Optimal Control of a Collective Migration Model

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Abstract

Collective migration of animals in a cohesive group is rendered possible by a strategic distribution of tasks among members: some track the travel route, which is time and energy-consuming, while the others follow the group by interacting among themselves. Here, we study a social dynamics system modeling collective migration. We consider a group of agents able to align their velocities to a global target velocity, or to follow the group via interaction with the other agents. The balance between these two attractive forces is our control for each agent, as we aim to drive the group to consensus at the target velocity.

Model

Inspired by the Cucker-Smale model, we study:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \alpha_i(V - v_i) + (1-\alpha_i)\frac{1}{N}\sum_{j=1}^{N} a_{ij}(v_j - v_i) \end{cases}$$

for $i \in \{1, \dots, N\}$, where:

- x_i and v_i are the position and velocity of agent i.
- a_{ij} characterizes the influence of agent j on agent i. For simplification purposes, $a_{ij} = 1$.
- V is the target velocity. WLOG, we set $V = 0$.
- α_i is the control, choosing whether the i-th agent follows the group ($\alpha_i = 0$), the target velocity ($\alpha_i = 1$), or compromises between the two ($0 < \alpha_i < 1$).

Projection of the dynamics

Notice that $e = \bar{v}/\|\bar{v}\|$ is constant and define $w_i = v_i - \langle v_i, e\rangle$. Then $\dot{w}_i = -w_i$, i.e. w_i cannot be controlled. Hence we define $\xi_i = \langle v_i, e\rangle$ and study the dynamics:

$$\dot{\xi}_i = -\xi_i + (1-\alpha_i)\bar{\xi}, \quad i \in \{1, \dots, N\},$$

where $\bar{\xi} = \frac{1}{N}\sum_i \xi_i$. *We reduced an Nd-dimensional system to an N-dimensional one.* Here onward we assume:

1. $\bar{v} \neq 0$
2. $\xi_1(0) \geq \xi_2(0) \geq \cdots \geq \xi_N(0)$

References

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Minimization Objectives

$$\mathbb{V} = \frac{1}{N}\sum_i \xi_i^2 = \underbrace{\bar{\xi}^2}_{\text{Steers to target velocity}} + \underbrace{\frac{1}{N}\sum_{i=1}^{N}(\xi_i - \bar{\xi})^2}_{\text{Drives to consensus}}$$

Let $T > 0$. We aim to solve the following problems:

1. **Instantaneous decrease**: Minimize $\dot{\mathbb{V}}(t)$ for all $t \in [0, T]$
2. **Final cost**: Minimize $\mathbb{V}(T)$
3. **Integral cost**: Minimize $\int_0^T \mathbb{V}(t)dt$

for $\alpha \in \mathcal{U} := \{\alpha : [0, T] \to [0, 1]^N \text{ measurable} \mid \sum_i \alpha_i \leq 1\}$.

Instantaneous Decrease

We compute: $\dot{\mathbb{V}} = -2\mathbb{V} + \frac{2}{N}\bar{\xi}\sum_i(1-\alpha_i)\xi_i$. Then,

$$\text{Find } \min_\alpha \dot{\mathbb{V}} \Leftrightarrow \text{Find } \min_\alpha \sum_i (1-\alpha_i)\xi_i.$$

Theorem 1 *Let $J(t) = \{i \in \{1, \dots, N\} | \xi_i(t) = \max_j \xi_j(t)\}$. Then*

$$\alpha_i(t) := \begin{cases} 1/|J(t)|, & i \in J(t) \\ 0, & i \notin J(t) \end{cases}$$

minimizes $\dot{\mathbb{V}}$ almost everywhere.

Remark 1 *Similar results are obtained for $\sum_i \alpha_i \leq M$, $M > 0$.*

Final Cost

We first solve the "full control" optimal control problem for $\alpha \in \mathcal{U}_F := \{\alpha : [0, T] \to [0, 1]^N \text{ measurable} \mid \sum_i \alpha_i \equiv 1\}$. Then the optimal control strategy is the same as for the instantaneous decrease.

Theorem 2 *Full Control Strategy.*
Let $t_l = \frac{N}{N-1}\ln\left((l-1)\frac{N-1}{N}\frac{\bar{\xi}_{1,l-1} - \xi_l}{\bar{\xi}}(0) + 1\right)$, $l \in \{1, \dots, N\}$,
where $\bar{\xi}_{1,l} = \frac{1}{l}\sum_{i=1}^{l}\xi_i$.
If $\in [t_l, t_{l+1}[$, *then any strategy satisfying:*

- $\xi_i(T) = \bar{\xi}_{1,l}(T)$ *for* $i \in \{1, \dots, l\}$ *and* $\sum_{i=1}^{l}\alpha_i \equiv 1$
- $\alpha_i \equiv 0$ *for* $i \in \{l+1, \dots, N\}$

is optimal in $\mathcal{U}_F$.
If $T \geq t_N$, *then any strategy satisfying*

- $\xi_i(T) = \bar{\xi}(T)$ *for* $i \in \{1, \dots, N\}$
- $\sum_{i=1}^{N}\alpha_i \equiv 1$

is optimal in $\mathcal{U}_F$.

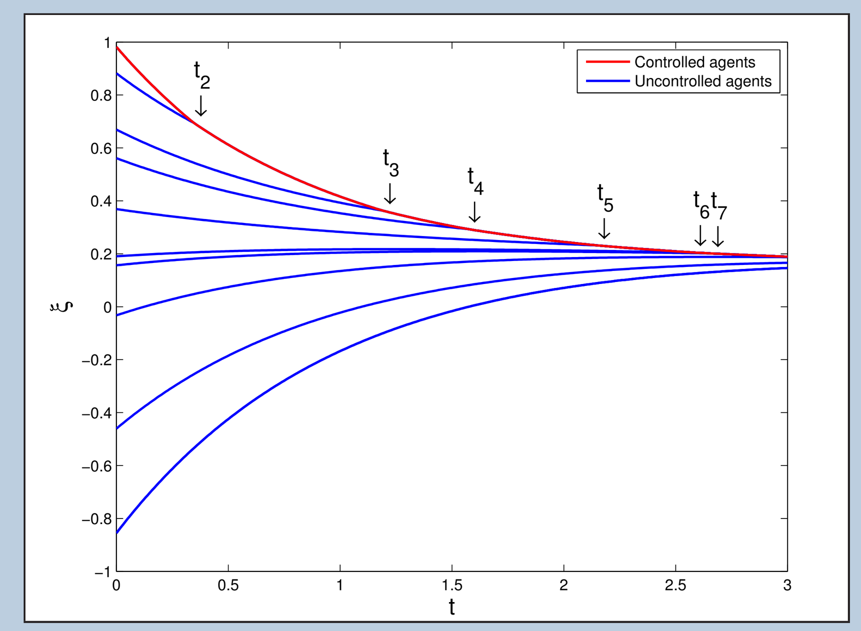

Figure 1: Evolution of ξ_i, $i \in \{1, \dots, 10\}$ with the full strength optimal control for a system of 10 agents, with $t_7 < T < t_8$.

Theorem 3 *Sufficient condition for full control.*
Let $\alpha \in \mathcal{U}$ be an optimal control. Define t_N as in Theorem 2. If $T \geq t_N$, then $\alpha \in \mathcal{U}_F$ and $\xi_i(T) = \bar{\xi}(T)$ for all $i \in \{1, \dots, N\}$.

Theorem 4 *Inactivation Principle.*
If $T < t_N$, then there exists some $\delta \in [0, T[$ such that $\alpha^{opt} \equiv 0$ on $[0, \delta]$, and $\sum_i \alpha_i^{opt} \equiv 1$ on $[\delta, T]$.

Simulations for Final Cost

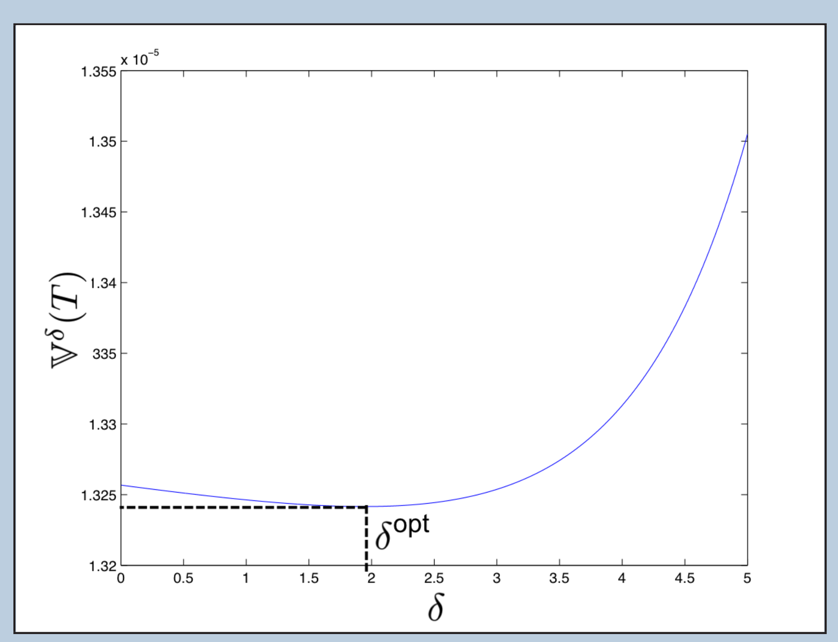

Figure 2: $\mathbb{V}^\delta(T)$ with respect to the inactivation time δ. Here the optimal inactivation time is $\delta^{\text{opt}} = 1.94 > 0$.

N	5	10	20	50
T=3	1.6 %	0.9 %	0%	0%
T=4	1.8 %	0.7 %	0.3 %	0%
T=5	1 %	0.2 %	0.2 %	0%
T=6	0.2 %	0.1 %	0%	0.1 %

Table 1: Number of cases in which $\delta > 0$ out of 1000 simulations. $\xi_i(0)$ chosen randomly in $[-1, 1]$.

N	5	10	20	50
T=3	0.073%	0.001%	-	-
T=4	0.27%	0.018%	0.001%	-
T=5	0.91%	0.056%	0.0069%	-
T=6	1.53%	0.2%	-	0.00003 %

Table 2: Gain in performance compared to full strength control.

Integral Cost

Theorem 5 *The optimal control strategies for the Integral Cost problem requires using full-strength control, i.e. $\alpha \in \mathcal{U}_F$. They are the same strategies as described in Theorem 2.*

Illustration

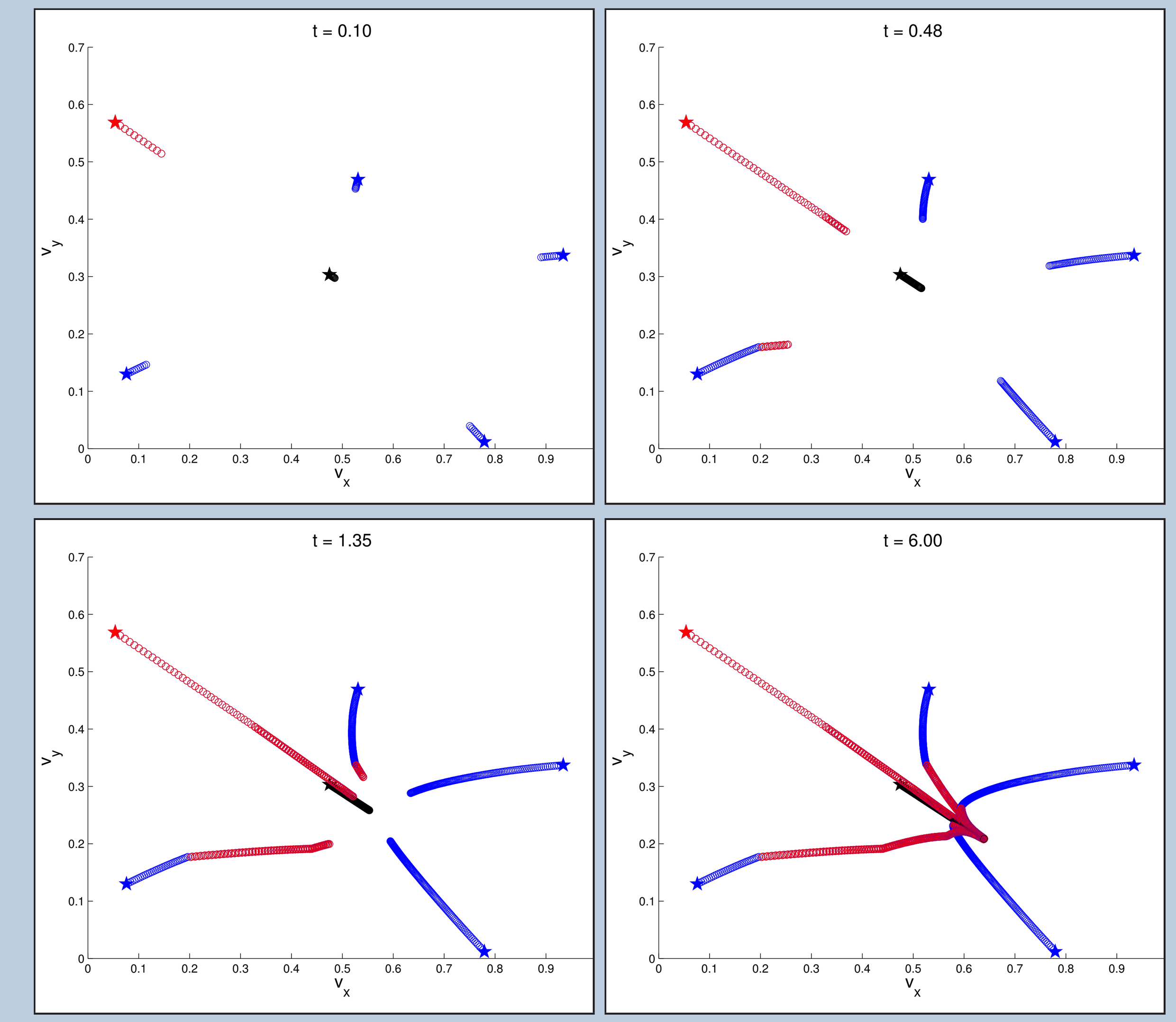

Figure 3: Full-strength control of 5 agents to reach $V = (1, 0)$. Agents are represented in the velocity space, controlled ones in red, uncontrolled ones in blue, and the mean velocity in black.

$$\forall k \leq l,\ \forall t \in [t_k, t_{k+1}[,\ \begin{cases} \alpha_i(t) = \frac{1}{k} \text{ if } i \leq k \\ \alpha_i(t) = 0 \text{ if } i > k. \end{cases}$$

Conclusions and Future Directions

- Counter-intuitively, in order to minimize a final cost, it can be beneficial to let the system evolve freely on some initial time interval. However, the number of cases in which Inactivation is advantageous is small, and even decreases as the number of agents in the group increases (see Table 1). Furthermore, the gain in performance compared to the strategy saturating the control at all time is minor (Table 2). For reasons of computational speed and complexity, it is very reasonable to neglect this phenomenon and to act with full control at all time.
- In the case of instantaneous decrease or integral cost, full control must be used at all time.
- A future direction is to exert *sparse control*, i.e. aim to control only a few agents spreading them optimally among the group.

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