well with numerical solution of the diffusion limiting eq. $\sim$ AP property

Example 3. (Marshak wave-2A) the same as Example 2 except $\sigma = \frac{10}{T^3} cm^2/g$. This case violates the equilibrium diffusion approximation.

Computed radiative temperature at $t = 0.2, 0.4, 0.6, 0.8, 1.0ns$:

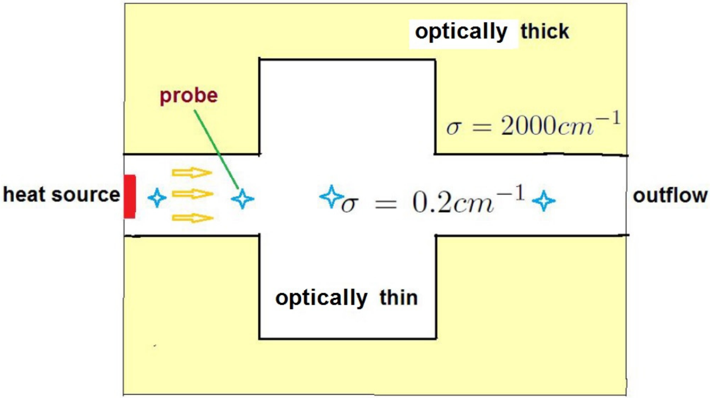


Material temperature distributions obtained by AP UGKS and the diffusion equation solution at $t = 1.0ns$: (clear difference, since not in the equilibrium diffusion approximation)



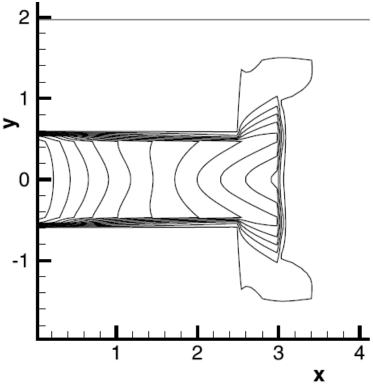
Example 4. (Tophat Test [N.A. Gentile,'01])

Domain: $[0,7] \times [-2,2]$, $\epsilon = 1$; Five probes: placed at $(0.25, 0), \cdots$ to monitor the change of the temperature in the thin opacity material.

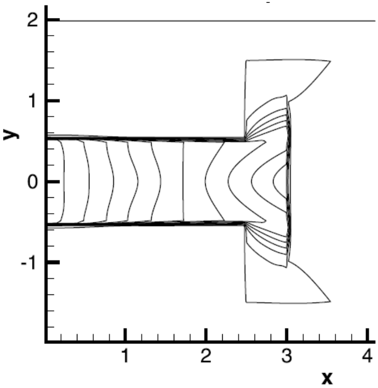


Computed material temperature at 8ns:

128 $\times$ 64 cells

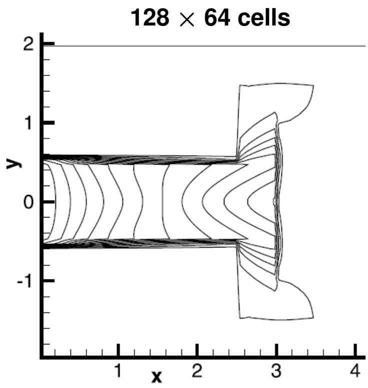


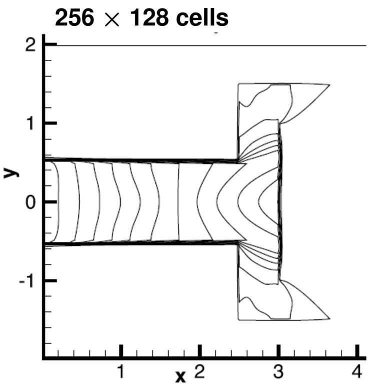
256 $\times$ 128 cells



Interface between optically thick and thin materials is captured sharply, compared with [Gentile ’01 (JCP)]

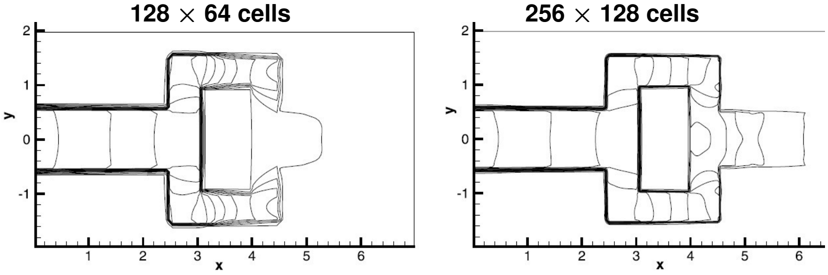
Computed radiation temperature at 8ns:



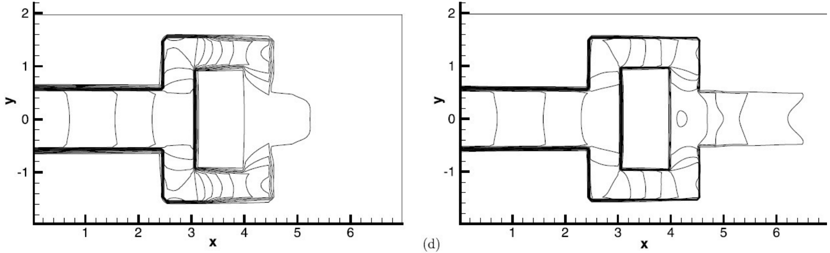


Interface between optically thick and thin materials is captured sharply.

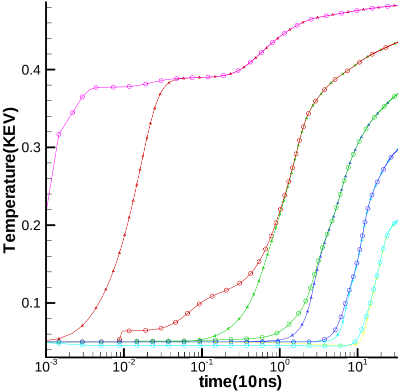
Computed material temperature at 94ns:



Computed radiation temperature



Time evolution of the material, radiation temperatures at 5 probes:



At the 5th probe, the temperature cools off slightly before being heated up by radiation, which is consistent with [Gentile].
But, the temperature curve here has a rapid growth initially, a slow increment, then a growth again. This is different from [Gentile], deserving further investigation.

The governing equations:

$$\begin{cases} \frac{\epsilon^2}{c}\frac{\partial I}{\partial t} + \epsilon\vec{\Omega}\cdot\nabla I = \sigma(B(\nu,T) - I), \\ \epsilon^2 C_v \frac{\partial T}{\partial t} \equiv \epsilon^2 \frac{\partial U}{\partial t} = \int_{4\pi}\int_0^\infty \sigma\big(I - B(\nu,T)\big)d\nu d\vec{\Omega}, \end{cases} \tag{15}$$

where $I \equiv I(t,\vec{r},\vec{\Omega},\nu)$, $\sigma \equiv \sigma(\vec{r},\nu,T)$, $B(\nu,T) = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1}$ is the Planck function with h and k being the Planck and Boltzmann constants; ν: frequency.

Normally, as $\epsilon \to 0$, Larsen et al.'83 showed: away from boundaries and initial time, the intensity I goes to a Planckian at the local temperature, i.e., $I^{(0)} = B(\nu, T^{(0)})$, where $T^{(0)}$ satisfies

$$\frac{\partial}{\partial t}U(T^{(0)}) + a\frac{\partial}{\partial t}(T^{(0)})^4 = \nabla \cdot \frac{ac}{3\sigma_R}\nabla(T^{(0)})^4, \qquad (16)$$

where $a = 8\pi k^4/(15h^3c^3)$ is the radiation constant and σ_R is the Rosseland mean opacity:

$$\frac{1}{\sigma_R} = \int_0^\infty \frac{1}{\sigma(\vec{r},\nu,T)}\frac{\partial B(\nu,T)}{\partial T}d\nu \Big/ \int_0^\infty \frac{\partial B(\nu,T)}{\partial T}d\nu.$$

The opacity is typically a decreasing function of frequency, thus in the same spatial region, the transport physics can be optically thick for the low frequency photons and optically thin for the high frequency ones $\sim$ additional numerical challenge, compared with the single frequency case.

5.1. Discretization of the frequency space

Discrete the continuous frequency space $(0,\infty)$ into G discrete frequency intervals $(\nu_{g-1/2},\nu_{g+1/2})$ $(g = 1,\cdots,G)$ and $\nu_{1/2} = 0$, $\nu_{G+1} = \infty$. We define the group variables:

$$I_g = \int_{\nu_{g-1/2}}^{\nu_{g+1/2}} I(t,\vec{r},\vec{\Omega},\nu)d\nu, \quad \phi_g = \int_{\nu_{g-1/2}}^{\nu_{g+1/2}} B(\nu,T)d\nu,$$

$$\sigma_g^e = \frac{\int_{\nu_{g-1/2}}^{\nu_{g+1/2}} \sigma B(\nu,T)d\nu}{\int_{\nu_{g-1/2}}^{\nu_{g+1/2}} B(\nu,T)d\nu}, \quad \sigma_g^a = \frac{\int_{\nu_{g-1/2}}^{\nu_{g+1/2}} \sigma I\, d\nu}{\int_{\nu_{g-1/2}}^{\nu_{g+1/2}} I\, d\nu}.$$

We can integrate (15) over the group frequencies $\Rightarrow$

$$\begin{cases} \dfrac{\epsilon^2}{c}\dfrac{\partial I_g}{\partial t} + \epsilon\vec{\Omega}\cdot\nabla I_g = (\sigma_g^e\phi_g - \sigma_g^a I_g), \qquad g = 1,\cdots,G, \\ \epsilon^2 C_v\dfrac{\partial T}{\partial t} \equiv \epsilon^2\dfrac{\partial U}{\partial t} = \displaystyle\sum_{g=1}^{G}\int_{4\pi}(\sigma_g^a I_g - \sigma_g^e\phi_g)d\vec{\Omega}. \end{cases} \tag{17}$$

Here for each g, (17) is a frequency-independent system. σ_g^a is a weighted integration of the unknown function I, which is usually replaced by the Planck function with the radiation temperature T_r instead of T. For this reason, we take

$$acT_r^4 = \int_{4\pi}\int_0^\infty I d\vec{\Omega} d\nu = \sum_{g=1}^{G}\int_{4\pi} I_g d\vec{\Omega},$$

$$\sigma_g^a = \int_{\nu_{g-\frac{1}{2}}}^{\nu_{g+\frac{1}{2}}} \sigma B(\nu, T_r) d\nu \Big/ \int_{\nu_{g-\frac{1}{2}}}^{\nu_{g+\frac{1}{2}}} B(\nu, T_r) d\nu$$

to complete (17). So far, we complete the frequency discretization.

Then, we can apply the above AP-UGKS to construct similarly an AP scheme for (17). The main idea works, although the construction and analysis are more complex and tedious.

5.2. A numerical test: Larsen's test problem

The frequency variable ν is logarithmically spaced with 50 groups between $h\nu_{\min} = 10^{-5}$keV and $h\nu_{\max} = 10$keV. Group g is defined by $\nu_{g-1/2} \leq \nu \leq \nu_{g+1/2}$, where

$$\nu_{1/2} = \nu_{\min}, \quad \nu_{g+1/2} = \left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/50} \nu_{g-1/2}.$$

The spatial domain is divided into 3 regions of different mesh sizes:

$$\Delta x = \begin{cases} 0.10cm, & 0cm < x < 1cm, \\ 0.02cm, & 1cm < x < 2cm, \\ 0.20cm, & 2cm < x < 4cm. \end{cases}$$

Photo-ionization absorption σ is corrected for simulated emission:

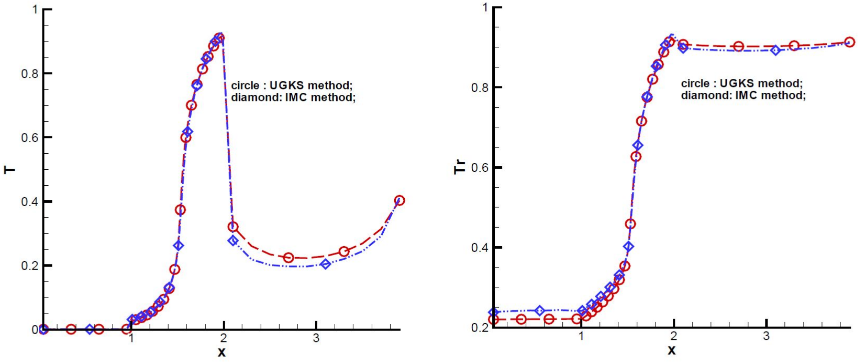
$$\sigma(\nu, T, x) = \gamma(x)(1 - e^{-h\nu/kT})(h\nu)^{-3},$$

$$\gamma(x) = \begin{cases} 1\,\mathrm{keV}^3/cm, & 0cm < x < 1cm, \\ 1000\,\mathrm{keV}^3/cm, & 1cm < x < 2cm, \\ 1\,\mathrm{keV}^3/cm, & 2cm < x < 4cm. \end{cases}$$

Heat capacity $C_v = 5.109 \times 10^{14}\mathrm{erg} \cdot \mathrm{keV}^{-1}\mathrm{cm}^{-3}$. $\epsilon = 1$.
$T|_{t=0} = 10^{-3}$keV which is in equilibrium with the initial intensity.

No photons enters from the left boundary, but a steady, direction-dependent, 1 keV Planckian distribution of photons enters from the right boundary.

The computed results (at time 900ps) by AP-UGKS agree well with those by IMC, particularly in the middle optically thick region. However, AP-UGKS is much cheaper.



6. Conclusions

- A unified gas scheme is constructed for the grey radiative transfer system, and the multi-group radiation transfer equations.
- The scheme has asymptotic preserving property and works well for both optically thin and optically thick regimes.
- We believe that it should work for more general problems.

7. Future studies

- Extension to quadrilateral meshes
- Extension to (multi-material) radiation hydrodynamics

THANK YOU !