Asymptotic stability of IMEX schemes for stiff hyperbolic PDE’s

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joint with

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Outline

1 Introduction
   - Stiff Hyperbolic PDE’s
   - Numerical Challenges
   - IMEX Schemes

2 Plan of the Talk

3 Examples
   - Unstable IMEX
   - Stable IMEX

4 Linear Stability Theory

5 RS-IMEX
   - Modified equation
   - Van der Pol Equation

6 Outlook
Introduction

- Stiff Hyperbolic PDE’s
- Numerical Challenges
- IMEX Schemes

Plan of the Talk

Examples

- Unstable IMEX
- Stable IMEX

Linear Stability Theory

RS-IMEX

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- Van der Pol Equation

Outlook
Isentropic gas dynamics

Dimensionless conservation laws for mass and momentum:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) &= 0.
\end{align*}
\]

\[p = p(\rho) = \text{pressure}. \quad c_{\text{ref}} = \sqrt{p'(\rho_{\text{ref}})} = \text{reference sound speed}.
\]

\[\varepsilon = \frac{u_{\text{ref}}}{c_{\text{ref}}} = \text{Mach number}\]
Zero Mach number limit

Asymptotic expansion (for general \( f(x, t; \varepsilon) \)):

\[
f(x, t) = f^{(0)}(x, t) + \varepsilon f^{(1)}(x, t) + \varepsilon^2 f^{(2)}(x, t) \ldots
\]

gives to leading order

\[
\rho = \rho^{(0)}(t) + \varepsilon^2 \rho^{(2)}(x, t)
\]
Leading order equations

Constraints for $\rho^{(0)}$ and $\nabla \cdot \mathbf{u}^{(0)}$:

$$(\nabla \cdot \mathbf{u}^{(0)})(t) = \frac{1}{|\Omega|} \int_{\partial \Omega} \mathbf{u}^{(0)}_{\text{bdry}} \cdot \mathbf{n} dS(x)$$

$$\frac{d}{dt} \rho^{(0)}(t) = -\rho^{(0)}(t) (\nabla \cdot \mathbf{u}^{(0)})(t)$$

Newton’s law for $\mathbf{u}^{(0)}$:

$$\partial_t \left( \rho^{(0)} \mathbf{u}^{(0)} \right) + \rho^{(0)} \nabla \cdot \left( \mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)} \right) + \nabla p^{(2)} = 0$$
Incompressible equations

Assumption: zero net flux across the boundary.
Consequence: \( \rho^{(0)} \) constant, \( \mathbf{u}^{(0)} \) divergence free.

Incompressible Euler (Klainerman/Majda 1981)

\[
\partial_t \mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} + \frac{\nabla p^{(2)}}{\rho^{(0)}} = 0
\]

Elliptic constraint

\[
\nabla \cdot \left( \mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} \right) + \frac{\Delta p^{(2)}}{\rho^{(0)}} = 0
\]
1 Introduction
   • Stiff Hyperbolic PDE’s
   • Numerical Challenges
   • IMEX Schemes

2 Plan of the Talk

3 Examples
   • Unstable IMEX
   • Stable IMEX

4 Linear Stability Theory

5 RS-IMEX
   • Modified equation
   • Van der Pol Equation

6 Outlook
Challenge I: stiffness for small Mach number

Propagation speeds in direction $\mathbf{n}$ ($u_n = \mathbf{u} \cdot \mathbf{n}$, $c = \sqrt{\frac{p'(\rho)}}{}}$):

$$u_n - \frac{c}{\varepsilon}, \quad u_n, \quad u_n + \frac{c}{\varepsilon}.$$ 

explicit schemes: inefficient ($\Delta t = O(\varepsilon \Delta x)$)

implicit schemes: excessively diffusive on advection wave

IMEX schemes: clever mix (Jin, Degond, . . .)
Challenge II: asymptotic behavior as $M \to 0$

Challenges:

**Asymptotic consistency:** for a sequence of well-prepared initial data, the numerical scheme should follow the low Mach number asymptotics

**Asymptotic stability:** the CFL number should be independent of $\varepsilon$
1 Introduction
   • Stiff Hyperbolic PDE’s
   • Numerical Challenges
   • IMEX Schemes

2 Plan of the Talk

3 Examples
   • Unstable IMEX
   • Stable IMEX

4 Linear Stability Theory

5 RS-IMEX
   • Modified equation
   • Van der Pol Equation

6 Outlook
Admissible Splittings

Definition

A splitting

\[ A = \tilde{A} + \hat{A}. \]

is admissible, if

(i) both \( \tilde{A} \) and \( \hat{A} \) induce a hyperbolic system

(ii) \[
\tilde{\lambda} := \rho(\tilde{A}) = O\left(\frac{1}{\varepsilon}\right) \\
\hat{\lambda} := \rho(\hat{A}) = O(1)
\]
CFL Conditions

$$\nu := \lambda_{\text{max}} \frac{\Delta t}{\Delta x} \quad \text{full CFL number}$$

$$\tilde{\nu} := \tilde{\lambda} \frac{\Delta t}{\Delta x} \quad \text{nonstiff CFL number}$$

$$\nu = O(1) \quad \Rightarrow \quad \tilde{\nu} = O(\varepsilon) \quad \text{stable \quad inefficient}$$

$$\nu = O\left(\frac{1}{\varepsilon}\right) \quad \Leftarrow \quad \tilde{\nu} = O(1) \quad \text{unstable \quad efficient}$$
Flux-Splitting & IMEX Time-Discretization

Implicit-explicit discretization

Klein 1996
Degond, Tang 2011
Haack, Jin, Liu 2011

\[ U^{n+1} = U^n + \tilde{A}U_x^{n+1} + \tilde{A}U_x^n \]
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1. Introduction
   - Stiff Hyperbolic PDE’s
   - Numerical Challenges
   - IMEX Schemes

2. Plan of the Talk

3. Examples
   - Unstable IMEX
   - Stable IMEX

4. Linear Stability Theory

5. RS-IMEX
   - Modified equation
   - Van der Pol Equation

6. Outlook
Plan of the talk

Noelle, Bispen, Arun, Lukacova, Munz  
• splitting A unstable  

Bispen, Arun, Lukacova, Noelle  
• splitting B stable  

Schütz, Noelle  
• linear stability theory  

Schütz, Kaiser, Noelle, Zakerzadeh  
• RS-IMEX splitting  

SISC 2014  
CiCP 2014  
JSC 2014  
(submitted 2015)
1. Introduction
   - Stiff Hyperbolic PDE’s
   - Numerical Challenges
   - IMEX Schemes

2. Plan of the Talk

3. Examples
   - Unstable IMEX
   - Stable IMEX

4. Linear Stability Theory

5. RS-IMEX
   - Modified equation
   - Van der Pol Equation

6. Outlook
Euler equations

\[ U_t + \nabla \cdot F(U) = 0, \]

\[ U = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho \mathbf{u}^T \\ \rho \mathbf{u} \otimes \mathbf{u} + \frac{p}{\varepsilon^2} \mathbf{I} \\ (\rho E + p) \mathbf{u}^T \end{pmatrix}, \]

Total energy \( \rho E \) and equation of state:

\[ p = (\gamma - 1) \left( E - \frac{\varepsilon^2}{2} \rho |\mathbf{u}|^2 \right), \]
Splitting A (Klein 1995)

\[ F(U) = \tilde{F}(U) + \bar{F}(U), \]

where
\[
\tilde{F}(U) = \begin{pmatrix}
0 \\
\frac{1-\varepsilon^2}{\varepsilon^2} p I \\
(p - \Pi)u^T
\end{pmatrix},
\bar{F}(U) = \begin{pmatrix}
\rho u^T \\
\rho u \otimes u + p I \\
(\rho E + \Pi)u^T
\end{pmatrix}.
\]

Auxiliary pressure
\[
\Pi(x, t) := \varepsilon^2 p(x, t) + (1 - \varepsilon^2)p_\infty(t),
\]

Reference pressure
\[
p_\infty(t) = \inf_x p(x, t)
\]
Eigenvalues of subsystems

Eigenvalues of $\tilde{A} := \tilde{F}'(U) \cdot n$

$$\tilde{\lambda} = 0, \pm \frac{1 - \varepsilon^2}{\varepsilon} \left( \frac{(\gamma - 1)(p - p_\infty)}{\rho} \right)^{1/2}$$

hyperbolicity. fast and slow waves. implicit timestep.

Eigenvalues of $\hat{A} := \hat{F}'(U) \cdot n$

$$\hat{\lambda} = u_n, \ u_n \pm c^*$$

hyperbolicity. only slow waves. explicit timestep.
Numerical experiment

- two colliding acoustic pulses (Klein 1995)
- weakly compressible

\[
\begin{align*}
\rho(x,0) &= \rho_0 + \frac{1}{2} \varepsilon \rho_1 \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad \rho_0 = 0.955, \quad \rho_1 = 2.0, \\
u(x,0) &= \frac{1}{2} u_0 \, \text{sign}(x) \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad u_0 = 2\sqrt{\gamma}, \\
p(x,0) &= p_0 + \frac{1}{2} \varepsilon p_1 \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad p_0 = 1.0, \quad p_1 = 2\gamma.
\end{align*}
\]
Stability for $\varepsilon = 1/11$

- two colliding pressure pulses
- $\varepsilon = 1/11$, $\hat{\nu} = 0.9$, $\nu = 9.9$
- stabilization constant $c_{stab} = 1/12$
Instability for $\varepsilon = 0.01$

difficulty:

- instability for $\varepsilon = 0.01$
- IMEX scheme needs reduced CFL number, $\hat{\nu} < 0.02$

first fix:

- high order pressure stabilization in elliptic equation
- asymptotic consistency only for $\Delta t = O(\varepsilon^{2/3})$
1 Introduction
   - Stiff Hyperbolic PDE’s
   - Numerical Challenges
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2 Plan of the Talk

3 Examples
   - Unstable IMEX
   - Stable IMEX

4 Linear Stability Theory

5 RS-IMEX
   - Modified equation
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6 Outlook
Shallow water equations

\[ U_t + \nabla \cdot F(U) = S(U) \]

\[ U = \begin{pmatrix} z \\ hu \end{pmatrix}, \quad F(U) = \begin{pmatrix} hu^T \\ hu \otimes u \end{pmatrix} + \frac{z^2 - 2zb}{2 \varepsilon^2} \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad S(U) = -\frac{z}{\varepsilon^2} \begin{pmatrix} 0 \\ \nabla^T b \end{pmatrix} \]

with

- \( b \) bottom topography
- \( z \) water surface
- \( h = z - b \) water height
- \( u = (u, v) \) horizontal velocity
- \( \varepsilon = \frac{u_{\text{ref}}}{\sqrt{gh_{\text{ref}}}} \) Froude number
Splitting B (Restelli, Giraldo 2009)

Linearize around \( z = 0, \ u = 0 \) (lake at rest):

\[
F(U) = \tilde{F}(U) + \hat{F}(U),
\]
\[
S(U) = \tilde{S}(U) + \hat{S}(U),
\]

where

\[
\tilde{F}(U) = \begin{pmatrix} hu^T & \frac{bz}{\varepsilon^2} \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \\ 0 \end{pmatrix}, \quad \tilde{S}(U) = S(U),
\]
\[
\hat{F}(U) = \begin{pmatrix} 0 \\ hu \otimes u \end{pmatrix} + \frac{z^2}{2\varepsilon^2} \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}, \quad \hat{S}(U) = 0.
\]
Eigenvalues of subsystems

Eigenvalues of $\tilde{A} := \tilde{F}'(U)$

$$\tilde{\lambda} = 0, \pm \frac{1}{\varepsilon} \sqrt{|b|}$$

hyperbolicity. fast and slow waves. implicit timestep.

Eigenvalues of $\hat{A} := \hat{F}'(U) \cdot n$

$$\hat{\lambda} = 0, u_n, 2u_n$$

hyperbolicity. only slow waves. explicit timestep.
Numerical experiment

- compactly supported smooth vortex
- transported to the right

\[
\begin{align*}
    h(x, y, 0) &= 110 + \left( \frac{\varepsilon \Gamma}{\omega} \right)^2 (k(\omega r_c) - k(\pi)) \\
    u(x, y, 0) &= 0.6 + \Gamma(1 + \cos(\omega r_c))(0.5 - y) \quad \text{if } \omega r_c \leq \pi \\
    v(x, y, 0) &= \Gamma(1 + \cos(\omega r_c))(x - 0.5) \quad \text{if } \omega r_c \leq \pi
\end{align*}
\]
Bispen 2014

- vortex, $\varepsilon = 0.8$ (top) and $\varepsilon = 0.01$ (bottom)
- Asymptotic Stability
**ε-uniform convergence**

Travelling Vortex, $L^1$-errors and order of convergence in $z$

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</table>

- $\hat{\nu} = 0.45$, $\nu = 0.9, 7.2, 35.$
Introduction
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Plan of the Talk

Examples
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Linear Stability Theory

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Outlook
The modified equation of the IMEX scheme is

$$w_t + A w_x = \frac{\Delta t}{2} C w_{xx}$$

with diffusion matrix

$$C := (\tilde{\alpha} + \bar{\alpha}) \frac{\Delta x}{\Delta t} I - (\tilde{A} - \bar{A}) (\tilde{A} + \bar{A})$$

and numerical viscosities $\tilde{\alpha}$, $\bar{\alpha}$. 
Is $C$ positive definite?

\[
C = \left( (\hat{\alpha} + \tilde{\alpha}) \frac{\Delta x}{\Delta t} \mathbb{I} - \hat{A}^2 \right) + (\hat{\mathbb{A}}\hat{\mathbb{A}} - \hat{A}\tilde{A}) + \hat{A}^2 \\
= O(1) + O\left(\frac{1}{\varepsilon}\right) + O\left(\frac{1}{\varepsilon^2}\right)
\]

Yes, if commutator $[\hat{\mathbb{A}}, \hat{\mathbb{A}}] = 0$
Example

Fourier stability analysis for prototype system

\[
A = \begin{pmatrix}
a & 1 & 0 \\
\frac{1}{\varepsilon^2} & a & \frac{1}{\varepsilon^2} \\
0 & 1 & a
\end{pmatrix}
\]

\(a > 0\), eigenvalues

\[
\lambda = a, \ a \pm \frac{\sqrt{2}}{\varepsilon}
\]
Euler: classical versus characteristic splitting

Comparison of classical versus characteristic splitting
How to recover stability

- Need e.g.

\[ \tilde{A}\tilde{A} - \tilde{A}\tilde{A} = O(1) \]

or

\[ \hat{A} = O(\varepsilon) \]

- Characteristic splitting is not possible in multi-D

- We need a nice piece of luck!!
Introduction
- Stiff Hyperbolic PDE’s
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Plan of the Talk

Examples
- Unstable IMEX
- Stable IMEX

Linear Stability Theory

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Outlook
Nonlinear hyperbolic system of balance laws

\[ \partial_t U(x, t; \varepsilon) + \nabla \cdot F(U, x, t; \varepsilon) = S(U, x, t; \varepsilon) \]

with

\[ U : \mathbb{R}^d \times \mathbb{R}_+ \times (0, 1] \to \mathbb{R}^m, \quad (x, t; \varepsilon) \mapsto U(x, t; \varepsilon) \]

- Challenge: Stiffness as \( \varepsilon \to 0 \)
- Goal: Asymptotic stability
Reference solution and scaled perturbation: \( U = \tilde{U} + D V \)

\[
\begin{align*}
\tilde{U} &: \quad \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^m, \quad (x, t) \mapsto \tilde{U}(x, t) \\
V &: \quad \mathbb{R}^d \times \mathbb{R}_+ \times (0, 1] \to \mathbb{R}^m, \quad (x, t; \varepsilon) \mapsto U(x, t; \varepsilon)
\end{align*}
\]

and

\[
D = \text{diag}(\varepsilon^{k_1}, \ldots, \varepsilon^{k_m})
\]

Taylor expansion with remainder of \( F \) and \( S \) around \( \tilde{U} \):

\[
\begin{align*}
F &= F(\tilde{U}) + A(\tilde{U}) \, D V + \bar{F}(\tilde{U}, V) = D(\bar{G} + \hat{G} + \bar{G}) \\
S &= S(\tilde{U}) + \tilde{S}' V \, D V + \bar{S}(\tilde{U}, V) = D(\bar{Z} + \hat{Z} + \bar{Z})
\end{align*}
\]
Reference solution and scaled perturbation: \( U = \tilde{U} + D \cdot V \)

\[
\begin{align*}
\tilde{U} : \mathbb{R}^d \times \mathbb{R}_+ & \rightarrow \mathbb{R}^m, \quad (x, t) \mapsto \tilde{U}(x, t) \\
V : \mathbb{R}^d \times \mathbb{R}_+ \times (0, 1] & \rightarrow \mathbb{R}^m, \quad (x, t; \varepsilon) \mapsto U(x, t; \varepsilon)
\end{align*}
\]

and

\[
D = \text{diag}(\varepsilon^{k_1}, \ldots, \varepsilon^{k_m})
\]

Taylor expansion with remainder of \( F \) and \( S \) around \( \tilde{U} \):

\[
F = F(\tilde{U}) + A(\tilde{U}) \cdot D V + \bar{F}(\tilde{U}, V) = D(\bar{G} + \hat{G} + \bar{G})
\]

\[
S = S(\tilde{U}) + \tilde{S}' V \cdot D V + \bar{S}(\tilde{U}, V) = D(\bar{Z} + \hat{Z} + \bar{Z})
\]

\[\text{RS+IM+EX}\]
Theorem (Modified equation for RS-IMEX (N. 2014))

\[ B_0 W_t = -\nabla \cdot B_1 + B_2 + \nabla \cdot (B_3 \cdot \nabla W) \]

with

\[ B_0 := I - \frac{\Delta t}{2} (\widehat{Z}' - \bar{Z}') , \]

\[ B_1 := \widehat{G} + \bar{G} + \frac{\Delta t}{2} ((\widehat{G}' - \bar{G}') (\widehat{Z}' + \bar{Z}' - \widehat{G}_x - \bar{G}_x)) , \]

\[ B_2 := \widehat{Z} + \bar{Z} + \frac{\Delta t}{2} (\widehat{Z}_t - \bar{Z}_t) , \]

\[ B_3 := \frac{(\widehat{\alpha} + \bar{\alpha}) \Delta x}{2} I + \frac{\Delta t}{2} (\widehat{G}' - \bar{G}') (\widehat{G}' + \bar{G}') . \]

Study this for each application!
1 Introduction
   - Stiff Hyperbolic PDE’s
   - Numerical Challenges
   - IMEX Schemes

2 Plan of the Talk

3 Examples
   - Unstable IMEX
   - Stable IMEX

4 Linear Stability Theory

5 RS-IMEX
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6 Outlook
van der Pol and IMEX (Schütz, Kaiser 2015)

- Prototype example
  \[
  \begin{pmatrix}
  y' \\
  z'
  \end{pmatrix} = \begin{pmatrix}
  \frac{z}{\varepsilon} \\
  g(y, z)
  \end{pmatrix}.
  \]

- 'Traditional' splitting: \[
  \begin{pmatrix}
  0 \\
  \frac{g(y, z)}{\varepsilon}
  \end{pmatrix} + \begin{pmatrix}
  z \\
  0
  \end{pmatrix}
  \]
van der Pol and IMEX

- ’Reference solution’ (RS) $\varepsilon \to 0$:

$$\begin{pmatrix} y'(0) \\ 0 \end{pmatrix} = \begin{pmatrix} z(0) \\ g(y(0), z(0)) \end{pmatrix}.$$

- **RS-IMEX** splitting based on $w(0)$:

$$f(w) = f(w(0)) + f'(w(0))(w - w(0)) + \text{Rest}$$

- Motivation: $w - w(0) = O(\varepsilon)$. 
RS-IMEX + Runge-Kutta

- IMEX Runge-Kutta (Pareschi, Russo, Boscarino ...)
- standard splitting looses convergence order
- RS-IMEX gives full order of accuracy
Outlook

**IMEX**
- Examples of uniform CFL stability and stability
- Linearized stability analysis

**RS-IMEX**
- A natural approach to stiff / non-stiff splitting
- Improves stability of IMEX schemes

**To do**
- Extend RS-IMEX to many more systems
- Test stability and efficiency
- Do rigorous stability analysis for modified equation
- Higher order accuracy
- Real life applications