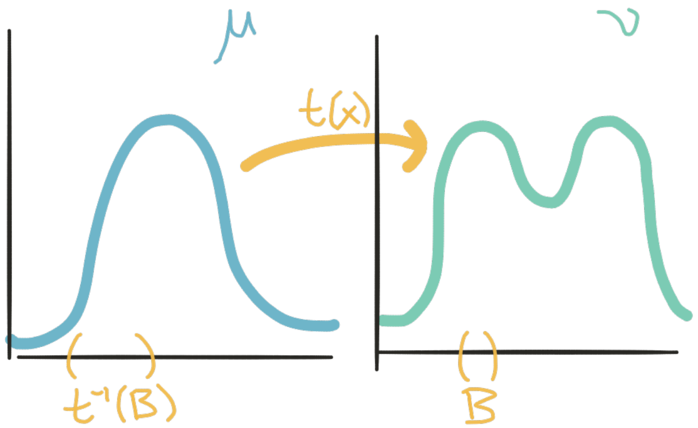


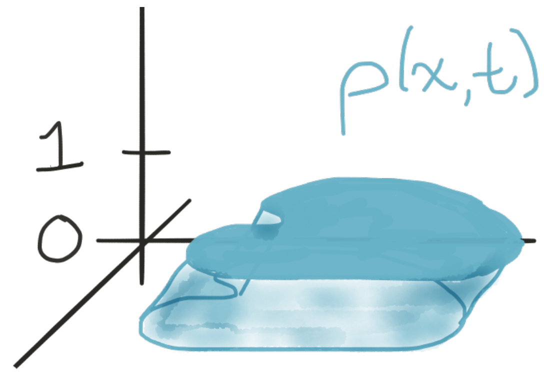


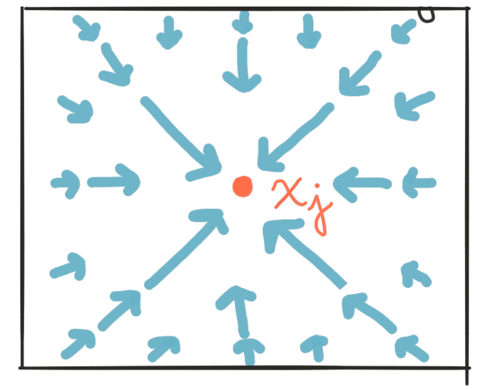


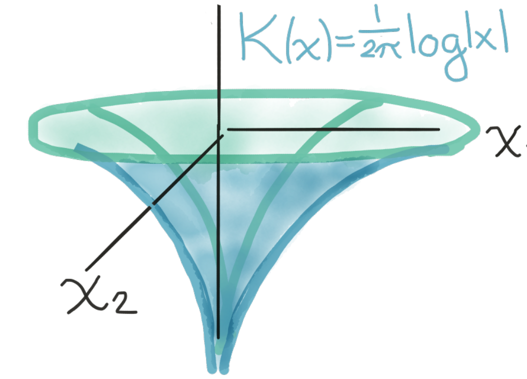


# slow diffusion and vanishing regularization: singular limits in aggregation diffusion equations


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Ki-Net Young Researchers Workshop, 10/22/18

# collective dynamics

# aggregation diffusion equations

- $\rho(x,t)$: $\mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ nonnegative density
- mass is conserved $\Rightarrow \int \rho(x)\, dx = M$

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho)}_{\text{aggregation}}\rho) + \underbrace{\Delta \rho^m}_{\text{(degenerate) diffusion}} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

interaction kernels:

- vortex motion/chemotaxis: $K(x) = \pm |x|^{2-d}/(2-d)$ $\quad |x|^0/0 = \log(|x|)$
- granular media: $K(x) = \quad |x|^3$
- swarming: $K(x) = |x|^a/a - |x|^b/b, \; -d < b \leq a$

degenerate diffusion: $\Delta \rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D} \nabla \rho)$

# agg diffusion → constrained agg

- $\rho(x,t)$: $\mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ nonnegative density
- mass is conserved ⇒ $\int \rho(x)\, dx = M$

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\underbrace{\nabla K * \rho}_{\text{aggregation}})\rho) + \underbrace{\Delta \rho^m}_{\text{(degenerate) diffusion}}$$

for $K(x) : \mathbb{R}^d \to \mathbb{R}$ and $m \geq 1$

constrained aggregation equation

“$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla (K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$”

for $K(x) : \mathbb{R}^d \to \mathbb{R}$

Formally, hard height constraint is singular limit of degenerate diffusion:

Idea: $\Delta \rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D} \nabla \rho)$, so as m→+∞, $D \to \begin{cases} +\infty & \text{if } \rho \geq 1 \\ 0 & \text{if } \rho < 1 \end{cases}$

# motivation #1: gradient flow structure

**Recall:** x(t): [0,+∞) → X is the gradient flow of an energy E: X → $\mathbb{R}$ if

$$“\;\boxed{\frac{d}{dt}x(t) = -\nabla_X E(x(t))}\;”$$

**Examples:**

| energy | gradient flow |
|---|---|
| $E(f) = \frac{1}{2}\int \lvert\nabla f\rvert^2$ | $\frac{d}{dt}f = \Delta f$ |
| $E(\rho) = \int \rho \log \rho$ | $\frac{d}{dt}\rho = \Delta \rho$ |
| $E_m(\rho) = \frac{1}{2}\int (K*\rho)\rho + \frac{1}{m-1}\int \rho^m$ | $\frac{d}{dt}\rho = \nabla\cdot((\nabla K*\rho)\rho) + \Delta\rho^m$ |
| $E_\infty(\rho) = \begin{cases} \frac{1}{2}\int (K*\rho)\rho & \text{if } \rho \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$ | $\begin{cases} \frac{d}{dt}\rho = \nabla\cdot(\nabla(K*\rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$ |

# motivation #2: shape optimization

$$\boxed{\textbf{Goal: } \min \left\{ \int_\Omega \int_\Omega K(x-y)\,dx\,dy : \Omega \subseteq \mathbb{R}^d,\ |\Omega| = M \right\}}$$

[Burchard,Choksi,Topaloglu '16]
[Frank, Lieb '17] [Lopes '17]

- For $K(x) = |x|^a/a - |x|^b/b$: $\displaystyle \int_\Omega \int_\Omega \frac{|x-y|^a}{a} dx\ dy - \int_\Omega \int_\Omega \frac{|x-y|^b}{b} dx\ dy$
- a>0, -d<b≤2-d: M

$m_1$ $m_2$

minimizers don't exist ? minimizers do exist

equivalent problem:

$$\min \left\{ \int (K * \rho)\rho : \rho = 1_\Omega \text{ for } \Omega \subseteq \mathbb{R}^d, |\Omega| = M \right\}$$

relaxed problem:

$$\min \left\{ \int (K * \rho)\rho : 0 \leq \rho \leq 1, \int \rho = M \right\} = \min \left\{ E_\infty(\rho) : \int \rho = M \right\}$$

liquid $M_1$ intermediate $M_2$ solid

a>0, b = 2-d: M

$|\{\rho = 1\}| = 0$ ? $|\{\rho = 1\}| = M$

# plan: two singular limits

1) slow diffusion limit

2) vanishing regularization limit (blob method)

3) numerical method for height constrained problems

# $W_2$ gradient flow

<u>**Def:**</u> $\rho(t):\mathbb{R}\to P_2(\mathbb{R}^d)$ is the **Wasserstein gradient flow** of $E:P_2(\mathbb{R}^d)\to\mathbb{R}$ if

$$E(\rho(t)) - E(\rho(0)) \leq -\frac{1}{2}\int_0^t |\partial E|(\rho(s))^2 ds - \frac{1}{2}\int_0^t |\rho'|(s)^2 ds$$

where

$$|\partial E|(\mu) := \limsup_{\nu\to\mu} \frac{(E(\mu)-E(\nu))^+}{W_2(\mu,\nu)} \quad \text{and} \quad |\rho'|(t) = \lim_{s\to t} \frac{W_2(\rho(s),\rho(t))}{|s-t|}$$

Analogy with **Euclidean gradient flow**:

$$\frac{d}{dt}x(t) = -\nabla E(x(t)) \iff \begin{cases} \left|\frac{d}{dt}x(t)\right| = |\nabla E(x(t))| \\ \frac{d}{dt}E(x(t)) = -|\nabla E(x(t))| \left|\frac{d}{dt}x(t)\right| \end{cases}$$

$$\iff \frac{d}{dt}E(x(t)) \leq -\frac{1}{2}|\partial E(x(t))|^2 - \frac{1}{2}|x'|^2(t)$$

# $\Gamma$-convergence of gradient flows

**Theorem:** (Serfaty 2010)**:** Let $\rho_m(x,t)$ be grad flows of $\mathcal{E}_m$ such that

$$\rho_m(x,t) \to \rho_\infty(x,t) \text{ and } \mathcal{E}_m(\rho_m(x,0)) \to \mathcal{E}_\infty(\rho_\infty(x,0))$$

If we have

1. $\liminf_{m \to +\infty} E_m(\rho_m(t)) \geq E_\infty(\rho_\infty(t))$
2. $\liminf_{m \to +\infty} \int_0^t |\rho_m'|^2(s)ds \geq \int_0^t |\rho_\infty'|^2(s)ds$
3. $\liminf_{m \to +\infty} \int_0^t |\partial E_m|^2(\rho_m(s))ds \geq \int_0^t |\partial E_\infty|^2(\rho_\infty(s))ds$

then $\rho_\infty(x,t)$ is a gradient flow of $\mathcal{E}_\infty$.

**Recall:** $\rho(t):\mathbb{R}\to P_2(\mathbb{R}^d)$ is the Wasserstein gradient flow of $E:P_2(\mathbb{R}^d)\to\mathbb{R}$ if

$$E(\rho(t)) - E(\rho(0)) \leq -\frac{1}{2}\int_0^t |\partial E|(\rho(s))^2 ds - \frac{1}{2}\int_0^t |\rho'|(s)^2 ds$$

# plan: two singular limits

1) slow diffusion limit
2) vanishing regularization limit (blob method)
3) numerical method for height constrained problems

# singular limit: slow diffusion

**Goal:**

$m \to +\infty$

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla (K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

**Equivalent Goal:**

$m \to +\infty$

gradient flows of

$$E_m(\rho) = \frac{1}{2}\int (K * \rho)\rho + \frac{1}{m-1}\int \rho^m$$

gradient flows of

$$E_\infty(\rho) = \begin{cases} \frac{1}{2}\int (K * \rho)\rho & \text{if } \rho \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

**Theorem:** [C., Topaloglu 2018]

- Suppose $\rho_m(x,t)$ are gradient flows of $E_m$ satisfying

$$\rho_m(x,0) \to \rho_\infty(x,0) \text{ and } E_m(\rho_m(x,0)) \to E_\infty(\rho_\infty(x,0))$$

Then $\rho_m(x,t) \to \rho_\infty(x,t)$, the gradient flow of $\mathcal{E}_\infty$.

# plan: two singular limits

1) slow diffusion limit

2) vanishing regularization limit (blob method)

3) numerical method for height constrained problems

# particle methods

**Goal:** Approximate a solution to $\frac{d}{dt}\rho(x,t) + \nabla$

## Benefits of particle methods

- positivity preserving
- inherently adaptive
- energy decreasing

…but what about when v(x,t) is not "nice"?

**A general recipe for a particle methods:**

(1) approximate $\rho_0(x)$ as a sum of Dirac masses

$$\rho_0 \approx \sum_{i=1}^{N} \delta_{x_i} m_i$$

(2) evolve the locations of the Dirac masses by

$$\boxed{\frac{d}{dt}x_i(t) = v(x_i(t),t) \quad \forall i}$$

(3) for v nice, $\rho_N(x,t) = \sum_{i=1}^{N} \delta_{x_i(t)} m_i$ is a weak solution of continuity eqn,

so stability estimates imply $\rho_N(x,t) \to \rho(x,t)$.

# particle methods

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m \qquad v = \nabla K * \rho + m\rho^{m-2}\nabla \rho$$

**Problem: v(x,t) is not nice**

- Degenerate diffusion term is worst: particles do not remain particles
- Even the interaction term can slow convergence if strong singularity

**Solution: regularize v(x,t) to make it nice**

- For interaction, regularize via convolution $K_\epsilon = K * \varphi_\epsilon$ [C, Bertozzi 2014]
- How to regularize diffusion term?

# blob method for diffusion

aggregation equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho)$$

energy:

$$E(\rho) = \frac{1}{2}\int (K * \rho)\rho$$

regularized energy:

$$E_\epsilon(\rho) = \frac{1}{2}\int (K_\epsilon * \rho)\rho$$

$$\rho_\epsilon = \rho * \varphi_\epsilon$$

$\Big\downarrow$ $W_2$ grad flow

regularized velocity:

$$v = -\nabla K_\epsilon * \rho$$

porous medium equation:

$$\frac{d}{dt}\rho = \Delta \rho^m$$

energy:

$$E(\rho) = \int \frac{\rho^{m-1}}{m-1}\rho$$

regularized energy:

$$E_\epsilon(\rho) = \int \frac{(\rho_\epsilon)^{m-1}}{m-1}\rho$$

regularized velocity:

$$v = -\nabla \varphi_\epsilon * (\rho_\epsilon^{m-2}\rho) - \rho_\epsilon^{m-2}\nabla \rho_\epsilon$$

# blob method for diffusion

**Previous work:**

- m=2: [Oelschläger 1998], [Lions, Mas-Gallic 2000]
- m=1: [Degond, Mustieles 1990], [Lacombe, Mas-Gallic 1999]
- [Jabin, in preparation]

porous medium equation:

$$\frac{d}{dt}\rho = \Delta \rho^m$$

energy:

$$E(\rho) = \int \frac{\rho^{m-1}}{m-1}\rho$$

regularized energy:

$$E_\epsilon(\rho) = \int \frac{(\rho_\epsilon)^{m-1}}{m-1}\rho$$

regularized velocity:

particles remain particles! $\longrightarrow$

$$v = -\nabla \varphi_\epsilon * (\rho_\epsilon^{m-2}\rho) - \rho_\epsilon^{m-2}\nabla\rho_\epsilon$$

# singular limit: blob method

$$\epsilon \to 0, \;\; N \to +\infty$$

**Goal:**

$$\frac{d}{dt}\rho + \nabla \cdot (v_\epsilon \rho) = 0$$

$$\rho_N(0) = \sum \delta_{x_i} m_i$$

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

$$\rho(0) = \rho_0$$

**Suffices to show:**

$$\epsilon \to 0$$

gradient flows of

$$E_\epsilon(\rho) = \frac{1}{2}\int (K * \rho)\rho + \int \frac{(\rho_\epsilon)^{m-1}}{m-1}\rho$$

gradient flows of

$$E(\rho) = \frac{1}{2}\int (K * \rho)\rho + \int \frac{\rho^m}{m-1}$$

**Theorem** [Carrillo, C., Patacchini 2017]: For fixed m ≥ 1,

- $E_\varepsilon$ Γ-converges to E.
- If K **confining** (K≥0, K(r) →∞ as r→∞), minimizers converge to minimizers.
- If K **semiconvex** ($D^2K$≥0) and gradient flows of $E_\varepsilon$ satisfy compactness estimates uniformly in ε, gradient flows converge to gradient flows.

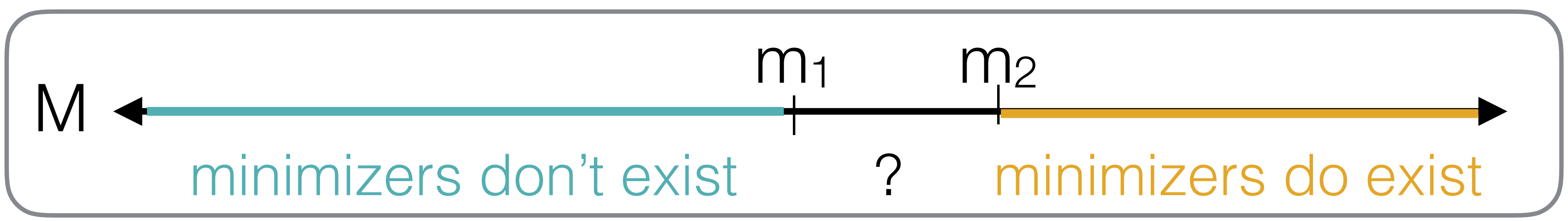
# plan: two singular limits

(1) slow diffusion limit

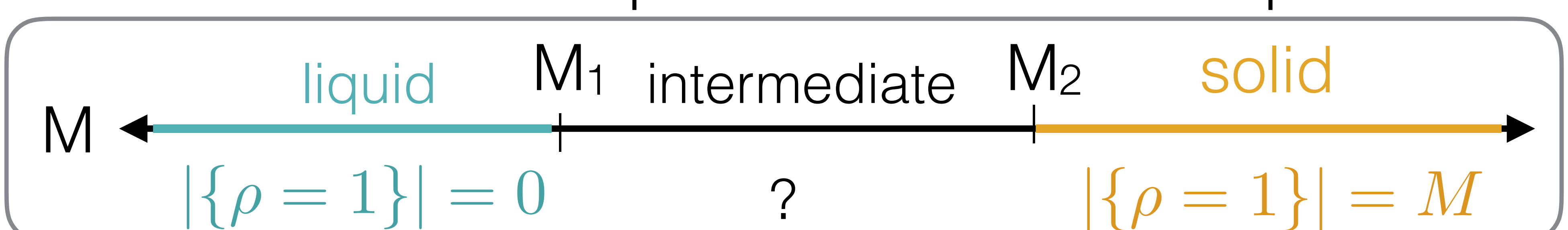
(2) vanishing regularization limit (blob method)

(3) numerical method for height constrained problems

- For singular repulsion and nonsingular interaction…
  - Are masses that allow set valued minimizers a connected interval?

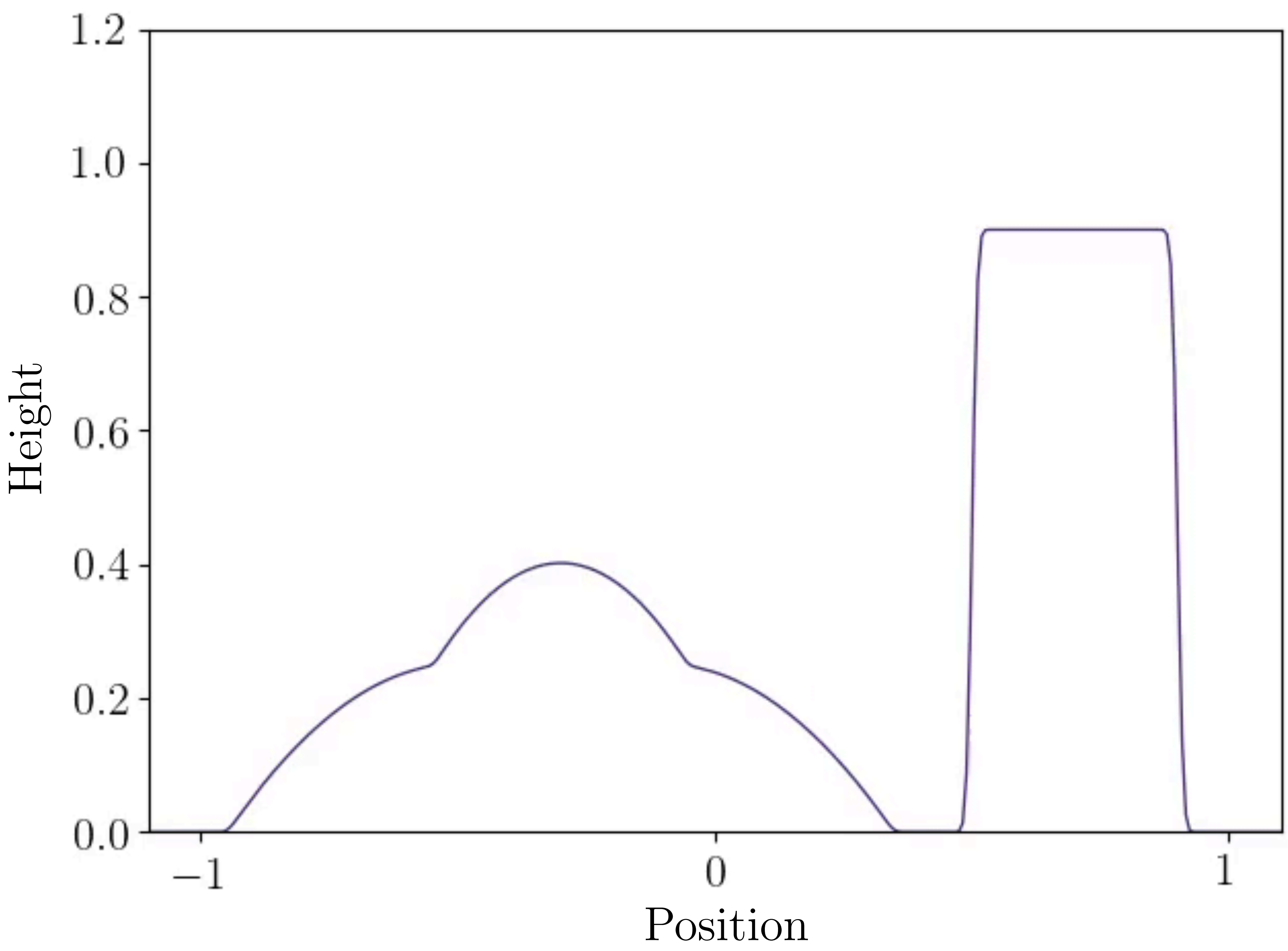



  - Does an intermediate phase exist between liquid and solid?




- What about for other types of repulsion and attraction?

# numerics: slow diff to hard height




K(x) = |x|
m = 800
Nx = 400
h =  0.006
ε = $h^{0.99}$

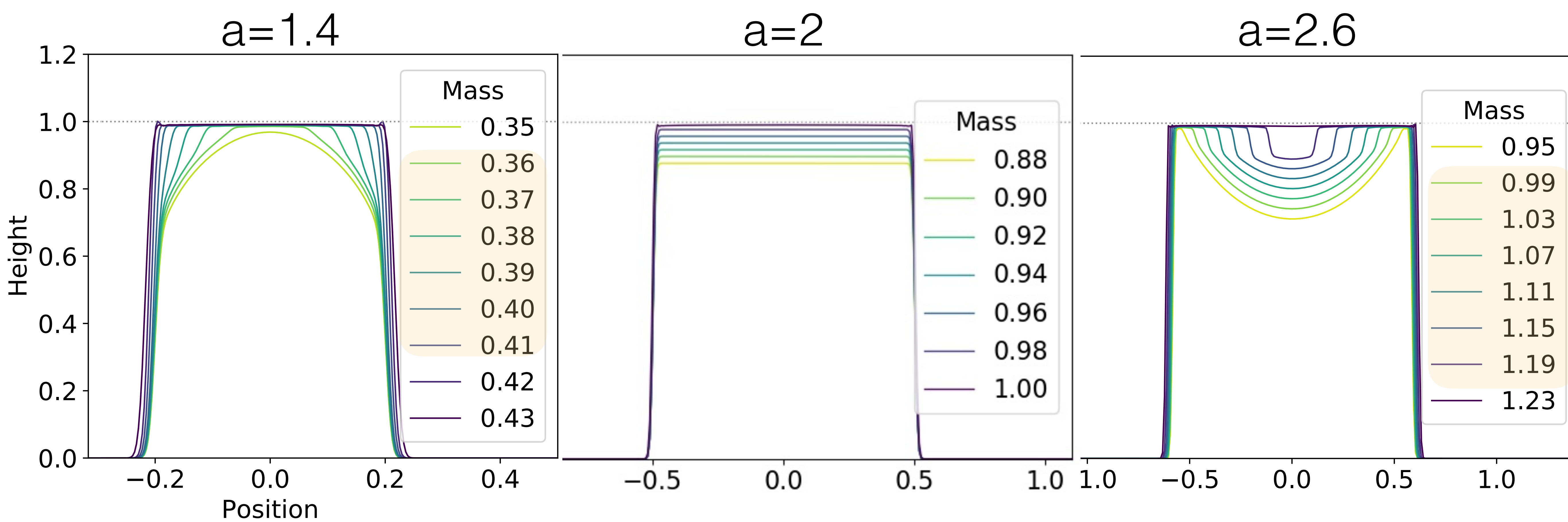
ODEs solved via BDF

remesh when particles>(1.5)h apart

plot $\varphi_\epsilon * \rho_\epsilon(t)$

# numerics: shape optimization

$$K(x) = \frac{|x|^a}{a} - |x|$$



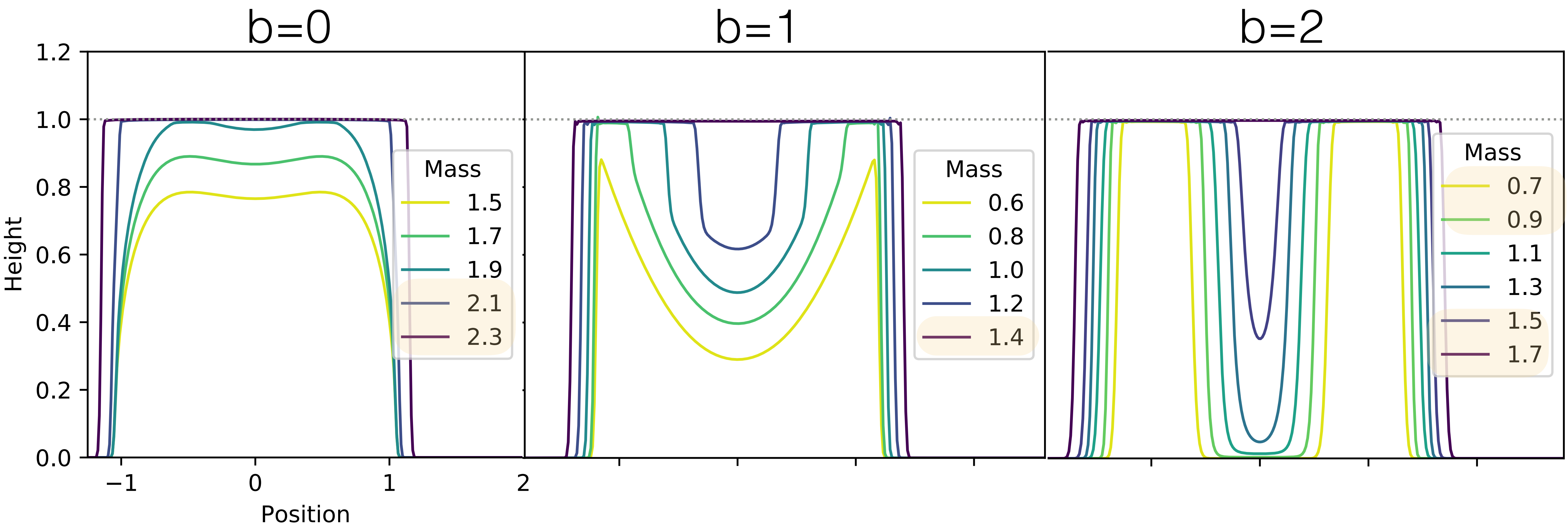

- a ≠ 2: intermediate phase ($M_1 \neq M_2$)
- b≤2-d: masses for set valued minimizers are connected interval ($m_1 = m_2$)

m = 800, Nx = 500

[C., Topaloglu 2018]

# numerics: shape optimization

$$K(x) = \frac{|x|^4}{4} - \frac{|x|^b}{b}$$




- b>2-d: masses for set valued minimizers are NOT connected interval
- singular repulsion key to phase transition results

m = 800, Nx = 300-600, ε = $h^p$, p = 0.85-0.99

[Carrillo, C., Yao 2018]

Thank you!

# Backup

# motivation #2: shape optimization

**Goal:** $$\min \left\{ \int_\Omega \int_\Omega K(x-y)\, dx\, dy : \Omega \subseteq \mathbb{R}^d,\ |\Omega| = M \right\}$$

Related problems:

- Poincaré's problem [Lieb 1977]

$$\min \left\{ -\int_\Omega \int_\Omega \frac{1}{|x-y|}\, dx\, dy : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = M \right\}$$

- Isoperimetric problem

$$\min \left\{ \mathrm{Perimeter}(\Omega) : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = M \right\}$$

- Nonlocal isoperimetric problem [Knüpfer, Moratov '13]. [Lu, Otto '14], [Frank, Lieb '15], …

$$\min \left\{ \mathrm{Perimeter}(\Omega) + \int_\Omega \int_\Omega \frac{1}{|x-y|}\, dx\, dy : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = M \right\}$$

# singular limit: blob method

**Theorem** [Carrillo, C., Patacchini 2017]: For fixed m ≥ 1,

- $E_\varepsilon$ Γ-converges to E.
- If K **confining** (K≥0, K(r) →∞ as r→∞), minimizers converge to minimizers.
- If K **semiconvex** ($D^2K \geq 0$) and gradient flows of $E_\varepsilon$ satisfy compactness estimates uniformly in ε, gradient flows converge to gradient flows.

compactness estimates:

A1 blobs converge in $L^m(\mathbb{R}^d)$ to something

A2 bounded (m-1)th moment

A3 bounded "BV norm" $\|\mu_\epsilon(t)\|_{BV_\epsilon^m} \sim \|\nabla \zeta_\epsilon * \mu_\epsilon^m(t)\|_{L^1(\mathbb{R}^d)}$

Note: (m=2) A1-3 hold if initial data has bdd entropy [Lions,Mas-Gallic 2000]

# singular limit: blob method

**Theorem** [Carrillo, C., Patacchini 2017]:

- For fixed m, $E_\varepsilon$ Γ-converges to E.
- If K **confining** (K≥0, K(r) →∞ as r→∞), minimizers converge to minimizers.

**Theorem**: Fix m ≥ 2 and K semiconvex. Suppose the GF $\rho_\varepsilon(t)$ of $E_\varepsilon$ satisfies

A1 $$\sup_{\epsilon>0} \int_0^t \int_{\mathbb{R}^d} |x|^{m-1} \rho_\epsilon(x,s)\,dx\,ds < +\infty$$

A2 $$\sup_{\epsilon>0} \int_0^t \|\rho_\epsilon(s)\|_{BV_\epsilon^m}\,ds < +\infty$$

A3 $$\zeta_\epsilon * \rho_\epsilon(t) \xrightarrow{L^m(\mathbb{R}^d)} \rho(t)$$

Then ρ(t) is a gradient flow of E.

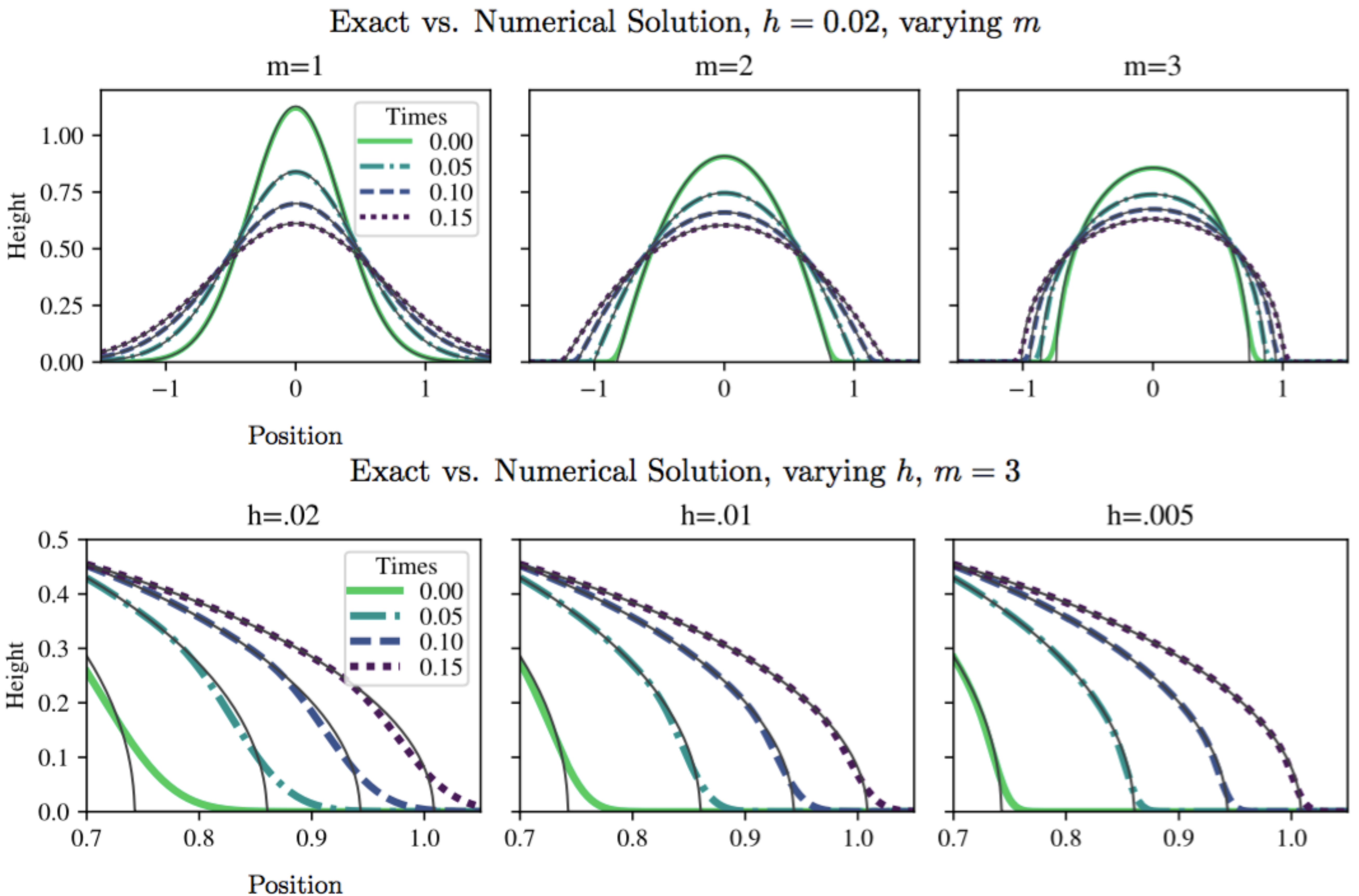
**Remarks:**

**1)** $\|\mu_\epsilon(t)\|_{BV_\epsilon^m} \sim \|\nabla \zeta_\epsilon * \mu_\epsilon^m(t)\|_{L^1(\mathbb{R}^d)}$

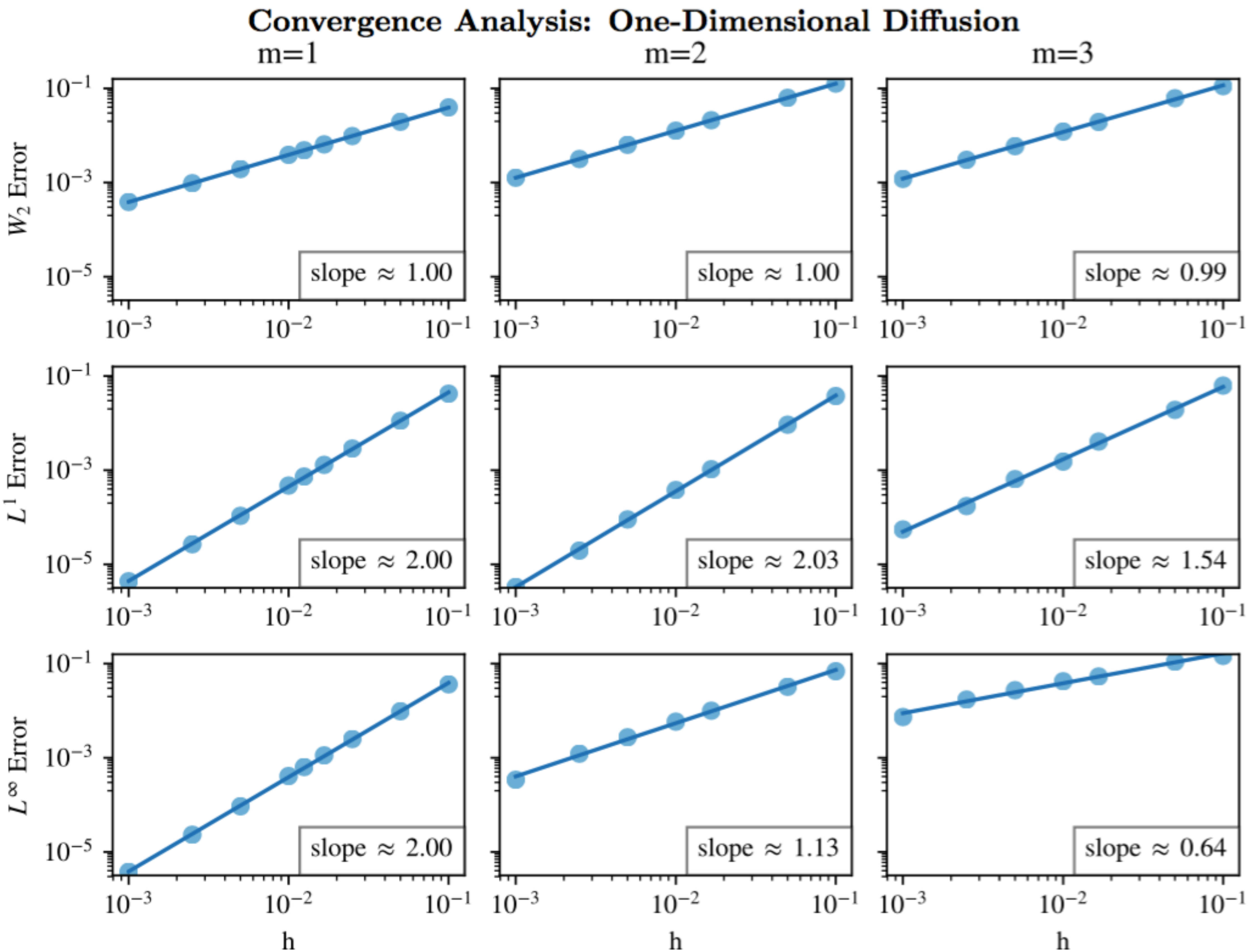
**2)** m=2: A1-3 hold if $\rho_\varepsilon(0)$ has uniformly bdd entropy [Lions, Mas-Gallic 2000]

# blob method: pure diffusion

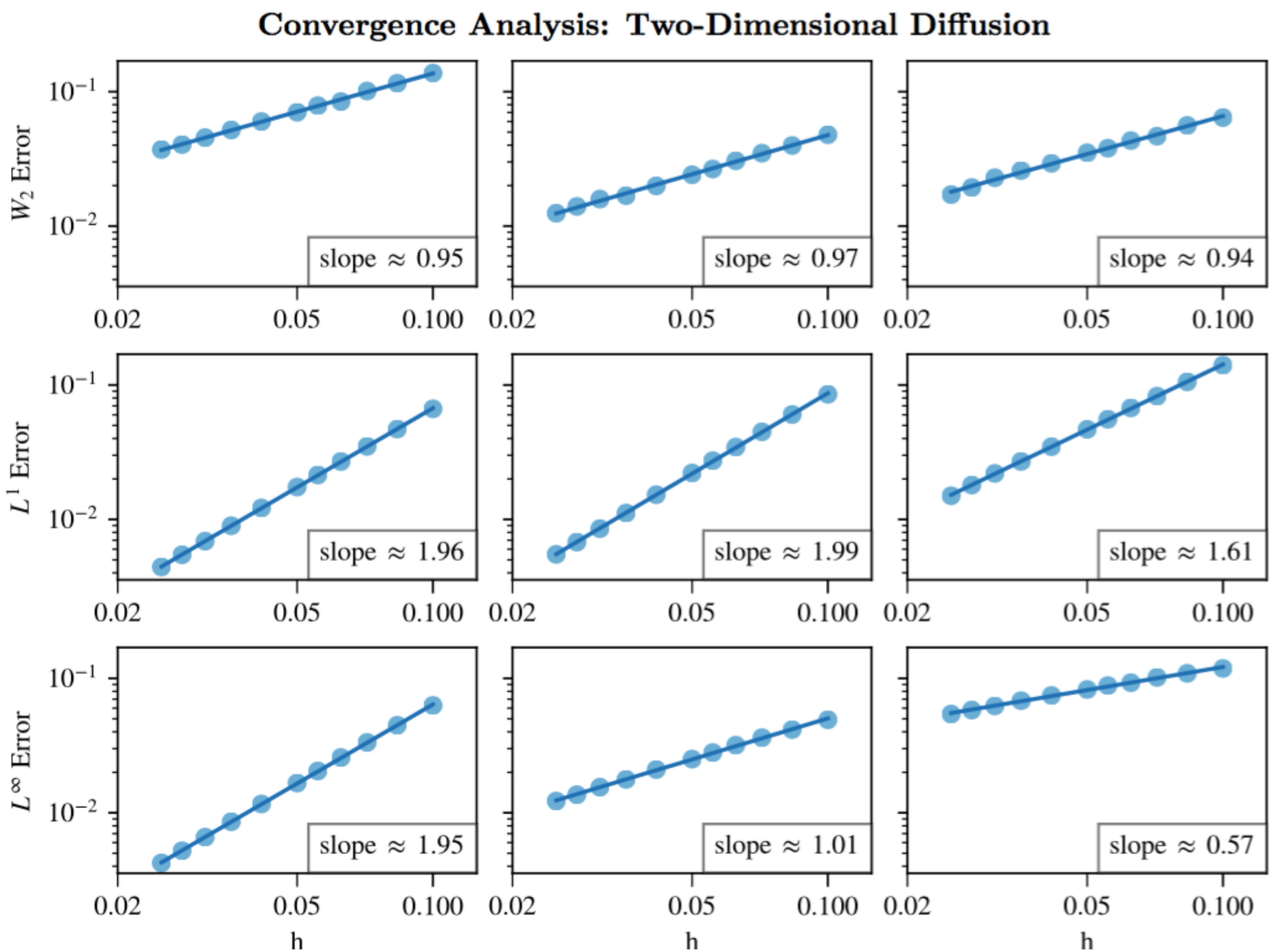
$\varepsilon = h^{.95}$



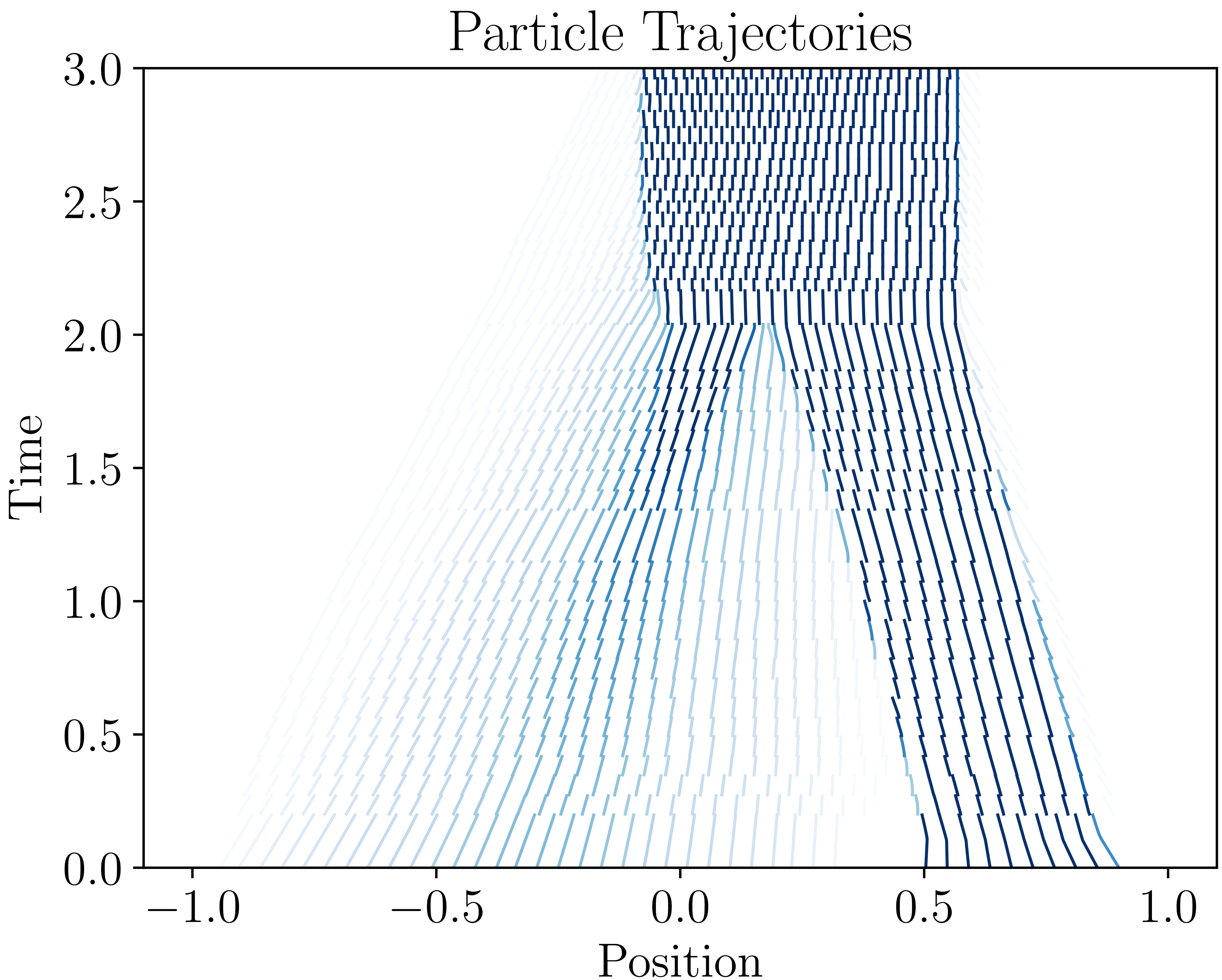
# numerics: rate of convergence (d=1)



# numerics: rate of convergence (d=2)



# numerics: slow diff to hard height

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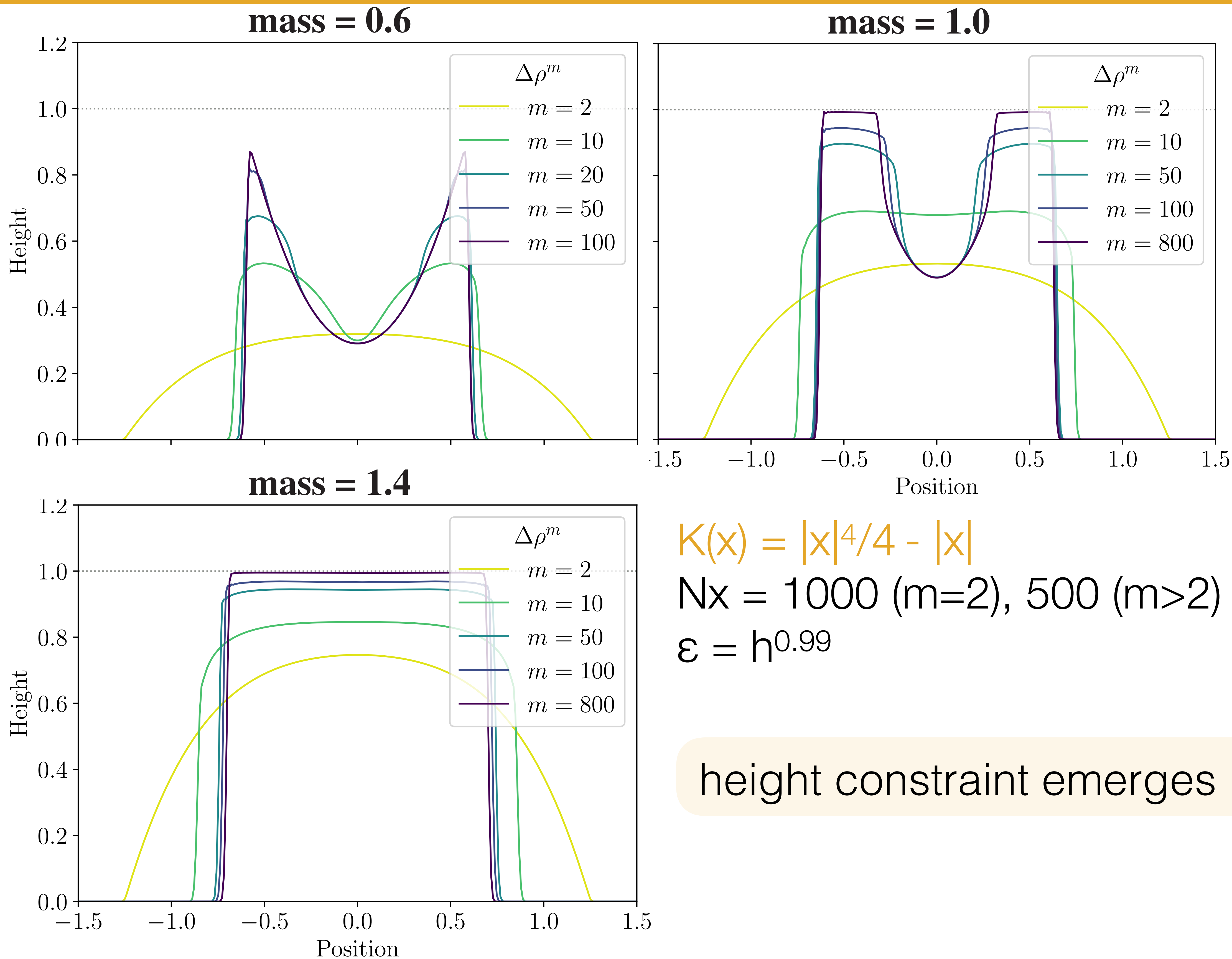
Nx = 400
h = 0.006
ε = $h^{0.99}$

ODEs solved via BDF

remesh when particles>(1.5)h apart

plot $\varphi_\epsilon * \rho_\epsilon(t)$

# numerics: rate of slow diffusion limit




$K(x) = |x|^4/4 - |x|$

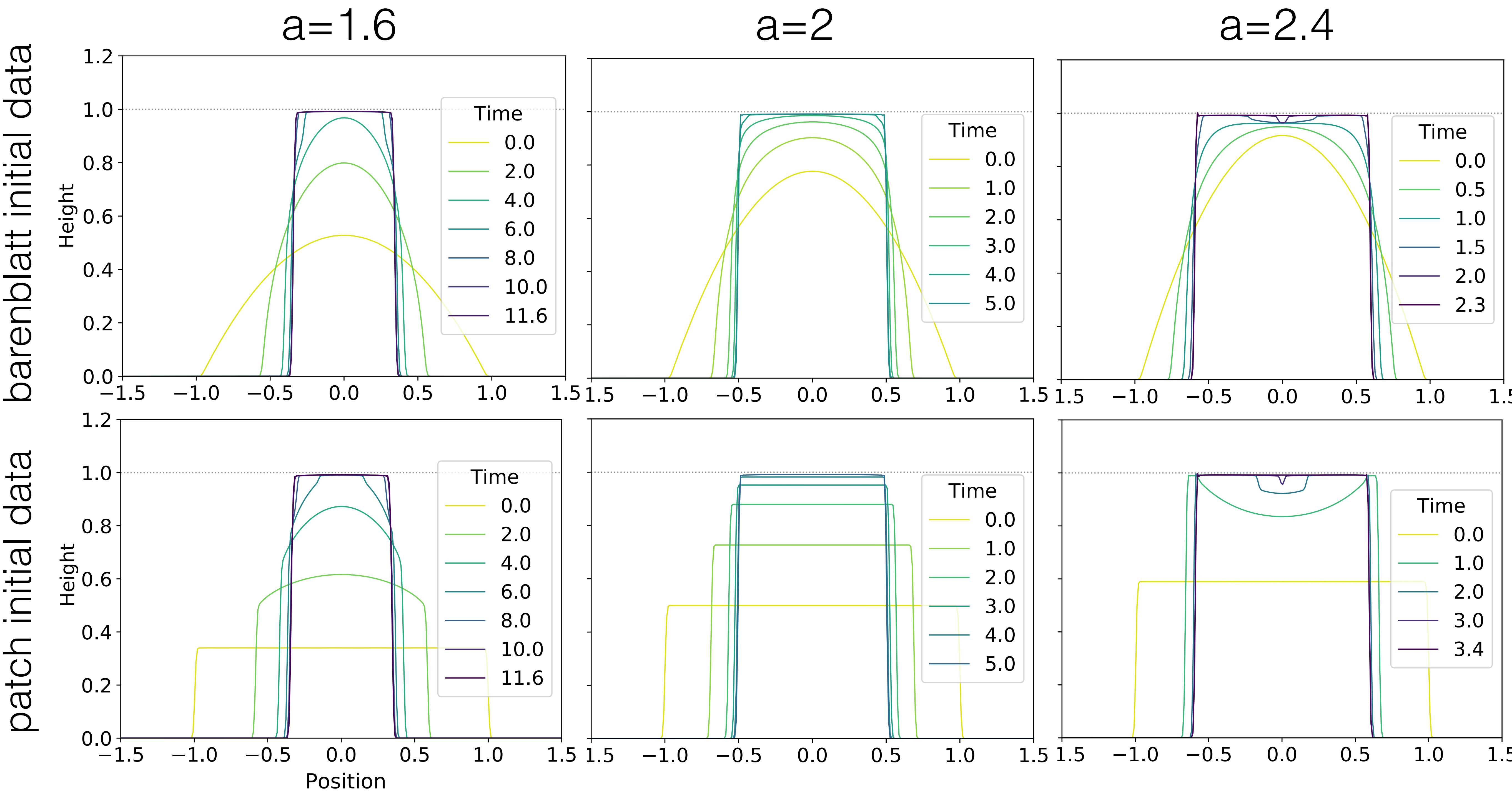
Nx = 1000 (m=2), 500 (m>2)

$\varepsilon = h^{0.99}$

height constraint emerges by m=800

# numerics: convergence to equilibrium

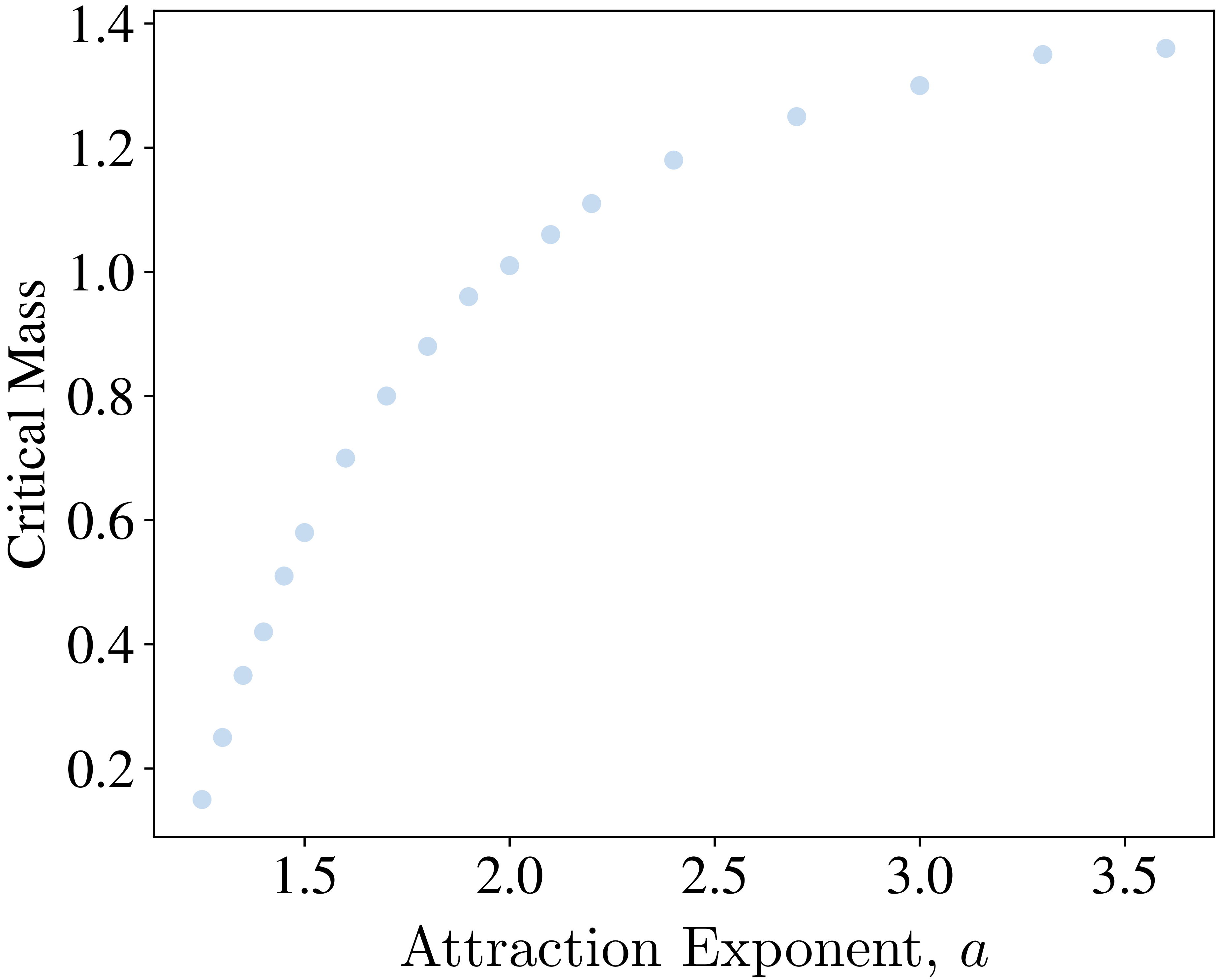
$$K(x) = |x|^a/a - |x|^b/b, \quad b = 1$$




Nx = 500, m = 800, M = critical mass

# numerics: critical mass for solid state

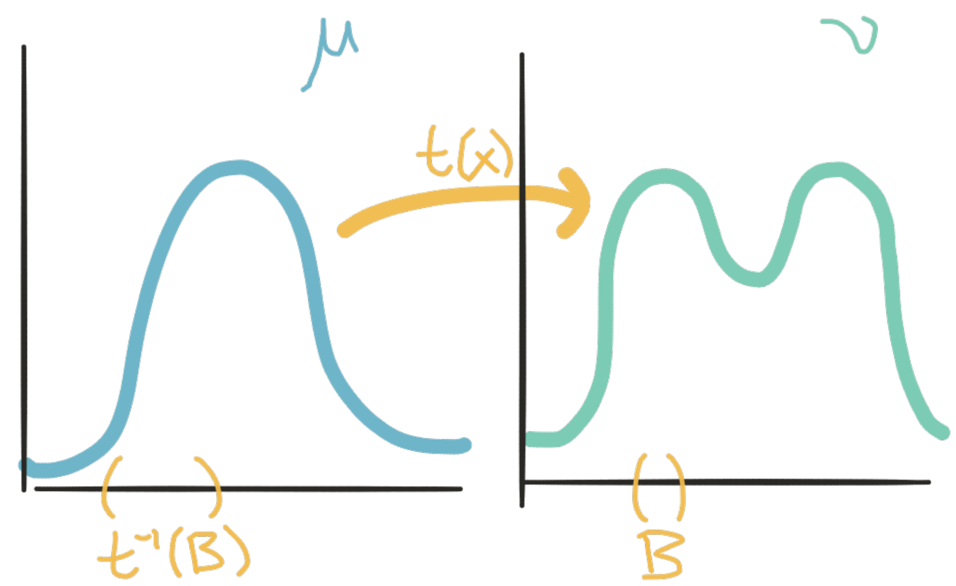
$$K(x) = \frac{|x|^a}{a} - |x|$$




numerical evidence for how critical mass scales with attraction exponent

# Wasserstein metric

Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $\mathbf{t} : \mathbb{R}^d \to \mathbb{R}^d$ *transports* $\mu$ *onto* $\nu$ if $\nu(B) = \mu(\mathbf{t}^{-1}(B))$.
Write $t\#\mu = \nu$ .




The *Wasserstein distance* between measures $\mu, \nu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$ is

$$W_2(\mu, \nu) := \inf \left\{ \left( \underbrace{\int |t(x) - x|^2 d\mu(x)}_{\text{effort to rearrange } \mu \text{ to look like v, using t(x)}} \right)^{1/2} : \underbrace{t\#\mu = \nu}_{\text{t sends } \mu \text{ to v}} \right\}$$

# geodesics

Not just a metric space… a geodesic metric space: there is a constant speed geodesic $\sigma : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ connecting any μ and ν.

$$\sigma(0) = \mu, \;\; \sigma(1) = \nu, \;\; W_2(\sigma(t), \sigma(s)) = |t - s| W_2(\mu, \nu)$$

Monge Kantorovich

$\mu$ Wasserstein geodesic $\sigma(t)$ $\nu$

$\mu$ linear interpolation $(1-t)\mu + t\nu$ $\nu$

[Peyré, Papadakis, Oudet 2013]

# convexity

Since the Wasserstein metric has geodesics, it has a notion of convexity.

Recall: in **Euclidean space**, E: $\mathbb{R}^d \to \mathbb{R}$ is…

convex

$D^2E \geq 0 \Longleftrightarrow E((1-t)x+ty) \leq (1-t)E(x)+tE(y)$

λ-convex

$D^2E \geq \lambda I_{d\times d} \Longleftrightarrow E((1-t)x+ty) \leq (1-t)E(x)+tE(y) - t(1-t)\frac{\lambda}{2}|x-y|^2$

Likewise, in the **Wasserstein metric**, E: $P_2(\mathbb{R}^d) \to \mathbb{R}$ is λ-convex if

$$E(\sigma(t)) \leq (1-t)E(\mu)+tE(\nu) - t(1-t)\frac{\lambda}{2}W_2^2(\mu,\nu)$$

# gradient flow

For λ-convex energies, the gradient flow theory is well-developed.

**Theorem** (Ambrosio, Gigli, Savaré 2005): If E is λ-convex, lower semicontinuous, and bounded below, solutions of its $W_2$ gradient flow

- exist
- are unique
- contract (λ>0)/expand (λ≤0) exponentially:

$$W_2(\rho_1(t), \rho_2(t)) \leq e^{-\lambda t} W_2(\rho_1(0), \rho_2(0))$$

# ω-convexity: well-posedness

For merely ω-convex energies, the gradient flow is well-posed.

**Theorem** (C. 2016): If E is ω-convex for ω(x) = x |log(x)|, lower semicontinuous, and bounded below, solutions of its $W_2$ gradient flow

- exist
- are unique
- contract (λ>0)/expand (λ≤0) double exponentially: for $W_2(\rho_1(0),\rho_2(0)) \leq 1$,

$$W_2(\rho_1(t), \rho_2(t)) \leq W_2(\rho_1(0), \rho_2(0))^{e^{2\lambda t}}$$

More generally, for ω(x) satisfying Osgood's condition, i.e.

$$\int_0^1 \frac{dx}{\omega(x)} = +\infty$$

we obtain the stability estimate

$$F_{2t}(W_2^2(\rho_1(t), \rho_2(t))) \leq W_2^2(\rho_1(0), \rho_2(0))$$

$$\frac{d}{dt} F_t(x) = \lambda\, \omega(F_t(x)), \quad F_0(x) = x$$

# gradient flow

**Good news:** the congested aggregation equation is the Wasserstein gradient flow of the constrained interaction energy:

$$\mathcal{E}_\infty(\rho) = \begin{cases} \frac{1}{2}\int (K * \rho)\rho & \text{if } \rho \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

"
$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$
"

**Fact:** If K: $\mathbb{R}^d \to \mathbb{R}$ is λ-convex, then $\mathcal{E}_\infty$ is λ-convex.

**Bad news:** $K(x) = |x|^a/a - |x|^b/b, \; 2-d \leq b \leq a < 2$ are not λ-convex.

$K(x) = |x|^a/a$

$E_\infty$ falls outside the scope of the existing theory.

# ω-convexity

**Solution:** Even though we don't have

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2}t(1-t)W_2^2(\mu,\nu)$$

λ-convexity

$E_\infty$ is ω-convex for ω(x) = x |log(x)|.

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2}\left[(1-t)\omega\left(t^2 W_2^2(\mu,\nu)\right) + t\omega\left((1-t)^2 W_2^2(\mu,\nu)\right)\right]$$

ω-convexity

**Examples:**

- $\omega(x) = x$, reduces to λ-convexity
- $\omega(x) = x|\log(x)|$, [Ambrosio Serfaty, 2008] [Carrillo Lisini Mainini, 2014]
- $\omega(x) = x^p, \;\; p > 1$, [Carrillo McCann Villani, 2006]

more generally: ω(x) is continuous, increasing, ω(0)=0, and $\displaystyle\int_0^1 \frac{dx}{\omega(x)} = +\infty$

# dynamics & long time behavior

$K(x) = -|x|^{2-d}/(2-d) = \Delta^{-1}$

[C. Kim, Yao 2018]

**Theorem** (dynamics): Given ρ(x,0) = $1_{\Omega(0)}(x)$, let p be a viscosity sol'n of

$$\begin{cases} -\Delta \mathbf{p} = 1 & \text{on} \quad \{\mathbf{p} > 0\} \\ V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}$$

Let Ω(t)={p(x,t)>0}. Then ρ(x,t)=$1_{\Omega(t)}(x)$ solves the congested agg equation.

**Theorem** (long time behavior): In two dimensions, given ρ(x,0) = $1_{\Omega(0)}(x)$,

$$\rho(x,t) \xrightarrow{L^p} 1_B(x) \text{ for all } 1 \leq p < +\infty$$

$$|E_\infty(\rho(\cdot,t)) - E_\infty(1_B)| \leq C_{\Omega(0)} t^{-1/6}$$

# previous work

Congested drift equation: “$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot ((\nabla V)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$”

[Maury, Roudneff-Chupin, Santambrogio 2010]
- introduced as a model of crowd motion in an evacuation scenario
- showed well-posedness as a $W_2$ gradient flow for V convex

[Alexander, Kim, Yao 2014]
- for V convex, proved slow diffusion limit
- for $\Delta V > 0$, characterized patch dynamics via free boundary problem

Challenges:
- $K * \rho$ not convex ⇒ $W_2$ gradient flow theory comparatively undeveloped
- $K * \rho$ nonlocal ⇒ no comparison principle

# Γ-convergence of gradient flows

**Goal:**

1. $\liminf_{m\to+\infty} E_m(\rho_m(t)) \geq E_\infty(\rho_\infty(t))$

3. $\liminf_{m\to+\infty} \int_0^t |\partial E_m|^2(\rho_m(s))ds \geq \int_0^t |\partial E_\infty|^2(\rho_\infty(s))ds$

$$\mathcal{E}_m(\rho) = \int K * \rho d\rho + \frac{1}{m-1}\int \rho^m$$

$$\mathcal{E}_\infty(\rho) = \begin{cases} \int K * \rho d\rho & \text{if } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

- (1) follows by interpolation of $L^p$ norms
- (3) is more difficult, due to the lack of convexity (or even ω-convexity) uniformly in m

instead, we must use specific structure of metric slope

$$|\partial \mathcal{E}_m|(\mu_m) = \left\| \nabla K * \mu_m + \frac{\nabla \mu_m^m}{\mu_m} \right\|_{L^2(\mu_m)}$$

# convexity

Since the Wasserstein metric has geodesics, it has a notion of convexity.

Recall: in **Euclidean space**, E: $\mathbb{R}^d \to \mathbb{R}$ is…

λ-convex $\quad E((1-t)x+ty) \leq (1-t)E(x) + tE(y) - t(1-t)\frac{\lambda}{2}|x-y|^2$

Likewise, in the **Wasserstein metric**, E: $P_2(\mathbb{R}^d) \to \mathbb{R}$ is…

λ-convex $\quad E(\sigma(t)) \leq (1-t)E(\mu) + tE(\nu) - t(1-t)\frac{\lambda}{2}W_2^2(\mu,\nu)$

ω-convex $\quad E(\sigma(t)) \leq (1-t)E(\mu) + tE(\nu)$

$$-\frac{\lambda}{2}\left[(1-t)\omega\left(t^2W_2^2(\mu,\nu)\right) + t\omega\left((1-t)^2W_2^2(\mu,\nu)\right)\right]$$

$$\int_0^1 \frac{dx}{\omega(x)} = +\infty, \qquad \text{e.g. } \omega(x) = x|\log(x)|$$

[Carrillo, McCann, Villani, '06] [Ambrosio, Serfaty, '08] [Carrillo, Lisini, Mainini, '14] [C. '17] [C., Kim, Yao '17]

# slow diffusion limit

**Sketch of proof (iii):** liminf $\int |\partial E_m|^2(\rho_m(t))\, dt \geq \int |\partial E_\infty|^2(\rho_\infty(t))$

$$E_m(\rho) = \frac{1}{2}\int K * \rho d\rho + \frac{1}{m-1}\int \rho^m$$

$$|\partial E_m|(\rho) = \left\| \nabla K * \rho + \frac{\nabla \rho^m}{\rho} \right\|_{L^2(\rho)}$$

$$\begin{tikzcd}
& \rho_m \text{ grad flow} \arrow[r, "E_m"] & \|\rho_m\|_m < C \arrow[r, "\text{HLS}"] \arrow[d] & \|\nabla K * \rho_m\|_{L^2(\rho_m)} < C \\
{} \arrow[r, "|\partial E_m|"] & \left\| \frac{\nabla \rho_m^m}{\rho_m} \right\|_{L^2(\rho_m)} < C \arrow[r] & \|\nabla \rho_m^m\|_{2-2/(m+1)} < C \arrow[r, "S"] & \|\rho_m^m\|_2 < C
\end{tikzcd}$$

With this compactness, we get

$$\nabla K * \rho_m \to \nabla K * \rho, \quad \frac{\nabla \rho_m^m}{\rho_m} \to \frac{\nabla \sigma}{\rho}, \quad \liminf |\partial E_m|(\rho_m) \geq \left\| \nabla K * \rho + \frac{\nabla \sigma}{\rho} \right\|_{L^2(\rho)}$$

We conclude by showing RHS $\geq |\partial E_\infty|(\rho_\infty)$.

# dynamics via free boundary problem

How does congested aggregation equation relate to free boundary problem?

“
$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$
”

Consider initial data: $\rho(x,0) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$

Since $\nabla K * \rho$ causes self-attraction, we expect $\rho(x,t)=1_{\Omega(t)}(x)$.

**Theorem** (C., Kim, Yao ’18):
Suppose $\rho(x,t)$ solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
Then $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t) = \{$**p(x,t)**$>0\}$, where **p** a viscosity solution of

$$\begin{cases} -\Delta \mathbf{p} = 1 & \text{on } \{\mathbf{p} > 0\} \\ V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}$$

# long time behavior

Using free boundary characterization, we can describe long time behavior:

- In any dimension, the Riesz Rearrangement Inequality guarantees that the unique minimizer of $E_\infty$ is $1_B(x)$.
- Need to show mass of ρ(x,t) doesn't escape to +∞. To accomplish this, we use an inequality due to Talenti, which holds in d=2.

**Theorem** (C., Kim, Yao 2016):

- Suppose ρ(x,t) solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
- Then, in two dimensions,

$$\rho(x,t) \xrightarrow{L^p} 1_B(x) \text{ for all } 1 \leq p < +\infty$$

and

$$|E_\infty(\rho(\cdot,t)) - E_\infty(1_B)| \leq C_{\Omega(0)} t^{-1/6}$$

# gradient flow

ρ(t): $\mathbb{R} \rightarrow P_2(\mathbb{R}^d)$ is the gradient flow of energy E: $P_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ if

“ $\boxed{\frac{d}{dt}x(t) = -\nabla_X E(x(t))}$ ”

More precisely, ρ(t) is the gradient flow of E if…

- $\exists\; v(t) \in L^2_{\text{loc}}((0,+\infty), L^2(\rho(t)))\; s.t.$ $\frac{d}{dt}\rho(x,t) + \nabla$
- $-v(t) \in \partial E(\rho(t))$ for a.e. t>0

The term brackets is analogous to ξ (ν - ρ)

Tangent space?

- ξ belongs to the subdifferential of E at ρ if as μ → ν,

$$E(\nu) - E(\rho) \geq \int \langle \xi, \mathbf{t}_\rho^\nu - \mathrm{id} \rangle \mathrm{d}\mu + \mathrm{o}(\mathrm{W}_2(\rho, \nu))$$

- If E and ρ are nice, $\partial E(\rho) = \left\{ \nabla \frac{\partial E}{\partial \rho} \right\}$
- Then solutions of the gradient flow can be characterized via a PDE.

# aside: ω-convexity & Euler equations

In fact, when $\omega(x) = x\,|\log(x)|$, ω-convexity is related to well-posedness of bounded solutions of the the Euler equations.

- λ-convexity in $W_2$ is analogous to $D^2E$ being bounded from below in Euclidean space, or that $\nabla E$ is one-sided Lipschitz.

- Likewise, ω-convexity in $W_2$ is analogous to $D^2E$ being BMO in Euclidean space, or that $\nabla E$ is log-Lipschitz.

- Log-Lipschitz regularity of the velocity field was precisely what allowed [Yudovich 1963] to prove uniqueness of bounded solutions of the two dimensional Euler equations.

# ω-convexity

**Examples:**

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

- Chemotaxis: $K(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } d = 2 \\ C_d |x|^{2-d} & \text{otherwise} \end{cases}$

ω-convex on $L^\infty$

- Swarming: $K(x) = |x|^a/a - |x|^b/b, \quad 2 - d \leq b < a$

ω-convex on $L^p$, $p \geq d/(b+d-2)$

- Granular media: $K(x) = |x|^3$

ω-convex on measures with fixed center of mass; $\omega(x) = x^{3/2}$

**Sufficient condition:**

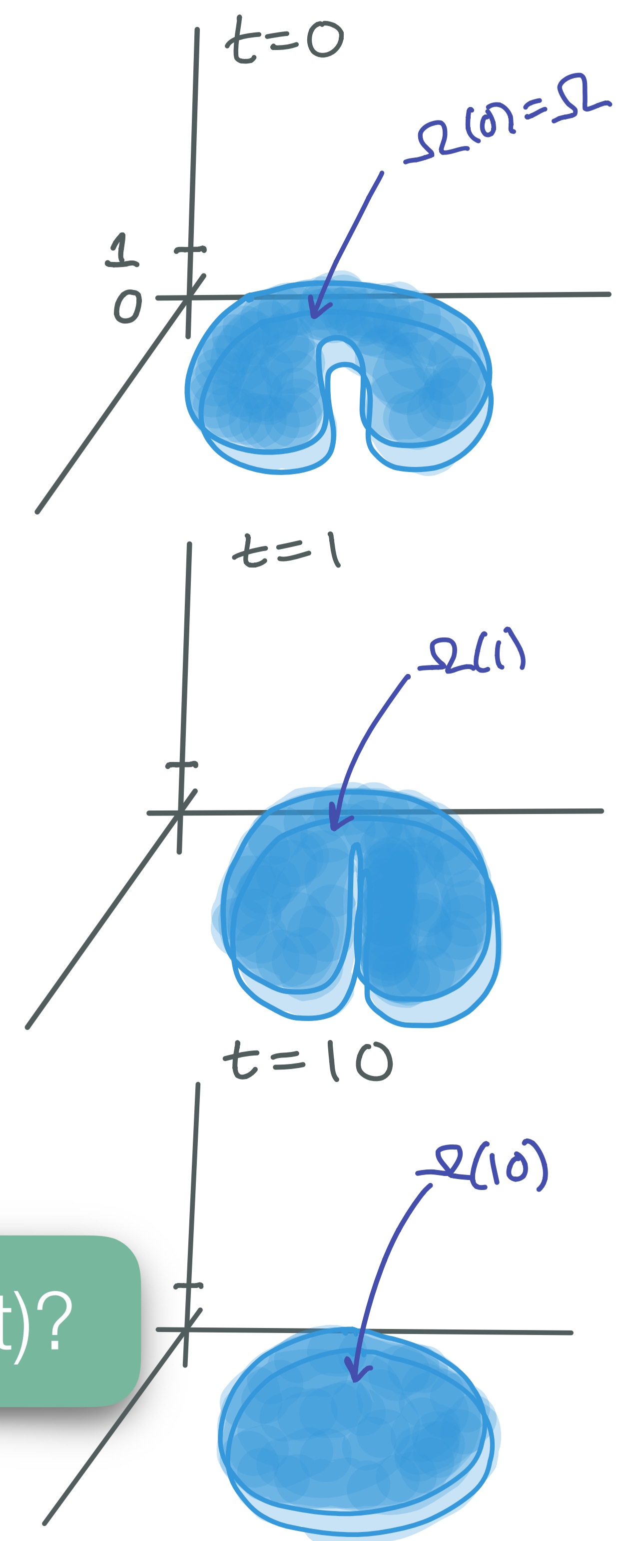
Above the tangent line inequality

$$E(\mu_1) - E(\mu_0) - \frac{d}{d\alpha} E(\mu_\alpha)|_{\alpha=0} \geq \frac{\lambda}{2} \omega(W_2^2(\mu_0, \mu_1))$$

# motivation for free boundary problem

How does congested aggregation equation relate to free boundary problem?

"
$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$
"




- Consider patch solutions. For a domain Ω, suppose that ρ(x,t) is a solution with initial data

$$\rho(x,0) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

- Since $K = \Delta^{-1}$, $\nabla K * \rho$ causes self-attraction. Thus, we expect ρ(x,t) to remain a characteristic function.

- Let Ω(t)={ρ=1} be *congested region*, so $\rho(x,t) = 1_{\Omega(t)}(x)$.

What free boundary problem describes evolution of Ω(t)?

# formal derivation

- Here is a **formal** derivation of the related free boundary problem.
- Suppose ρ(x,t) solves "$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$"
- Since mass is conserved, we expect ρ(x,t) satisfies a continuity equation

$$\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho)$$

where **∇p(x,t)** is the pressure arising from the height constraint.

Height constraint is active on the congested region {**p**>0} = Ω(t).

Height constraint is inactive outside the congested region {**p**=0}= $\Omega(t)^c$.

# formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot ((\underbrace{\nabla K * \rho + \nabla \mathbf{p})}_{v}\, \rho)}$ what happens on congested region?

- Because of hard height constraint, on the congested region Ω(t)={ρ=1}, the velocity field is incompressible, **∇**·v=0.

- Since K= $\Delta^{-1}$, $\nabla \cdot v = \Delta K * \rho + \Delta \mathbf{p} = \rho + \Delta \mathbf{p}$, so incompressibility means

$$\boxed{-\Delta \mathbf{p} = \rho \text{ on } \Omega(t) = \{\rho = 1\}}$$

- Using that the height constraint is active on the congested region, Ω(t)={**p**>0}, we obtain the following equation for the pressure:

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

# formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho)}$ what about bdy of congested region?

outward normal velocity of ∂Ω(t)

- By conservation of mass,

$$0 = \frac{d}{dt}\int_{\Omega(t)} \rho = \int_{\Omega(t)} \frac{d}{dt}\rho + \int_{\partial\Omega(t)} V\rho$$

- Using that ρ(x,t) solves the above continuity equation, this equals

$$= \int_{\Omega(t)} \nabla \cdot ((\nabla K * \rho + \nabla \mathbf{p})\rho) + \int_{\partial\Omega(t)} V\rho = \int_{\partial\Omega(t)} (\partial_\nu K * \rho + \partial_\nu \mathbf{p} + V)\rho$$

- Since $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t)=\{\mathbf{p}>0\}$, we again obtain an equation for $\mathbf{p}$,

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

# free boundary problem

Combining the observations that…

- on the congested region,

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

- and on the boundary of the congested region,

outward normal velocity of $\partial\Omega(t)$

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

Remind myself the hoops we had to jump through to even define viscosity solutions

**Theorem** (C., Kim, Yao 2016):

- Suppose ρ(x,t) solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
- Then $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t) = \{\mathbf{p(x,t)}>0\}$, where **p** a viscosity solution of

$$\boxed{\begin{cases} -\Delta \mathbf{p} = 1 & \text{on} \quad \{\mathbf{p} > 0\} \\ \quad V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}}$$