Kinetic models in information percolation and mean-field games

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"Kinetic theory for the emergence of complex behavior in social and economic systems"
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Vast body of literature dedicated to study of information flows (through networks).

- "Information" = wealth, opinions, spins, disease, productivity, ...

- Finite Markov Information Exchange (FMIE) processes are examples of interacting particle systems on graphs.

D. Aldous, "Interacting Particle Systems as Stochastic Social Dynamics," preprint (submitted to the Bernoulli).

Components of model are

- Agents: $i \in G$

- Strength of relationship (i.e., rate of random matching): $\lambda_{ij} \geq 0$

- Informational state of agent i: $\theta_i \in \Theta$

A mean-field game model of information aggregation

Joint work with M. Sirbu (UT Austin) and I. Gamba (UT Austin)

Background:

- Agents are not particles!

- Each agent chooses a strategy to maximize/minimize their individual utility/cost, given present state.

Optimal control (or stopping)

- Solutions are *Nash equilibria*, which are typically not unique. Of interest to determine which of these are stable and Pareto optimal.

Model setup:

• (Binary) information aggregation—prior types add to give posterior types

$$(\theta_i, \theta_j) \mapsto (\theta_i + \theta_j, \theta_i + \theta_j)$$

Bayesian framework:

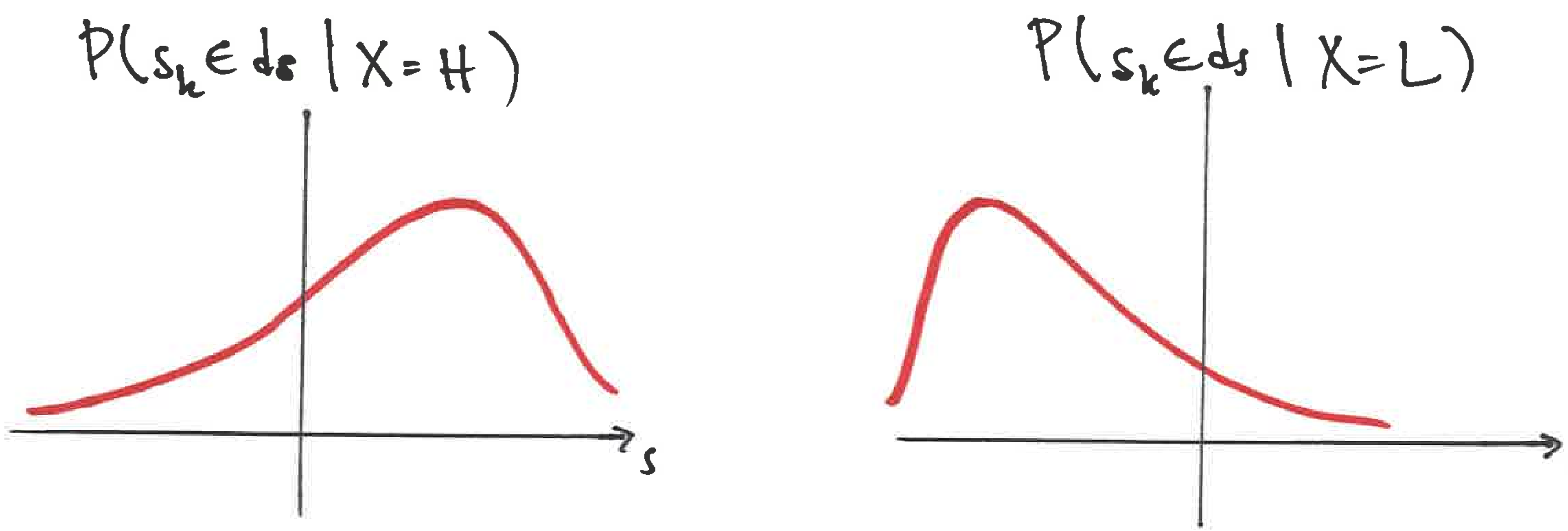
• Consider a good with value given by a binary random variable *X* which is either H (high) or L (low)

• Assume prior probabilities of value are unbiased:

$$\mathbb{P}(X = H) = P(X = L) = 1/2.$$

D. Duffie, G. Giroux, G. Manso, "Information percolation." American Economic Journal: Microeconomics, 2010.

- X unknown, must be estimated by agents based on acquired signals
- Signals $\{s_k\}_{k \in K}$ are conditionally iid r.v.'s given X
- Signals correlated to X (i.e., informative)



- Types = log-likelihoods:

$$\log \frac{\mathbb{P}(X = H|s_1, \dots, s_n)}{\mathbb{P}(X = L|s_1, \dots, s_n)} = \sum_{k=1}^{n} \log \frac{dF^H}{dF^L}(s_k) := \theta(s_1, \dots, s_n)$$

- Types are additive if signal sets are disjoint

- Law of large numbers implies *X* is known a.s. if (countably) infinite number of signals are observed:

$$\lim_{n \to \infty} \theta(s_1, \dots, s_n) = \begin{cases} +\infty & \text{if } X = H \\ -\infty & \text{if } X = L \end{cases} \qquad \mathbb{P}\text{-a.s.}$$

Dynamics:

- (Uncountably) infinite number of agents

- Initial condition: Agents given disjoint subsets $S_i \subset S = \{s_k\}_{k \in K}$ of total signal set

- Randomly matched with others according to Poisson process with common rate λ across agents.

Note: Matched agents have non-intersecting interaction trees up to time of meeting (a.s.). Requires construction of random matching mechanism on appropriate probability space.

- At meetings, agents share all of their acquired signals exactly

Type of agent i at time t: $\theta_i(t) = \theta(\mathcal{F}_i(t))$
Proportion of agents with type in dw at time t: $\mu^X(t, dw)$

• Without further modeling, complete description given by simple aggregation equation

$$\partial_t \mu^X = \lambda \left(\mu^X \star \mu^X - \mu^X \right)$$

• More complicated interactions lead to interesting kinetic equations (derived from Kac-type models), which we do not discuss here.

Information acquisition in an economic/financial context is costly.
What does this imply?

Costly information aggregation:

• Agents participate in information market, in which they have an opportunity to acquire information from others. After leaving market, agents purchase contract based on their estimate of *X*.

• Individuals must balance the cost of acquiring additional information with the cost of having a wrong estimate of *X*.

Cost for agent i:

$$C_i(\tau) = \mathbb{E}\left[\int_0^\tau e^{-\gamma s} c(s) ds + e^{-\gamma\tau} g\left(\theta_i(\tau)\right)\right],$$

Note: Cost function is given by expectation under total probability (i.e., not conditional on X).

- Agents choose their individual stopping times τ in order to minimize their total expected cost $C_i(\tau)$.

- Given *X*, $\theta_i^X(t)$ is a pure-jump (compound Poisson) Markov process with jump size distribution $\mu^X(t, dw)$.

- Generator of process given *X*:

$$\left(\mathcal{L}_\mu^X \eta\right)(t, v) = \lambda \int_{\mathbb{R}} (\eta(v + w) - \eta(v))\mu^X(t, dw)$$

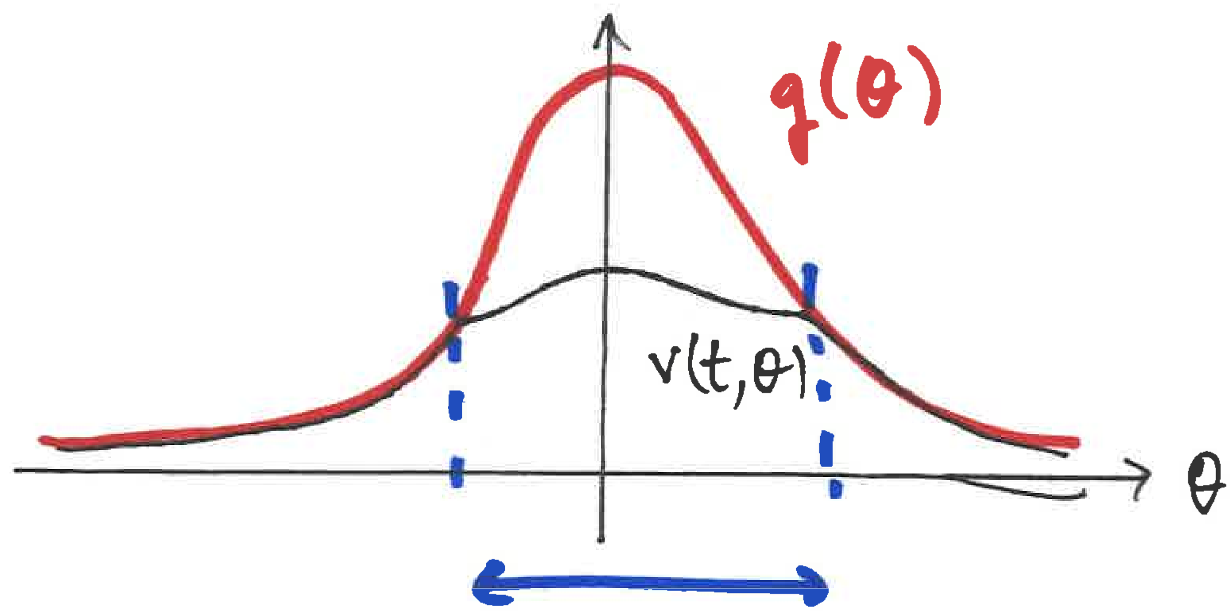
- The unconditional process has generator

$$\bar{\mathcal{L}}_\mu = \frac{1}{2}\mathcal{L}_\mu^H + \frac{1}{2}\mathcal{L}_\mu^L.$$

Obstacle problem determines stopping region:

$$\max\left\{\partial_t v - \bar{\mathcal{L}}_\mu v + \gamma v - c, v - g\right\} = 0$$

where we are solving for the value function $v(t, w)$



$\mathcal{R}_t \doteq$ CONTINUATION REGION $=$ Supp $(v-g)$.

This allow us to find the continuation region $\mathcal{R}_t = \text{supp}(v - g)$
(the part of the state space in which agents remain active).

Forward Kolmogorov equation determines evolution of mean-field:

$$P_i^X(t, dw) = \mathbb{P}(\theta_i^X(t) \in dw | X)$$

$$\partial_t P_i^X = \left(\mathcal{L}_\mu^X\right)^\dagger P_i^X \quad \text{in } \mathcal{R}_t, \qquad \text{supp}\left(P_i^X\right) \subseteq \mathcal{R}_t$$

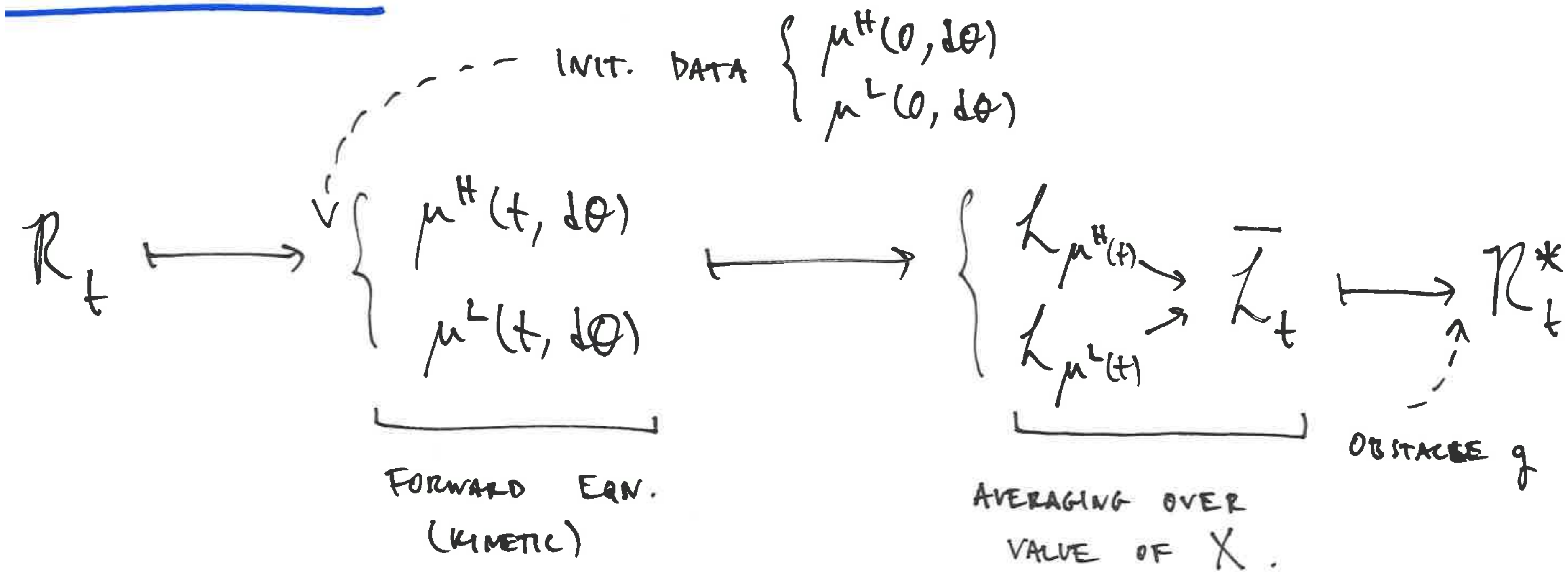
- Law of large numbers (in number of agents) implies

$$\partial_t \mu^X = \left(\mathcal{L}_\mu^X\right)^\dagger \mu^X \quad \text{in } \mathcal{R}_t, \qquad \text{supp}\left(\mu^X\right) \subseteq \mathcal{R}_t$$

- This is an aggregation model on a bounded domain:

$$\partial_t \mu^X = \lambda \left(\mu^X \star \mu^X - \mu^X \left(\mathcal{R}_t\right) \mu^X\right) \quad \text{in } \mathcal{R}_t$$

In summary:



$$\partial_t \mu^X = \lambda \left(\mu^X \star \mu^X - \mu^X\left(\mathcal{R}_t\right)\mu^X\right) \quad \text{in } \mathcal{R}_t$$

$$\max\left\{\partial_t v - \bar{\mathcal{L}}_\mu v + \gamma v - c, v - g\right\} = 0$$

- Nash/MFG equilibria are fixed points of this map!

Related literature:

Optimal control of agents' matching rates

D. Duffie, S. Malamud, G. Manso, "Information percolation with equilibrium search dynamics," Econometrica, 2010.

Role of network geometry

D. Aldous, "When Knowing Early Matters: Gossip, Percolation and Nash Equilibria," ArXiv:1005.4846 (2010).

S. Chatterjee, R. Durrett, "Asymptotic behavior of Aldous' gossip process," The Annals of Applied Probability (2010).

Related literature:

Mean-field game theory

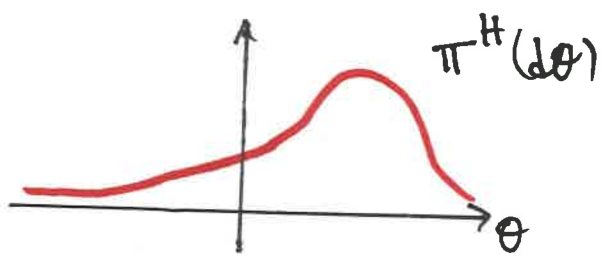
J.M. Lasry, P.L. Lions, O. Guéant, and coauthors (2006-)

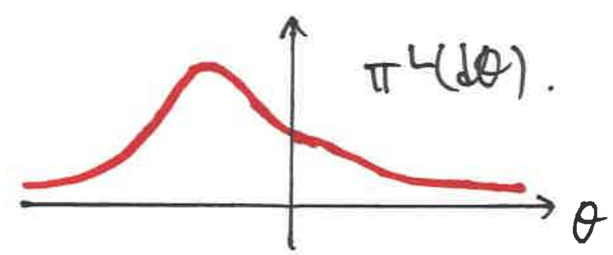
$$-\nu\nabla^2 m - \operatorname{div}\left(\frac{\partial H}{\partial p}(x,\nabla u)m\right) = 0$$

$$-\nu\nabla^2 u + H(x,\nabla u) + \lambda = V(x,m)$$

Stationary problem:

• Agent i replaced with new agent drawn from entrance distribution $\tau_i \sim Exp(\beta)$ after exponentially distributed time (we assume these distributions are symmetric w.r.t. X)





Nash/MFG equilibria: $(\mathcal{R}, \mu^H)$

$$\max\left\{-\bar{\mathcal{L}}_\mu v + (\gamma+\beta)v - (c+\beta g), v-g\right\} = 0 \quad \Longrightarrow \quad \mathcal{R} = \mathrm{supp}(v-g)$$

$$0 = \lambda\left(\mu^H * \mu^H - \mu^H(\mathcal{R})\mu^H\right) + \beta(\pi^H - \mu^H) \quad \text{in } \mathcal{R}, \qquad \mathrm{supp}(\mu^H) \subseteq \mathcal{R}.$$

Stationary problem:

- $(\mathcal{R}, \mu^H) = (\emptyset, 0)$ is a trivial equilibrium (sometimes stable!)

- This means that to find nontrivial Nash/MFG equilibria (particularly the Pareto optimal one) we need to establish more than well-posedness. This is typically difficult, but becomes more tractable if we have some monotonicity in the system.

- Nontrivial equilibria depend on rates, costs, and entrance measure

Can we numerically compute Nash/MFG equilibria:?

• One idea: Derive exit cost by putting quadratic cost for wrong guess for X (in terms of posterior *probabilities*, not posterior *types*!)

16a

• NOTE: COST FUNCTION IS GIVEN BY EXPECTATION UNDER TOTAL PROBABILITY (I.E., NOT CONDITIONAL ON X).

$$\mathbb{E}\left[\left(p_i(\tau) - \mathbb{1}_{\{X=H\}}\right)^2\right]$$

$$= \mathbb{E}\Big[(p_i(\tau) - 0)^2 \underbrace{(1-p_i(\tau))}_{=P(X\neq H \mid \text{INFO. TO TIME } \tau)} + (p_i(\tau)-1)^2 \underbrace{(p_i(\tau))}_{=P(X=H \mid \text{INFO. TO TIME } t)}$$

$$= \mathbb{E}\Big[\underbrace{p_i(\tau)(1-p_i(\tau))}_{=g(\theta_i(\tau))}\Big].$$

$$p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}}.$$

⇝ COST PENALIZES "FENCE-SITTING," NOT WRONG ANSWER GIVEN X.

(14)

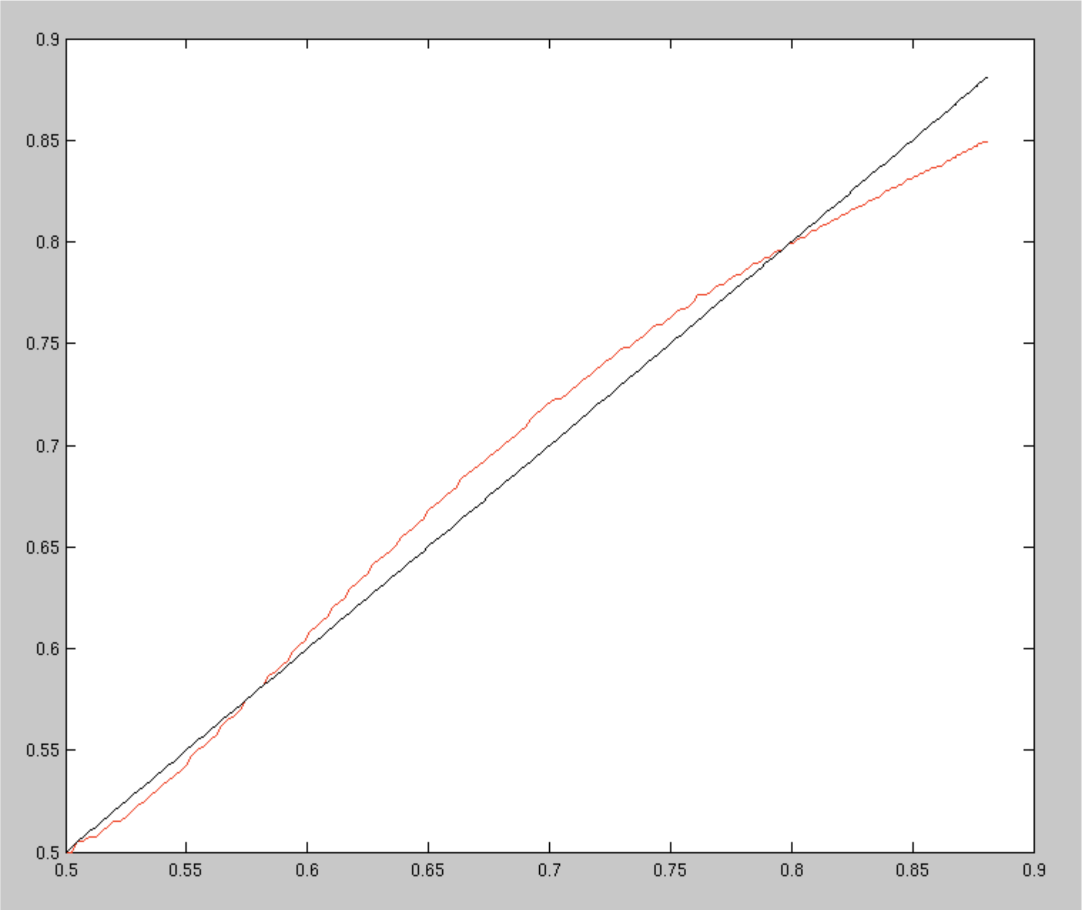
Numerics: Make ansatz $R = (-r, r)$ for some $r \geq 0$.

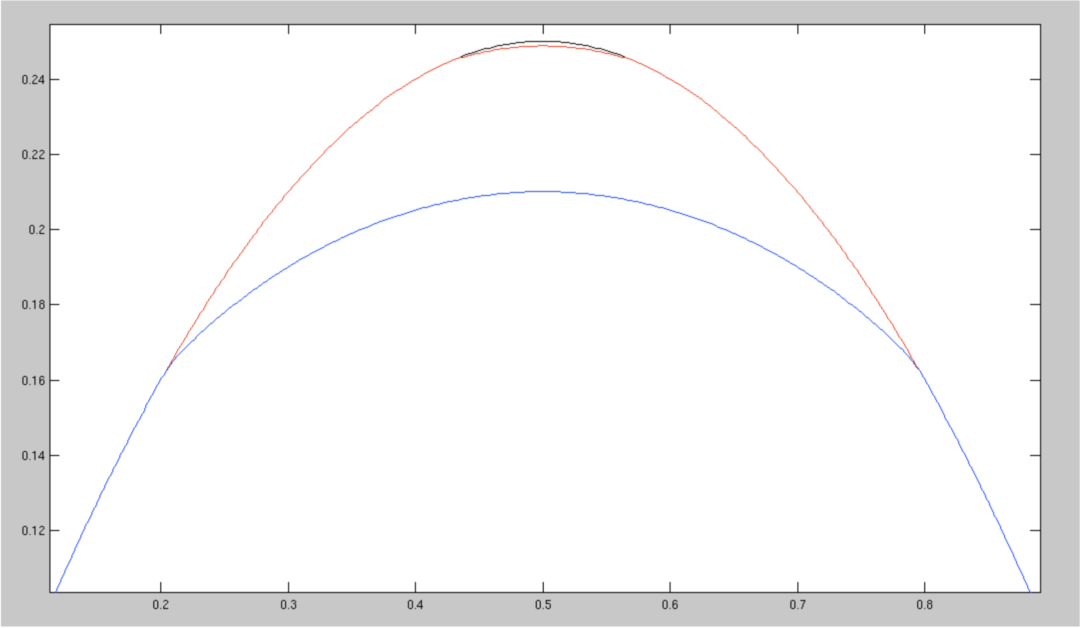
- With $\lambda = 2$, $\beta = 0.05$, $\gamma = 0.05$, $c = 0.0125$,

  $\rightsquigarrow$ 40 meetings / agent on ave.

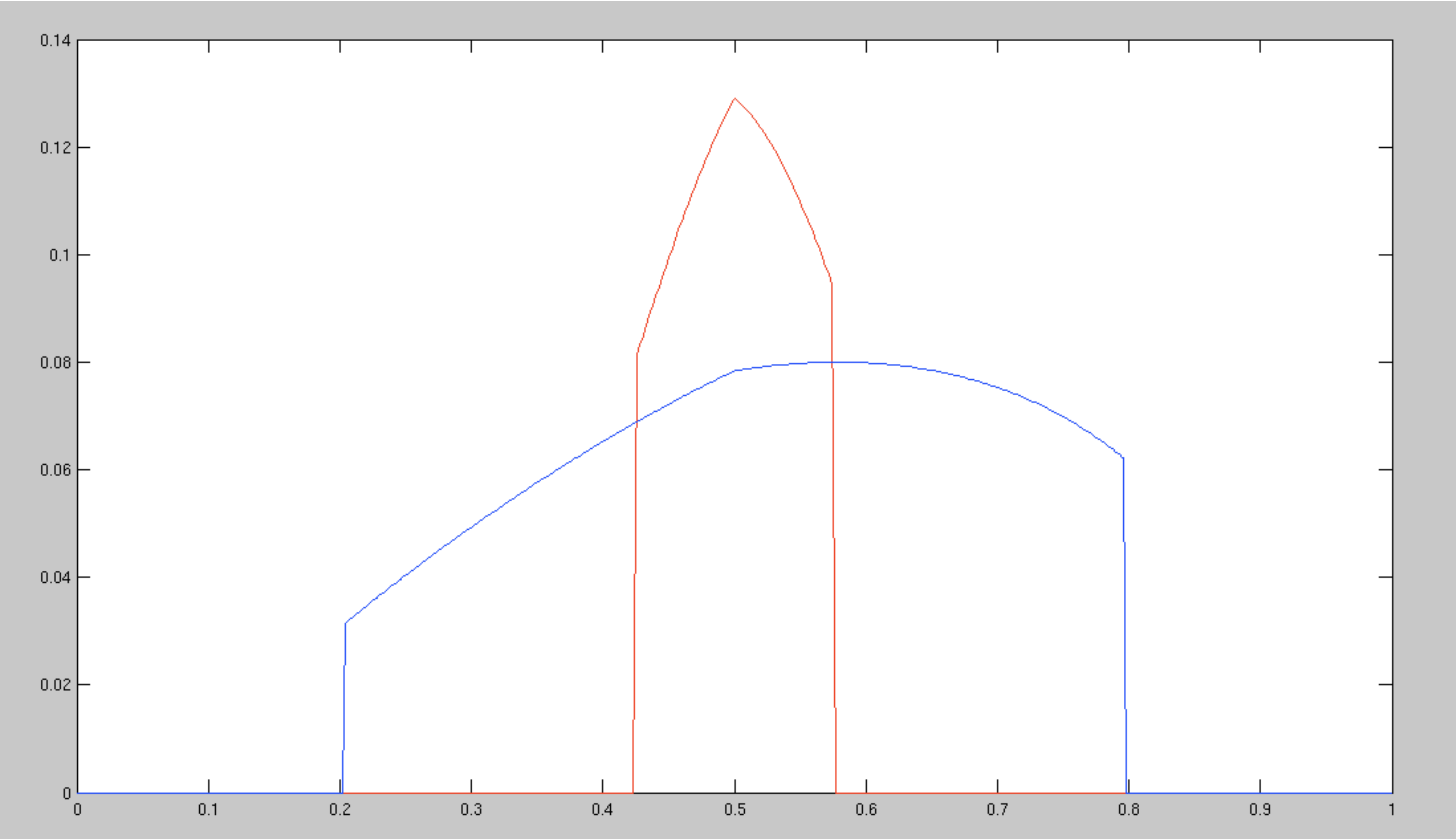
$$\pi^H(d\theta) = \frac{\exp(-(\theta-1)^2/8)\, d\theta}{\int \exp(-(\theta-1)^2/8)\, d\theta},$$

$R \mapsto R^*$ $(r \mapsto r^*)$:





- Largest equilibrium value is Pareto-optimal!


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

FORWARD SCHEME:

$$\begin{cases} 0 = \lambda\left(\mu * \mu - \mu\left(\int_{-r}^{r} \mu\right)\right) + \beta(\pi - \mu) \quad \text{IN } (-r,r) \\ \operatorname{supp} \mu \subset (-r,r). \end{cases}$$

⇝ CONTRACTIVE MAP FOR μ WHEN $\beta > 4\lambda$.

- FOR GENERAL $\beta, \lambda > 0$ WRITE AS $\mu = \Phi(\mu)$,

$$\Phi(\mu) = \frac{1}{\beta + \lambda\left(\int_{-r}^{r} \mu\right)} \left(\lambda(\mu * \mu) + \beta\pi\right) \mathbb{1}_{(-r,r)}$$

- $\begin{cases} \mu^{(n+1)} = \Phi(\mu^{(n)}) \\ \mu^{(0)} = \pi\, \mathbb{1}_{(-r,r)} \end{cases}$ CONVERGES FOR ANY $\beta, \lambda > 0$

(PROOF IN PROGRESS)

OBSTACLE PROBLEM SCHEME:

- RANDOM MATCHING ⇝ PURE-JUMP PROCESS (COMPOUND POISSON) FOR INDIVIDUAL'S TYPE.

  ⇝ AGENTS ONLY STOP IMMEDIATELY AFTER JUMP (UNLESS REPLACED BEFORE THEN)

$\Longrightarrow$ TIME-DISCRETE PROBLEM!

LET

$$T\varphi(\theta) = \mathbb{E}\left[\int_0^{\tau_1} e^{-(\gamma+\beta)s}\left(c + \beta g(\theta_i(s))\right) ds + e^{-(\gamma+\beta)\tau_1}\varphi(\theta_i(\tau_1))\right]$$

WHERE $\tau_1 \sim \mathrm{Exp}(\lambda)$ IS FIRST JUMP TIME OF $\theta_i(t)$.

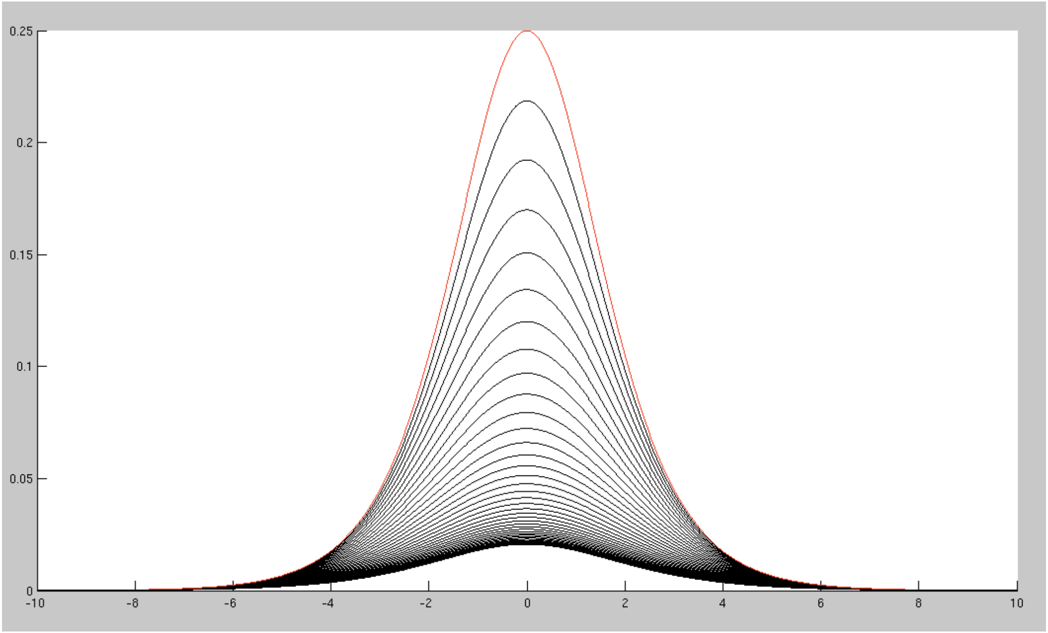
17

$$\max\left\{-\mathcal{L}v+(\gamma+\beta)v-(c+\beta g),\; v-g\right\}=0$$ **OBSTACLE PROBLEM**

$$\Updownarrow$$

$$\min\left\{Tv,\; g\right\}=v$$ **WALD-BELLMAN EQN.**

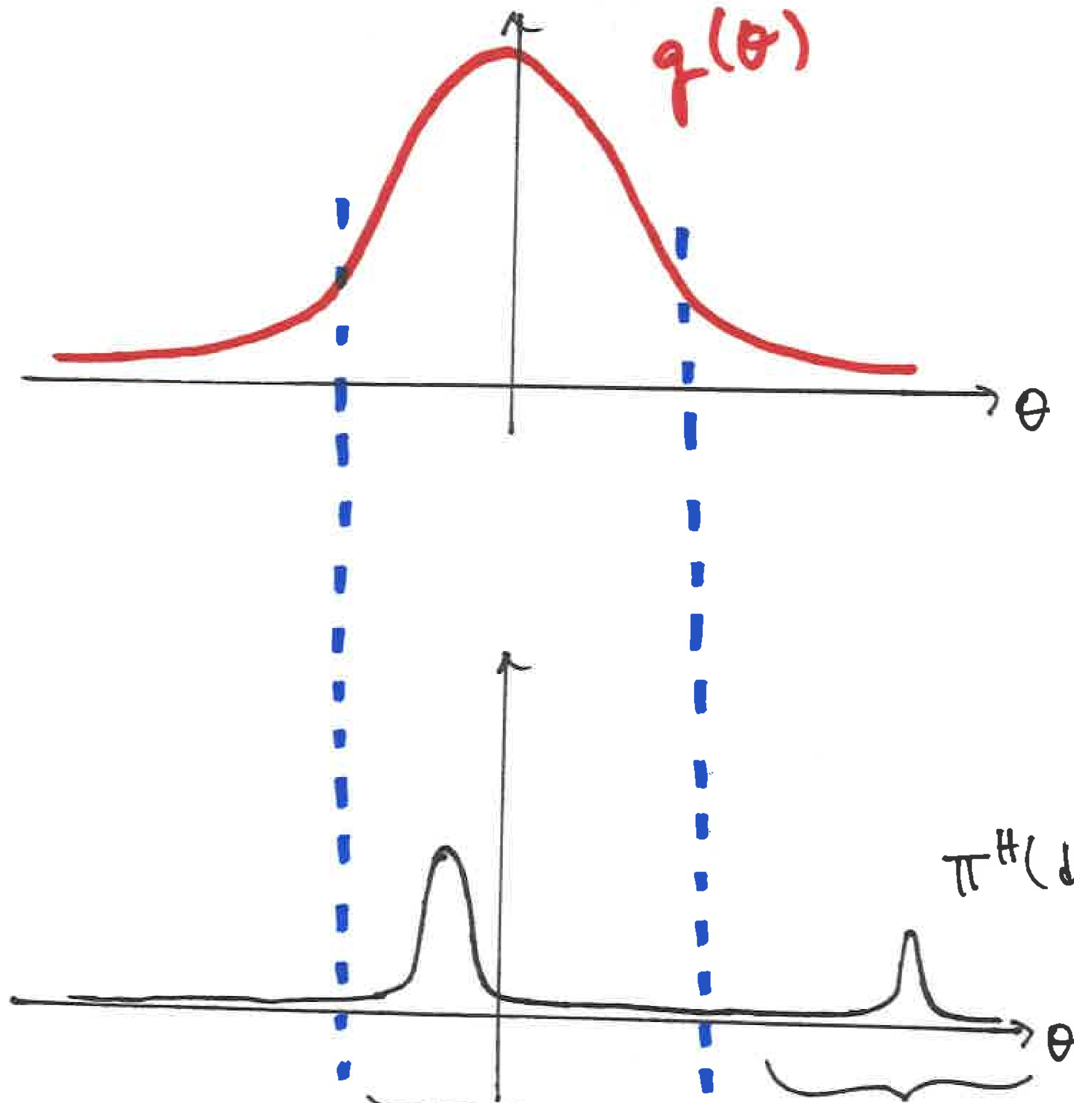
- SCHEME : $\begin{cases} v^{(n+1)} = \min\left\{Tv^{(n)}, g\right\} \\ v^{(0)} = g \end{cases} \Rightarrow v^{(n)} \searrow v$ UNIF.



- SAME PROPERTY THAT LEADS TO KINETIC FORWARD EQN. SIMPLIFIES OPTIMAL STOPPING PROBLEM.

- MODEL ALLOWS FOR RICH CLASS OF EQUILIBRIA

EX. (EDUCATED GET RICHER, POORLY/BADLY EDUCATED GET POORER)



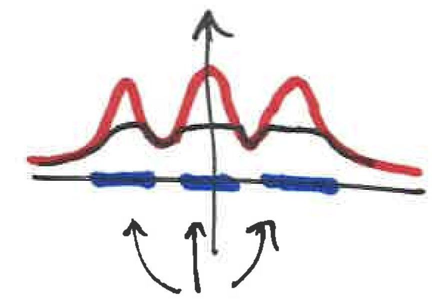
BADLY EDUCATED REMAIN AND LEAVE WITH WRONG INFORMATION.

EDUCATED LEAVE MARKET IMMEDIATELY

WORK IN PROGRESS:

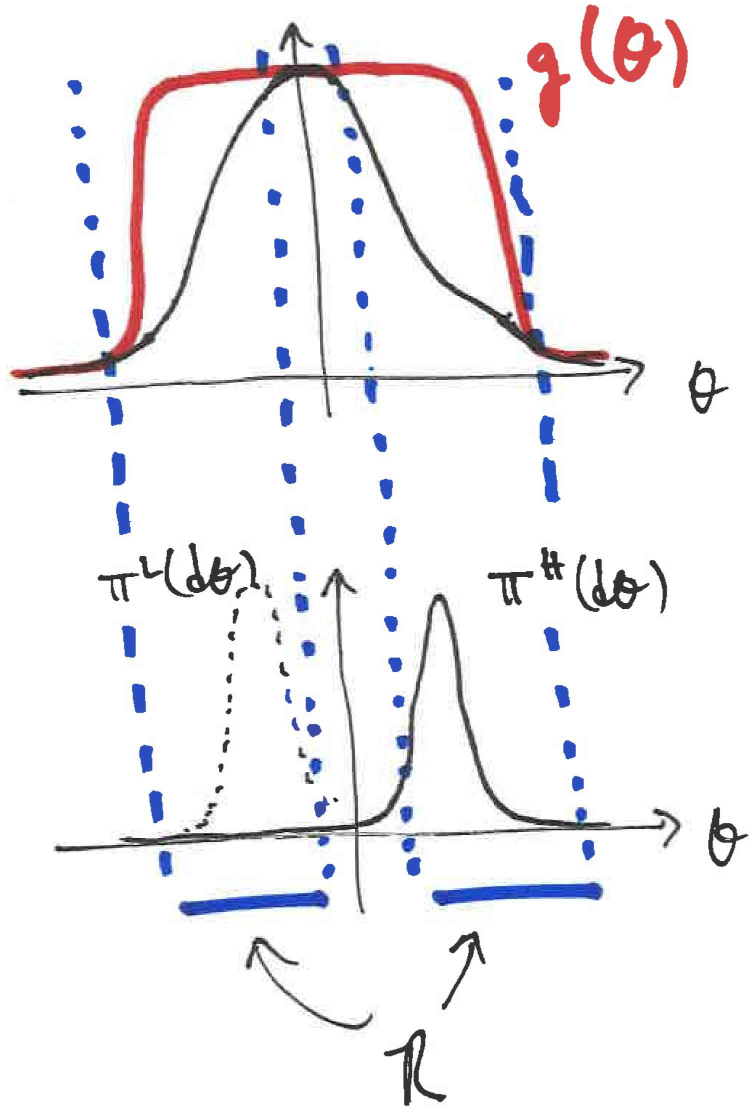
- EXISTENCE THEOREM FOR NASH/MFG EQUILIBRIA.
- HOW "BAD" CAN $\partial \mathcal{R}$ BE?

  ⇝ AS "BAD" AS OBSTACLE g



- FOR g SIMILAR TO ONE ORIGINALLY CONSIDERED (g'' HAS ONLY ONE CHANGE IN SIGN FOR $\theta > 0$), HYPOTHESIZE THAT $\mathcal{R} = (-r, r) \setminus (-r_{inner}, r_{inner})$ FOR SOME $r, r_{inner} \geq 0$.

INTUITION:



(g'' CHANGES SIGN <u>ONCE</u> FOR $\theta > 0$)

- THOSE AGENTS CLOSE TO SUDDEN DROP IN EXIT COST STAY ACTIVE IN MARKET.
- THOSE COMPLETELY UNINFORMED GIVE UP IMMEDIATELY.

A MFG model of economic growth due to innovation diffusion

R. Lucas, B. Moll, "Knowledge growth and the allocation of time," preprint.

Main idea: Should agents devote their time to production or learning (for improved future production)?

- State of the economy completely described by distribution of productivity levels. More easily described in terms of cost level distribution:

There is a constant population of infinitely-lived agents of measure one. We identify each person at each date as a realization of a draw $\tilde{z}$ from a cost distribution, described by its cdf

$$F(z,t) = \Pr\{\tilde{z} \leq z \text{ at date } t\},$$

or equivalently by its density function $f(z,t)$. This function $f(\cdot,t)$ fully describes the state of the economy at t. A person with cost draw $\tilde{z}$ can produce $\tilde{a} = \tilde{z}^{-\theta}$ units of a single consumption good, where $\theta \in (0,1)$.[6]

- Agent allocates fraction of time for goods production, and other time for improving knowledge (by searching for more productive agents):

Every person has one unit of labor per year. He allocates his time between a fraction $1 - s(z,t)$ devoted to goods production and $s(z,t)$ devoted to improving his production-related knowledge. His goods production is

$$[1 - s(z,t)]\, z^{-\theta}. \tag{1}$$

Individual preferences are

$$V(z,t) = \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\tau - t)} \left[1 - s(\tilde{z}(\tau),\tau)\right] \tilde{z}(\tau)^{-\theta} d\tau \middle| \tilde{z}(t) = z \right\}. \tag{2}$$

• Agent learns by meeting another who is more knowledgeable (asymmetric interaction):

We model the evolution of the distribution $f(z,t)$ as a process of individuals meeting others from the same economy, comparing ideas, improving their own productivity. The details of this meeting and learning process are as follows.[7] A person z allocating the fraction $s(z,t)$ to learning observes the cost z' of one other person with probability $\alpha\left[s(z,t)\right]\Delta$ over an interval $(t, t+\Delta)$, where α is a given function. He compares his own cost level z with the cost z' of the person he meets, and leaves the meeting with the best of the two costs, $\min(z,z')$. (These meetings are not assumed to be symmetric: z learns from and perhaps imitates z' but z' does not learn from z and in fact he may not be searching himself at all.)

- Therefore, forward equation is

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t))f(z,t)\int_0^z f(y,t)dy + f(z,t)\int_z^\infty \alpha(s(y,t))f(y,t)dy.$$

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} = -\alpha(s(z,t))\int_0^z f(y,t)dyf(z,t).$$

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}} = f(z,t)\int_z^\infty \alpha(s(y,t))f(y,t)dy.$$

- HJB (backward) equation is

$$\rho V(z,t) = \max_{s\in[0,1]} \left\{(1-s)z^{-\theta} + \frac{\partial V(z,t)}{\partial t} + \alpha(s)\int_0^z [V(y,t) - V(z,t)]f(y,t)dy\right\}.$$

• Self-similar solutions are "balanced growth paths" (states of the economy where total production grows at a constant rate):

$$f(z,t) = e^{\gamma t}\phi(ze^{\gamma t}),$$

$$V(z,t) = e^{\theta\gamma t}v(ze^{\gamma t}),$$

$$s(z,t) = \sigma(ze^{\gamma t})$$

- The "stationary" MFG:

  - Restate (BE), (LM) for BGP only. Use $x = ze^{\gamma t}$

$$(\rho - \theta\gamma)\, v(x) - v'(x)\gamma x = \max_{\sigma \in [0,1]} \left\{ (1-\sigma)x^{-\theta} + \alpha(\sigma) \int_0^x [v(y) - v(x)]\phi(y)\, dy \right\}$$

$$\phi(x)\gamma + \phi'(x)\gamma x = \phi(x) \int_x^\infty \alpha(\sigma(y))\phi(y) dy - \alpha(\sigma(x))\phi(x) \int_0^x \phi(y) dy.$$

  - Growth rate?

$$\phi(0)\gamma = \phi(0) \int_0^\infty \alpha(\sigma(y))\phi(y) dy.$$

$$\gamma = \int_0^\infty \alpha(\sigma(y))\phi(y) dy$$

- No analytical results (all numerical)! Can one prove existence of Nash equilibria and classify Pareto optimal ones?