Collective migration of animals in a cohesive group is rendered possible by a stratified structuring of tasks among members some track the travel route, which is time and energy-consuming, while the others follow the group by interacting among themselves. Here, we study a social dynamics system modeling collective migration. We consider a group of agents able to align their velocities to a global target velocity, or to follow the group via interaction with the other agents. The balance between these two attractive forces is our control for each agent, as we aim to drive the group to consensus at the target velocity.

**Minimization Objectives**

Let \( t > 0 \). We aim to solve the following problems:

1. **Instantaneous decrease**. Minimize \( V(t) \) for all \( t \in [0, T] \).
2. **Final cost**. Minimize \( V(T) \).
3. **Integral cost**. Minimize \( \int_0^T V(t) \) dt for \( \alpha \in U := \{ \alpha \mid [0, T] \to [0, 1]^N \text{ measurable} \} \).

**Optimal Control Strategy.**

Let \( t_i = \frac{\xi_i}{\sum_{j=1}^N \xi_j} \) in \( (1 - \alpha_i) \frac{\sum_{j=1}^N \alpha_j (v_j - v_i)}{N} t \), \( i \in \{1, \ldots, N\} \), where

\[
\xi_i = \sum_{j=1}^N \xi_j \quad \text{and} \quad \alpha_i = 1 - \alpha_i. 
\]

Then \( \xi_i \neq 0 \), i.e., \( \alpha_i \) cannot be controlled. Hence we define \( \xi_i = (v_i, e_i) \) and study the dynamics:

\[
\dot{\xi}_i = (1 - \alpha_i) (v_i - v_i).\]

They are the same strategies as described in Theorem 2.

**Theorem 3**

Sufficient condition for full control.

\( \alpha \in U \) is an optimal control. Define \( \alpha_{\text{opt}} \) as in Theorem 2. If \( T \geq T_N \), then \( \alpha \in U \) and \( \xi(T) = (0, \ldots, 0) \) for all \( i \in \{1, \ldots, N\} \).

**Theorem 4**

Inactivation Principle.

If \( T < T_N \), then there exists some \( \delta \in [0, T] \) such that \( \alpha^\text{opt} \equiv 0 \) on \([0, \delta]\), and \( \sum_{i=1}^N \alpha_i = 1 \) on \([\delta, T]\).

**Conclusions and Future Directions**

- Counter-intuitively, in order to minimize a final cost, it can be beneficial to let the system evolve freely on some initial time interval. However, the number of cases in which Inactivation is advantageous is small, and even decreases as the number of agents in the group increases (see Table 1). Furthermore, the gain in performance compared to the strategy saturating the control at all time is minor (Table 2). For reasons of computational speed and feasibility, it is very common to neglect this phenomenon and to act with full control at all time.

- In the case of instantaneous decrease or integral cost, full control must be used at all time.

- A future direction is to exert sparse control, i.e. aim to control only a few agents spreading them optimally among the group.

**Simulations for Final Cost**

Table 1: \# of agents in the group increases (see Table 1). Furthermore, the gain in performance compared to the strategy saturating the control at all time is minor (Table 2). For reasons of computational speed and feasibility, it is very common to neglect this phenomenon and to act with full control at all time.

- In the case of instantaneous decrease or integral cost, full control must be used at all time.

- A future direction is to exert sparse control, i.e. aim to control only a few agents spreading them optimally among the group.

**Models**

Inspired by the Cucker-Smale model, we study:

\[
\begin{align*}
\dot{v}_i &= a_i (V - v_i) + (1 - a_i) \frac{1}{N} \sum_{j=1}^N a_j (v_j - v_i) \\
\end{align*}
\]

for \( i \in \{1, \ldots, N\} \), where

- \( v_i \) and \( v_i \) are the position and velocity of agent \( i \),
- \( a_i \) characterizes the influence of agent \( j \) on agent \( i \). For simplification purposes, \( a_j = 1 \).
- \( V \) is the target velocity. WLOG, we act \( V = 0 \).
- \( \alpha_i \) is the control, choosing whether the \( j \)-th agent follows the group (\( \alpha_j = 0 \)), the target velocity (\( \alpha_j = 1 \)), or compromises between the two (\( 0 < \alpha_j < 1 \)).

**Theorem 1**

Let \( J(t) = \frac{1}{T} \int_0^T V(t) dt \), \( i \in \{1, \ldots, N\} \), and \( \sum_{i=1}^N \alpha_i = 1 \).

Find \( \min_{\alpha \in U} J(t) \). The optimal control strategy is the same as for the instantaneous decrease.

**Theorem 2**

Full Control Strategy.

Let \( t_i = \frac{\xi_i(T)}{\sum_{j=1}^N \xi_j} \) in \( (1 - \alpha_i) \frac{\sum_{j=1}^N \alpha_j (v_j - v_i)}{N} t \), \( i \in \{1, \ldots, N\} \), where

\[
\xi_i = \sum_{j=1}^N \xi_j. 
\]

Then, any strategy satisfying:

- \( \xi_i(T) = 0 \) for \( i \in \{1, \ldots, N\} \), and \( \sum_{i=1}^N \alpha_i = 1 \)
- \( \alpha_i = 0 \) for \( i \in \{1, \ldots, N\} \)
- \( \alpha_i = 0 \) for \( i \in \{1, \ldots, N\} \)

is optimal in \( U \).

**Figure 3:** Evolution of \( \xi_i, i \in \{1, \ldots, 10\} \) with the full strength optimal control for a system of 10 agents, with \( T = 5 < T_N \).

**Table 1:** Number of cases in which \( T > 0 \) out of 1000 simulations. 

<table>
<thead>
<tr>
<th># of agents</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td># of cases</td>
<td>1.6%</td>
<td>0.9%</td>
<td>0.9%</td>
<td>0.9%</td>
</tr>
</tbody>
</table>

**Table 2:** Gain in performance compared to full strength control.

<table>
<thead>
<tr>
<th># of agents</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain (%)</td>
<td>0.9%</td>
<td>0.9%</td>
<td>0.9%</td>
<td>0.9%</td>
</tr>
</tbody>
</table>