Activated fluids: continuum description, analysis and computational results

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Continuum Thermodynamics - concept of continuum

- balance equations
  - conservation of mass, energy
  - principles of classical Newtonian mechanics applied to subsets of the body: \( \frac{d}{dt}(m\mathbf{v}) = \mathbf{F} \) with \( \mathbf{v} = \frac{d\mathbf{\chi}}{dt} \)
  - principles of classical thermodynamics applied to subsets of the body assumed to be at local equilibrium
- boundary conditions
- initial conditions

Insufficient to describe mechanical and thermal processes inside the body
Initial and boundary value problems

- balance equations

\[
\begin{align*}
\dot{\rho} &= \rho \text{div } \mathbf{v} \\
\rho \dot{\mathbf{v}} &= \text{div } \mathbf{T} \\
\rho \dot{E} &= \text{div}(\mathbf{T} \mathbf{v} - \mathbf{j}_e) \\
E &= e + \frac{1}{2}|\mathbf{v}|^2
\end{align*}
\]

- the density \( \rho \)
- the velocity \( \mathbf{v} = (v_1, v_2, v_3) \)
- the internal energy \( e \)
- the Cauchy stress tensor \( \mathbf{T} = (T_{11}, T_{12}, T_{13}, T_{22}, T_{23}, T_{33}) \)
- the energy flux \( \mathbf{j}_e = (j_{e1}, j_{e2}, j_{e3}) \)

- boundary conditions
- initial conditions

Insufficient to predict mechanical processes inside the body.

Closure - constitutive (material) equations involving \( \mathbf{T} \) and \( \mathbf{j}_e \).
Section 1

Balance equations and stress power
General form of the balance equations

\[ x = \chi(t, X) \]

Balance equation for \( z \)

\[
\frac{d}{dt} \int_{\mathcal{P}_t} z(t, x) \, dx = \int_{\partial\mathcal{P}_t} j_z(t, x) \cdot n(t, x) \, dS + \int_{\mathcal{P}_t} s_z(t, x) \, dx
\]

Incompressibility:

\[
\frac{d}{dt} \text{Vol}(\mathcal{P}_t) = 0 \quad \iff \quad \frac{d}{dt} \int_{\mathcal{P}_t} dx = 0
\]
General form of the balance equations

For all $P_t \subset \Omega$:

$$\dot{z} := \frac{\partial z}{\partial t} + \mathbf{v} \cdot \nabla z$$

$$\mathbf{v} := \frac{\partial \chi}{\partial t}$$

$$\int_{P_t} \{ \dot{z} + z \ \text{div} \ \mathbf{v} - \text{div} \mathbf{j}_z + s_z \} \ dx = 0$$

$$\dot{z} + z \ \text{div} \ \mathbf{v} - \text{div} \mathbf{j}_z + s_z = 0$$

For mass density $\rho$:

$$\dot{\rho} = -\rho \ \text{div} \ \mathbf{v}$$

For linear momentum $\rho \mathbf{v}$:

$$\frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \ \text{div} \ \mathbf{v} = \rho \mathbf{\dot{v}} = \text{div} \ \mathbf{T}$$

For total energy $\rho E$:

$$\frac{\partial E}{\partial t} + \rho E \ \text{div} \ \mathbf{v} = \rho \dot{E} = \text{div}(\mathbf{T} \mathbf{v} - \mathbf{j}_e)$$

Incompressibility:

$$\text{div} \ \mathbf{v} = 0 \implies \dot{\rho} = 0 \iff \rho(t, \chi(t, X)) = \rho_0(X)$$
Multiplying $\rho \dot{\mathbf{v}} = \text{div} \mathbb{T}$ scalarly by $\mathbf{v}$:

$$\mathcal{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

$$\frac{1}{2} \rho |\mathbf{v}|^2 = \text{div}(\mathbb{T}\mathbf{v}) - \mathbb{T} : \mathcal{D}$$

Subtracting this from $\rho \dot{\mathcal{E}} = \text{div}(\mathbb{T}\mathbf{v} - \mathbf{j}_e)$

$$\rho \dot{\mathbf{e}} = \text{div} \mathbf{j}_e + \mathbb{T} : \mathcal{D}$$

**Stress power - source $s_e$**

$$\mathbb{T} : \mathcal{D} = \mathbb{S} : \mathbb{D}_\delta + m \text{div} \mathbf{v} = \mathbb{S} : \mathbb{D}_\delta \text{ if } \text{div} \mathbf{v} = 0$$

where

$$A_\delta := A - \frac{1}{3}(\text{Tr} A) \mathbb{I}$$

and

$$\mathbb{T} = \mathbb{S} + m \mathbb{I} \quad \mathbb{S} := \mathbb{T}_\delta \text{ and } m := \frac{1}{3} \text{Tr} \mathbb{T}$$
Stress power and the 2nd law of thermodynamics

So far, continuum thermodynamics entered only through the conservation of energy (First law of thermodynamics). For classical compressible fluids the rate of entropy production takes the form (Second law of thermodynamics)

\[
\theta \xi = S : D_\delta + (m + p_{th}) \text{div} \, \mathbf{v} - j_e \cdot \frac{\nabla \theta}{\theta} \quad \text{and} \quad \xi \geq 0
\]  

(1)

Remarks

- \( S : D_\delta + (m + p_{th}) \text{div} \, \mathbf{v} \neq S : D_\delta + m \text{div} \, \mathbf{v} \)
- For incompressible fluids and isothermal processes: \( \xi = S : D \geq 0 \)
- Represents gain/loss for internal/kinetic energy
- A purely mechanical systems (isothermal processes) are merely approximation
- Classification of incompressible fluids based on stress power - towards model with activation (mixing)
- Constitutive theory for \( \mathbb{T} \) and \( j \) stemming from (1) - towards geo-physical models
Section 2

Classification of incompressible fluids
Josiah Williard Gibbs (1839-1903): One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity.
Internal isothermal flows of Incompressible fluids

Incompressible fluids with constant density \( \rho_* \)

\[
\text{div } \mathbf{v} = 0
\]

\[
\rho_* \left( \frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} \right) = \nabla m + \text{div } \mathbf{S} \quad \text{in } (0, T) \times \Omega
\]

Internal flows

\[
\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } (0, T) \times \partial \Omega
\]

Balance of energy

\[
\mathbf{s} := (-\mathbf{S} \mathbf{n})_\tau
\]

\[
\frac{d}{dt} \int_\Omega \rho_* \frac{|\mathbf{v}|^2}{2} \, dx + \int_\Omega \mathbf{S} : \mathbf{D} \, dx + \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}_\tau \, dS = 0
\]
Two dissipative mechanisms

\( \mathbf{S} : \mathbf{D} \) mechanical energy due to friction between layers of the fluid in the bulk and due to further microstructural changes, transformed into heat: growth of the internal energy
- \( \mathbf{D} \) the symmetric part of the velocity gradient
- \( \mathbf{S} \) the traceless part of the Cauchy stress

\( \mathbf{S} \cdot \mathbf{v}_\tau \) mechanical energy due to mutual interaction of the fluid in bulk and the solid that forms the boundary; transformed into the heat: growth of internal energy
- \( \mathbf{v}_\tau \) tangential part of the velocity on \( \partial \Omega \)
- \( \mathbf{s} \) projection of the normal traction to the tangent plane

Requirements

\[
\mathbf{S} : \mathbf{D} \geq 0 \quad \text{and} \quad \mathbf{s} \cdot \mathbf{v}_\tau \geq 0
\]

We formulate the whole cascade of models in bulk (i.e. constitutive equations relating \( \mathbf{S} \) and \( \mathbf{D} \)) and the whole cascade of boundary conditions, for internal flows (i.e. constitutive equations relating \( \mathbf{s} \) and \( \mathbf{v}_\tau \))
A linear relation \( S = 2\nu_* D \) \( \iff \) \( S = \alpha_* D \)

Navier-Stokes fluid

- \( \nu_* > 0 \) is the shear viscosity
- \( \alpha_* > 0 \) is the fluidity \( \alpha_* = \frac{1}{2\nu_*} \)

Two remarkable trivial situations

- \( S = 0 \) \( \iff \) \( T = mI \)
  - Euler fluid

- \( D = 0 \) \( \iff \) \( \mathbf{v}(t, x) = a(t) \times x + b(t) \)
  - rigid body motion
**Figure:** Response of Euler fluid, Navier-Stokes fluid, and rigid body.

Implicit constitutive relations

\[ G(S, D) = 0 \]
Power-law fluids

\[ S = |D|r^{-2}D \quad \iff \quad D = |S|r'-2S \]

Meaningful for \( r > 1 \) \( r' = r/(r - 1) \)

**Figure:** Response of the power-law model for various values of \( r \).
A generalization of power-law fluid

\[ S = 2\nu_* \left( \frac{1}{2} + \frac{1}{2} \frac{|D|^2}{d_*^2} \right)^{\frac{r-2}{2}} D \]

\[ D = \frac{1}{2\nu_*} \left( \frac{1}{2} + \frac{1}{2} \frac{|S|^2}{(2\nu_* d_*)^2} \right)^{\frac{r'-2}{2}} S \]

Diagram with various values of \( r \) and \( r' \) for different orientations.
A generalization of power-law fluid

\[ S = \left( 1 + A (1 + |D|^2)^{\frac{r-2}{2}} \right) D \]

\[ D = \left( 1 + A (1 + |S|^2)^{\frac{r'-2}{2}} \right) S \]

Both models can be simplified by making the response *monotone* (dashed line). Note that only on the left \( S \) is a function of \( D \); on the right, \( D \) is a function of \( S \).
**Figure:** Response of the Bingham fluid, the Navier-Stokes fluid, and activated Euler-Navier-Stokes fluid.

**Bingham fluid**
- mixes rigid body behaviour with fluid behaviour
- a key model of viscoplasticity
- a special issue of IJNonNFM (2015)

**Euler/Navier-Stokes fluid**
- connects behavior of fluids where shear effects are negligible in parts of the fluid domain
- a possible model in boundary layer theory
- superfluids
Activated power-law fluids

\[ \mathbb{D} = 0 \iff |S| \leq \sigma_* \]
\[ \mathbb{D} \neq 0 \iff S = \sigma_* \frac{\mathbb{D}}{|\mathbb{D}|} + 2\nu_g \left( |\mathbb{D}|^2 \right) \mathbb{D} \]

\[ \mathbb{S} = 0 \iff |\mathbb{D}| \leq \delta_* \]
\[ \mathbb{S} \neq 0 \iff \mathbb{D} = \delta_* \frac{\mathbb{S}}{|\mathbb{S}|} + \frac{1}{2\nu_g \left( |\mathbb{D}|^2 \right)} \mathbb{S} \]

\[ \mathbb{D} = \frac{1}{2\nu_g \left( |\mathbb{D}|^2 \right)} \frac{(|S| - \sigma_*)^+}{|S|} \mathbb{S} \]
\[ \mathbb{S} = 2\nu_g \left( |\mathbb{D}|^2 \right) \frac{(|\mathbb{D}| - \delta_*)^+}{|\mathbb{D}|} \mathbb{D} \]
Workshop Viscoplastic fluids: from theory to application (2013) (Xavier Chateau, Antony Wachs)

- The realistic and accurate modeling of viscoplastic and thixotropic materials still remains an unsolved question in the field
- Efforts in designing new numerical approaches with enhanced accuracy and fast convergence have seemed to slow down and the workshop was an occasion to acknowledge that this research should be revived

A novel approach

\[ G(S, D) = \emptyset \]

Continuous curve over the Cartesian product \( \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \) (replaces viewpoints through "multivalued" or "discontinuus" functions, or variational inequalities)
Fluids with limiting shear-stress/shear-rate

\[ S = 2\nu_* \left( 1 + \frac{|\mathcal{D}|^2}{d_*^2} \right)^{-\frac{1}{2}} \mathcal{D} \]

\[ \mathcal{D} = \frac{1}{2\nu_*} \left( 1 + \frac{|S|^2}{d_*^2} \right)^{-\frac{1}{2}} S \]
Fluids with limiting shear-stress/shear-rate

\[ S = 2\nu_* \left(1 + \frac{|D|^a}{d_*^a}\right)^{-\frac{1}{a}} D \]

\[ D = \frac{1}{2\nu_*} \left(1 + \frac{|S|^b}{d_*^b}\right)^{-\frac{1}{b}} S \]
Fluids with limiting shear-stress/shear-rate

\[ S = 2\nu^* \left(1 + \frac{|\mathcal{D}|^a}{d^a_*}\right)^{-\frac{1}{a}} \mathcal{D} \]

\[ \mathcal{D} = \frac{1}{2\nu^*} \left(1 + \frac{|S|^b}{d^b_*}\right)^{-\frac{1}{b}} S \]
<table>
<thead>
<tr>
<th>Euler/limiting shear-rate</th>
<th>limiting shear-rate</th>
<th>rigid body</th>
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</tr>
<tr>
<td>Euler</td>
<td>limiting shear stress</td>
<td>perfect plastic</td>
</tr>
</tbody>
</table>

\[ |\mathcal{D}| \leq \delta_* \iff |\mathcal{S}| = 0 \]
\[ |\mathcal{S}| \leq \sigma_* \iff |\mathcal{D}| = 0 \]

Summary of systematic classification of fluid-like responses with corresponding $|\mathcal{S}|$ vs $|\mathcal{D}|$ diagrams.
“Mixing” two of the above given fluids

\[ S = S_1 + S_2 \]

\[ D = D_1 + D_2 \]
From slip through Navier’s slip to no-slip

\[ s \cdot v_\tau \geq 0 \]

A linear relation \[ s = \gamma_* v_\tau \] \iff \n
\[ v_\tau = \frac{1}{\gamma_*} s \]

Navier’s slip

Two remarkable trivial situations

- \[ s = 0 \] (slip)
- \[ v_\tau = 0 \] (no-slip)

\[ v_\tau = 0 \iff |s| \leq s_* \]

\[ v_\tau \neq 0 \iff s = s_* \frac{v_\tau}{|v_\tau|} + \gamma_* v_\tau \]

stick-slip

\[ s = 0 \iff |v_\tau| \leq v_* \]

\[ s \neq 0 \iff v_\tau = v_* \frac{s}{|s|} + \frac{1}{\gamma_*} v_\tau \]

slip/Navier’s slip

\[ v_\tau = \frac{1}{\gamma_*} \frac{|s| - s_*}{|s|} s \]

\[ s = \gamma_* \frac{|v_\tau| - v_*}{|v_\tau|} v_\tau \]
Summary of systematic classification of boundary conditions with corresponding $|s|$ vs $|v_\tau|$ diagrams.
Section 3

Is the developed framework useful?
**NAVIER-STOKES FLUID** can not describe several phenomena that have been observed and documented experimentally:

- **shear thinning, shear thickening** - $\nu_g$ depends on $|\mathcal{D}|^2$ and/or $|\mathcal{S}|^2$
- **pressure thickening** - $\nu_g$ depends on $p$
- **the presence of activation or deactivation criteria** - “jump" singularities
- **the presence of the normal stress differences at simple shear flows**
- **stress relaxation**
- **non-linear creep**
- **responses of anisotropic fluids**
- **thixotropy**

$G(\mathcal{T}, \mathcal{L}) = 0$ has potential to describe four of them - rich structure.

Models connected with names like Ostwald (1925), de Waele (1923), Carreau (1972), Yasuda (1979), Eyring (1958), Cross (1965), Sisko (1958), Matsuhisa and Bird (1965), Glen (1955), Blatter (1995), Barus (1893), Bingham (1922) etc.

- Ununknowns \((\mathbf{v}, p, S)\):

  \[- \text{div} S = -\nabla p + \mathbf{b} \]

  \[G(S, D) = 0 \]

  \[D(\mathbf{v}) = D \]

  improves convergence for larger \(\tau_*\)

Shear stress and shear rate

\[ V = V_{\text{top}} \hat{e}_z \]

\[ V = -V_{\text{top}} \hat{e}_z \]

\[ \frac{\partial p}{\partial z} \hat{e}_z \]

\[ V = V_{\text{top}} \hat{e}_z \]

\[ \hat{e}_x \]

\[ \hat{e}_y \]

\[ \hat{e}_z \]
Nonmonotone response

Gradient banding

 shear thinning

 $$\bar{T}_{xy}$$

 $$\dot{\gamma}$$

 shear thickening

 $$\bar{T}_{xy}$$

 $$\dot{\gamma}$$

Vorticity banding

 $$\bar{T}_{xy}$$

 $$\dot{\gamma}$$

 $$\bar{T}_{xy}$$

 $$\dot{\gamma}$$

---

Nonmonotone response – gradient and vorticity banding


Nonmonotone response – gradient and vorticity banding

Equilibrium properties and shear banding transitions

Nonmonotone response – gradient and vorticity banding

J. Málek, V. Pruša, G. Tierra: Numerical scheme for simulation of transient flows of non-Newtonian fluids characterized by a non-monotone relation between $\mathcal{D}$ and $S$, in preparation (2016)
Can one describe such non-monotone response of fluid-like materials?

![Graph showing shear stress (σ) versus shear rate (γ) for different particle volume fractions (V)].

Stress-controlled and strain-controlled data

Tris (2-hydroxyethyl) ammonium acetate (TTAA) surfactant dissolved in water with addition of sodium salicylate (NaSal)

\[ \sigma [\text{Pa}] \quad 10^0 \]
\[ 10^{-0.5} \]
\[ 10^{1.2} \quad 10^{1.4} \quad 10^{1.6} \quad 10^{1.8} \]
\[ \dot{\gamma} [\text{s}^{-1}] \]

\( \Delta \) stress-controlled
\( \bullet \) strain-controlled

\[ \dot{\gamma} = (\alpha (1 + \beta (\sigma - \sigma_{\text{yield}})^2)^n + \delta) (\sigma - \sigma_{\text{yield}}) \]

\[ \begin{align*}
\alpha &= 181.2401 \\
\beta &= 0.0019862 \\
n &= -2324.5725 \\
\delta &= 32.4491 \\
\sigma_{\text{yield}} &= 0.22756
\end{align*} \]

\[ \dot{\gamma} = (\alpha + \beta (\sigma - \sigma_{\text{yield}})^2)^n + \delta) (\sigma - \sigma_{\text{yield}}) \]

\[ \begin{align*}
\alpha &= 157.5779 \\
\beta &= 0.40146 \\
n &= -15.515 \\
\delta &= 31.1746 \\
\sigma_{\text{yield}} &= 0.19502
\end{align*} \]


Some fluids exhibit new qualitative phenomena (shear banding, vorticity banding).

Experimental data can be explained by nonmonotone shear stress/shear rate relation.

The framework of implicit constitutive relations seems suitable to described fluids with activation.

A new way to look at the problems from perspective of PDE analysis and numerical simulations.
Activated fluids: continuum description, analysis and computational results

II. A continuum thermodynamic approach

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May 23, 2016, University of Maryland
A Modern Thermodynamic Approach to Constitutive Theory

Classical equilibrium thermodynamics

- \( E = E(S, V) \)
- \( T = \text{def } \frac{\partial E}{\partial S}, \ P = \text{def } - \frac{\partial E}{\partial V} \)
- \( dS \geq \frac{dQ}{T} \) or \( dS = \frac{dQ}{T} \) for reversible processes

Continuum mechanics equilibrium thermodynamics

- \( e = e(\eta, \rho) \)
- \( \theta = \text{def } \frac{\partial e}{\partial \eta}, \ p_{\text{th}} = \text{def } - \frac{\partial e}{\partial \left(\frac{1}{\rho}\right)} = \rho^2 \frac{\partial e}{\partial \rho} \)
- \( \rho \dot{\eta} + \text{div} \left( \frac{j_{\eta}}{\theta} \right) \geq 0 \)

\[
\rho \dot{\xi} = \text{def } \rho \dot{\eta} + \text{div} \left( \frac{j_{\eta}}{\theta} \right) \geq 0 \quad \text{and} \quad \xi \geq 0
\]
Navier–Stokes–Fourier Fluid

- \( e = e(\eta, \rho) \implies \rho \dot{e} = \rho \frac{\partial e}{\partial \eta} \dot{\eta} + \rho \frac{\partial e}{\partial \rho} \dot{\rho} \)

- Use the balance equations

\[
\rho \dot{\eta} + \text{div} (\frac{\mathbf{j}_e}{\theta}) = \frac{1}{\theta} \left[ \mathbb{S} : \mathbb{D}_\delta + (m + p_{th}) \text{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla \theta}{\theta} \right]
\]

\[
\rho \theta \dot{\xi} = \text{def} \left[ \mathbb{S} : \mathbb{D}_\delta + (m + p_{th}) \text{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla \theta}{\theta} \right] > 0
\]
\[
P_\theta \xi = \text{def} \left[ S : D_\delta + (m + p_{\text{th}}) \text{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla \theta}{\theta} \right] > 0
\]

- \[ S = 2\nu D_\delta, \quad \nu > 0 \] (a)
- \[ m + p_{\text{th}} = \tilde{\lambda} \text{div} \mathbf{v}, \quad \tilde{\lambda} > 0 \] (b)
- \[ \mathbf{j}_e = -k \nabla \theta, \quad k > 0 \] (c)

From (a) and (b) follows:

\[
T = S + m I = 2\nu D_\delta + \left( \tilde{\lambda} \text{div} \mathbf{v} - p_{\text{th}} \right) I
\]

\[
= 2\nu D - p_{\text{th}} I + \left( \tilde{\lambda} - \frac{2\nu}{3} \right) (\text{div} \mathbf{v}) I
\]

\[
= -p_{\text{th}} I + 2\nu D + \lambda (\text{div} \mathbf{v}) I
\]

\[
\lambda = \text{def} \quad \tilde{\lambda} - \frac{2\nu}{3} \iff \tilde{\lambda} = \frac{2\nu + 3\lambda}{3}
\]
General Thermodynamic Framework

1. \( e = e(\eta, y_1, \ldots, y_n) \) is increasing function w.r.t. \( \eta \)

2. \( \rho \dot{e} = \rho \frac{\partial e}{\partial \eta} \dot{\eta} + \rho \sum_j \frac{\partial e}{\partial y_j} \dot{y}_j \)

We need to know \( \dot{y}_j \) from balance equations or kinematics

3. \( \theta = \frac{\partial e}{\partial \eta} > 0 \)

4. \( \rho \dot{\eta} + \text{div} \left( \frac{j_n}{\theta} \right) = s_\eta \), where \( s_\eta = \frac{1}{\theta} \sum_\alpha J_\alpha A_\alpha \)

   each \( J_\alpha A_\alpha \) represents independent dissipative mechanism

5. Identify \( s_\eta \) with \( \rho \xi \)

   \[
   \rho \xi = \text{def} \frac{1}{\theta} \sum_\alpha J_\alpha A_\alpha \quad \text{and} \quad \xi \geq 0 \quad (1)
   \]

6. 1. Linear non-equilibrium thermodynamics: \( J_\alpha = \gamma_\alpha A_\alpha, \gamma_\alpha > 0 \)

   2. Non-linear non-equilibrium thermodynamics: specification of constitutive equation for \( \xi \) and its maximization with the constraint (1)

Korteweg–NSF Fluid

- Korteweg (1901)

\[ T = -p\mathbb{I} + 2\nu(\rho)D + \left( \lambda(\rho) \text{div} \mathbf{v} + \alpha(\rho) |\nabla \rho|^2 + \beta(\rho) \Delta \rho \right) \mathbb{I} + \gamma(\rho) \nabla \rho \otimes \nabla \rho \]

- **Q:** Is this model compatible with 2nd law of thermodynamics?
- **Q:** How to extend this model to include thermal processes?

\[ e = e_{\text{NSF}}(\eta, \rho) + \frac{\sigma}{2\rho} |\nabla \rho|^2 \]

\[ \frac{\dot{\nabla}}{\nabla} = -\nabla (\rho \text{div} \mathbf{v}) + [\nabla \mathbf{v}]^\top \nabla \rho \]

\[ \rho \xi = \frac{1}{\theta} \left[ (\mathbb{T}_\delta - \sigma (\nabla \rho \otimes \nabla \rho)_\delta) : \mathbb{D}_\delta \right. \]

\[ + \left( m + p_{\text{th}}^K - \frac{\sigma}{3} |\nabla \rho|^2 - \sigma \rho \Delta \rho + (1 - \delta) \sigma \rho (\nabla \rho) \cdot \frac{\nabla \theta}{\theta} \right) \text{div} \mathbf{v} \]

\[ - (\mathbf{j}_e - \delta \sigma \rho (\text{div} \mathbf{v}) \nabla \rho) \cdot \frac{\nabla \theta}{\theta} \]

\[ p_{\text{th}}^K = p_{\text{th}}^{\text{NSF}} - \frac{\sigma}{2} |\nabla \rho|^2, \quad \delta \in [0, 1] \]
Rate Type Fluid Models

- Popular class of phenomenological models in visco-elasticity
- Broad applications of visco-elasticity:
  - Bio-mechanics (soft tissues, bio-fluids)
  - Polymer industry, glass technology
  - Food industry
  - Geo-mechanics (Earth’s mantle, tectonic plates, glacier, soil)
- Derivation of complete 3D models that are consistent with the second law of thermodynamics is very recent


Example of a Visco-Elastic Material

Asphalt binders

- Widely used
- Microstructure and chemistry are not well understood $\implies$ macroscopic description is the only possible choice
- An example of a complex material with complicated microstructure exhibiting—with clear evidence—visco-elastic phenomena (stress relaxation, non-linear creep, normal stress differences) $\implies$ their response cannot be described by standard models
- Good access to available experimental data

Asphalt binder

- Glue in the asphalt concrete (very sticky)
- Almost incompressible (compared to asphalt concrete)
- Mixture of a large number of hydrocarbons
- Exhibits viscoelastic behavior
## Solid- and Fluid-Like Materials

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<tr>
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<td>5th drop</td>
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<td>2000</td>
<td>8th drop</td>
</tr>
<tr>
<td>2014</td>
<td>9th drop</td>
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</table>
Incompressible Rate-Type Fluid Models

• Balance equations for compressible fluids

\[ \dot{\rho} = -\rho \text{div} \mathbf{v} \]
\[ \rho \mathbf{\dot{v}} = \text{div} \mathbf{T}, \quad \mathbf{T} = \mathbf{T}^\top \]

• Balance equations for incompressible fluids

\[ \text{div} \mathbf{v} = 0 \]
\[ \rho_\star \mathbf{\dot{v}} = \text{div} \mathbf{T}_\delta + \nabla \mathbf{m}, \quad \mathbf{T}_\delta = \mathbf{T}_\delta^\top \]

• **Goal**: To find an additional evolution equation for a part of the stress
Standard Viscoelastic Rate-Type Fluid Models

Cauchy stress $\mathbf{T} = -p \mathbb{I} + \mathbf{S}$, 

$\mathbf{S} = \text{def} \frac{d\mathbf{S}}{dt} - L\mathbf{S} - S\mathbf{L}^T$, \quad $L = \text{def} \nabla \mathbf{v}$

- **Maxwell (1867)**

  $\mathbf{S} + \lambda \mathbf{S} = 2\mu \mathbf{D}$

- **Oldroyd-B (1950)**

  $\mathbf{S} + \lambda \mathbf{S} = 2\eta_1 \mathbf{D} + 2\eta_2 \mathbf{D}$

- **Burgers (1939)**

  $\mathbf{S} + \lambda_1 \mathbf{S} + \lambda_2 \mathbf{S} = 2\eta_1 \mathbf{D} + 2\eta_2 \mathbf{D}$

- **Giesekus (1982)**

  $\mathbf{S} + \lambda_1 \mathbf{S} - \frac{\alpha \lambda_2}{\mu} \mathbf{S}^2 = -2\mu \mathbf{D}$

- **Models due to Phan-Thien–Tanner (1977), Johnson–Segelman (1977), White–Metzer (1977), etc.**
Basic Questions

• **Q:** Are these models compatible with 2nd law of thermodynamics?

• **Q:** How to extend these models to include thermal processes?
• Deformation gradient \( \mathbf{F} = \text{def} \mathbf{F}_{\kappa R} \) is split into the elastic and the dissipative part: \( \mathbf{F}_{\kappa p(t)} \) and \( \mathbf{G} \)

\[
\mathbf{F}_{\kappa R} \rightarrow \mathbf{F}_{\kappa R} \\
X_{\kappa R} \rightarrow X_{\kappa R} \\
\kappa_R(B) \rightarrow \kappa_t(B) \\
X_{\kappa p(t)} \rightarrow X_{\kappa p(t)} \\
\kappa_p(t)(B) \\
\mathbf{G} \\
\mathbf{F}_{\kappa p(t)} \rightarrow \mathbf{F}_{\kappa p(t)}
\]

• \( \mathbf{F} = \mathbf{F}_{\kappa p(t)} \mathbf{G} \)
• Left and right Cauchy–Green tensors:

\[ B_{\kappa_p(t)} = \text{def} \ F_{\kappa_p(t)} F_{\kappa_p(t)}^T, \quad C_{\kappa_p(t)} = \text{def} \ F_{\kappa_p(t)}^T F_{\kappa_p(t)} \]

• Right and left Green–Lagrange strain tensors:

\[ G = \dot{\kappa}_R(X) \]

\[ L_{\kappa_p(t)} = \dot{G} G^{-1}, \quad D_{\kappa_p(t)} = \text{def} \ \frac{1}{2} \left( L_{\kappa_p(t)} + L_{\kappa_p(t)}^T \right) \]

\[ \begin{aligned}
\dot{B}_{\kappa_p(t)} & = LB_{\kappa_p(t)} + B_{\kappa_p(t)} L^T - 2F_{\kappa_p(t)} D_{\kappa_p(t)} F_{\kappa_p(t)}^T \\
\corner B_{\kappa_p(t)} & = -2F_{\kappa_p(t)} D_{\kappa_p(t)} F_{\kappa_p(t)}^T
\end{aligned} \]
Constitutive equations

- **Internal energy** $e$ for compressible neo-Hookean solid

\[
e = e_{NSF}(\eta, \rho) + \frac{\mu}{2\rho} \left( \text{Tr} \mathcal{B}_{\kappa_p(t)} - 3 - \log \det \mathcal{B}_{\kappa_p(t)} \right)
\]

\[
\rho \dot{\xi} = \frac{1}{\theta} \left[ \left( \mathbb{T} - \mu \mathcal{B}_{\kappa_p(t)} \right) \delta : \mathbb{D} \delta + \mu \left( \mathbb{C}_{\kappa_p(t)} - \mathbb{I} \right) : \mathbb{D} \kappa_p(t) - \mathbf{j}_e \cdot \nabla \theta \right.
\]

\[
+ \left( m + p_{th}^M - \mu \left( \frac{1}{3} \text{Tr} \mathcal{B}_{\kappa_p(t)} - 1 \right) \right) \text{div } \mathbf{v}
\]

\[
p_{th}^M \overset{\text{def}}{=} p_{th}^{NSF} - \frac{\mu}{2} \left( \text{Tr} \mathcal{B}_{\kappa_p(t)} - 3 - \log \det \mathcal{B}_{\kappa_p(t)} \right)
\]

- **Linearity**

\[
\left( \mathbb{T} - \mu \mathcal{B}_{\kappa_p(t)} \right) \delta = 2\nu \mathbb{D} \delta, \quad \nu > 0
\]

\[
m + p_{th}^M - \mu \left( \frac{1}{3} \text{Tr} \mathcal{B}_{\kappa_p(t)} - 1 \right) = \frac{2\nu + 3\lambda}{3} \text{div } \mathbf{v}, \quad 2\nu + 3\lambda > 0
\]

\[
\mu \left( \mathbb{C}_{\kappa_p(t)} - \mathbb{I} \right) = 2\nu_1 \mathbb{D}_{\kappa_p(t)}, \quad \nu_1 > 0
\]

\[\mathbf{j}_e = -k \nabla \theta, \quad k > 0\]
Compressible Giesekus Fluid

1  \[
(T - \mu \mathbb{B}_{\kappa_p(t)})_\delta = 2\nu \mathbb{D}_\delta \\
m + p_{\text{th}}^M - \mu \left( \frac{1}{3} \text{Tr} \mathbb{B}_{\kappa_p(t)} - 1 \right) = \frac{2\nu + 3\lambda}{3} \text{div} \mathbf{v}
\]

imply

\[
T = T_\delta + m\mathbb{I} = -p_{\text{th}}^M\mathbb{I} + 2\nu \mathbb{D} + \lambda (\text{div} \mathbf{v}) \mathbb{I} + \mu \left( \mathbb{B}_{\kappa_p(t)} - \mathbb{I} \right)
\]

2  \[
\mu \left( \mathbb{C}_{\kappa_p(t)} - \mathbb{I} \right) = 2\nu_1 \mathbb{D}_{\kappa_p(t)} \quad \text{and} \quad \mathbb{B}_{\kappa_p(t)} = -2\mathbb{F}_{\kappa_p(t)} \mathbb{D}_{\kappa_p(t)} \mathbb{F}_{\kappa_p(t)}^T
\]

imply

\[
\mu \mathbb{B}_{\kappa_p(t)}^2 - \mu \mathbb{B}_{\kappa_p(t)} = \nu_1 \mathbb{B}_{\kappa_p(t)}
\]
**Incompressible Giesekus Fluid**

- **Internal energy** \( e \) for compressible neo-Hookean solid

\[
e = e_{NSF}(\eta, \rho) + \frac{\mu}{2\rho} \left( \text{Tr} \mathbb{B}_\kappa(t) - 3 - \log \det \mathbb{B}_\kappa(t) \right)
\]

\[
\rho \xi = \frac{1}{\theta} \left[ (T - \mu \mathbb{B}_\kappa(t))_\delta : \mathbb{D} \delta + \mu (C_\kappa(t) - \mathbb{I}) : \mathbb{D}_\kappa(t) - j_e \cdot \frac{\nabla \theta}{\theta} \right]
\]

- **Linearity** implies

\[
T = T_\delta + m \mathbb{I} = m \mathbb{I} + 2\nu \mathbb{D} + \mu (\mathbb{B}_\kappa(t) - \mathbb{I})
\]

- \( \mu (C_\kappa(t) - \mathbb{I}) = 2\nu_1 \mathbb{D}_\kappa(t) \) and \( \mathbb{B}_\kappa(t) = -2F_{\kappa(t)} \mathbb{D}_\kappa(t) F^T_{\kappa(t)} \)

imply

\[
\mu \mathbb{B}^2_{\kappa(t)} - \mu \mathbb{B}_\kappa(t) = \nu_1 \mathbb{B}^\nabla_{\kappa(t)}
\]
Compressible Maxwell and Oldroyd-B Fluid

- **Compressible Maxwell fluid**
  - Internal energy $e$

  $$
e = e_{NSF}(\eta, \rho) + \frac{\mu}{2\rho} \left( \text{Tr} B_{\kappa_p(t)} - 3 - \log \det B_{\kappa_p(t)} \right)$$

  - Rate of entropy production $\xi$:

  $$\xi = 2\mu_1 \mathbb{D}_{\kappa_p(t)} : \mathbb{C}_{\kappa_p(t)} \mathbb{D}_{\kappa_p(t)} \geq 0$$

- **Compressible Oldroyd-B fluid**
  - Internal energy $e$

  $$
e = e_{NSF}(\eta, \rho) + \frac{\mu}{2\rho} \left( \text{Tr} B_{\kappa_p(t)} - 3 - \log \det B_{\kappa_p(t)} \right)$$

  - Rate of entropy production $\xi$:

  $$\xi = 2\mu_1 \mathbb{D}_{\kappa_p(t)} : \mathbb{C}_{\kappa_p(t)} \mathbb{D}_{\kappa_p(t)} + 2\mu_2 \mathbb{D} : \mathbb{D} \geq 0$$

Josef Málek
Continuum thermodynamic approach
\[ S + \lambda_1 \nabla S + \lambda_2 \nabla \nabla S = 2\eta_1 \mathcal{D} + 2\eta_2 \mathcal{D} \]
Summary

• An important step towards analysis of initial and boundary value problems (a priori estimates) – specifying the object for relevant computer simulations

• Material coefficients may, in general, depend on state variables

• Compressible and incompressible Navier–Stokes–Fourier (NSF) fluids, Korteweg NSF fluids, Rate type fluids


Summary

- **Cahn–Hilliard NSF fluids**
  

- **Allen–Cahn NSF fluids**


- **Binary mixtures with and without chemical reactions**

Activated fluids: continuum description, analysis and computational results

Josef Málek

Nečas Center for Mathematical Modeling
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Faculty of Mathematics and Physics

May 24, 2016

ERC-CZ project LL1202 - MORE
Implicitly constituted material models: from theory through model reduction to efficient numerical methods
http://more.karlin.mff.cuni.cz/
1. Incompressible fluids and boundary conditions with activation

2. Structure of implicit relations

3. Weak stability of Problem

4. Bingham fluids with threshold slip - existence of unsteady flows for large data

5. Implicitly constituted fluids described by maximal monotone $\psi$-graph - existence of unsteady flows subject to Navier’s slip for large data
Part #1

Incompressible fluids and boundary conditions with activation
### Summary of systematic classification of fluid-like responses

with corresponding $|S|$ vs $|D|$ diagrams.

<table>
<thead>
<tr>
<th>Euler/limiting shear-rate</th>
<th>limiting shear-rate</th>
<th>rigid body</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler/shear-thickening</td>
<td>shear-thickening</td>
<td>rigid/shear-thickening</td>
</tr>
<tr>
<td>Euler/Navier-Stokes</td>
<td>Navier-Stokes</td>
<td>Bingham = rigid/Navier-Stokes</td>
</tr>
<tr>
<td>Euler/shear-thinning</td>
<td>shear-thinning</td>
<td>rigid/shear-thinning</td>
</tr>
<tr>
<td>Euler</td>
<td>limiting shear stress</td>
<td>perfect plastic</td>
</tr>
</tbody>
</table>

$|D| \leq \delta_\ast \iff S = \emptyset$

$|S| \leq \sigma_\ast \iff D = \emptyset$
| Incompressible fluids and boundary conditions with activation |

| Summary of systematic classification of boundary conditions with corresponding $|s|$ vs $|v_\tau|$ diagrams. |

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>slip/Navier’s slip</td>
<td>$v_\tau \leq \delta_*$ $\iff$ $s = 0$ no activation</td>
</tr>
<tr>
<td>Navier’s slip</td>
<td>$v_\tau = 0$</td>
</tr>
<tr>
<td>stick-slip</td>
<td>$s \leq \sigma_*$ $\iff$ $v_\tau = 0$</td>
</tr>
<tr>
<td>slip</td>
<td>$s = 0$</td>
</tr>
</tbody>
</table>

---
Formulation of the problem

PROBLEM

\[
\begin{align*}
\text{div } \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \text{div}(\mathbf{v} \otimes \mathbf{v}) - \text{div} S &= -\nabla p + \mathbf{b} \\
S(\mathbf{S}, \mathbf{D}) &= 0 \\
\mathbf{v} \cdot \mathbf{n} &= 0 \\
S := -\left(\mathbf{S} \cdot \mathbf{n}\right)_\tau & \quad g(\mathbf{s}, \mathbf{v}_\tau) = 0 \\
\mathbf{v}(0, \cdot) &= \mathbf{v}_0
\end{align*}
\]

\[
\left\{ \begin{array}{l}
\text{in } Q_T \\
\text{on } \Sigma_T \\
\text{in } \Omega
\end{array} \right.
\]

DATA

- $\Omega \subset \mathbb{R}^d$ bounded, open set with $\partial \Omega \in C^{1,1}$ and $\mathbf{n} : \partial \Omega \to \mathbb{R}^d$
- $T > 0$ and $Q_T := (0, T) \times \Omega$, $\Sigma_T := (0, T) \times \partial \Omega$
- $\mathbf{v}_0, \mathbf{b}$
- $\mathcal{G}$ and $g$ - constitutive functions in the bulk and on the boundary
Main questions addressed

**UNKNOWN** triplet \((v, p, S)\) defined on \(Q_T\) and \(s\) defined on \(\Sigma_T\)

\[
\begin{align*}
\text{div} \, v &= 0 \\
\partial_t v + \text{div}(v \otimes v) - \text{div} S &= -\nabla p + b \\
G(S, D) &= 0 \\
v \cdot n &= 0 \\
g(s, v_\tau) &= 0 \\
v(0, \cdot) &= v_0
\end{align*}
\]

in \(Q_T\) on \(\Sigma_T\) in \(\Omega\)

**AIM**

- To establish large data existence of solution for any set of data \((\Omega, T, v_0, b)\) and for robust class of constitutive equations described by \(G\) and \(g\)
- To develop a theory with \(p \in L^1(Q_T)\) - important
  - heat-conducting incompressible fluids (M. Bulíček, E. Feireisl - G. Schimperna)
  - one/two equation turbulence model (M. Bulíček, R. Lewandowski)
  - incompressible fluids with pressure and shear-rate dependent viscosity (J. Nečas, KR Rajagopal, M. Bulíček, M. Majdoub, A. Hirn, J. Stebel, M. Lanzendörfer, ...)
  - corresponding numerical methods and their analysis
**Theoretical results**

- Existence of WS to NSEs in 2d and 3d (Leray (1929-1934), Oseen (1922))

- Existence of WS to NSEs in bounded domains, its 2d uniqueness and 3d conditional uniqueness and existence (Hopf (1952), Kiselev & Ladyzhenskaya (1959), Prodi (1959), Serrin (1963))

- Existence of WS to $S = 2(\nu_0 + \nu_1|\nabla|^r-2)|\nabla|$ for $r \geq \frac{11}{5}$ and its uniqueness if $r \geq \frac{5}{2}$ (Ladyzhenskaya (1967-1972), J.-L. Lions (1969))
  - Nečas, Bellout, Bloom, Málek, Růžička (1993-2000): $r \geq \frac{9}{5}$
  - Diening, Růžička, Wolf (2010), Breit, Diening, Schwarzacher (2015): $r > \frac{6}{5}$
  - Bulíček, Ettwein, Kaplický, Pražák (2010): uniqueness for $r > \frac{11}{5}$

- Existence of WS to monotone (rather than strictly monotone) response, Orlicz function-type response (Bulíček, Gwiazda, Málek, Świerczewska-Gwiazda (2012): $r > \frac{6}{5}$)

- Existence of WS to activated fluids with activated boundary conditions (Bulíček, Málek (2016): $r > \frac{6}{5}$)
Part #2

Structure of implicit relations
Basic information

A PRIORI ESTIMATES

Multiplying the 2nd Eq. by \( \mathbf{v} \) \((b) \equiv 0\)

\[
\frac{1}{2} \frac{\partial |\mathbf{v}|^2}{\partial t} + \text{div}(\frac{1}{2} |\mathbf{v}|^2 \mathbf{v}) - \text{div}(\mathbf{S}\mathbf{v}) + \mathbf{S} \cdot \mathbb{D} = - \text{div}(\rho \mathbf{v})
\]

Since \( \mathbf{v} \cdot \mathbf{n} = 0 \), integrating it over \( \Omega \) leads to

\[
\frac{1}{2} \frac{d}{dt} \|\mathbf{v}\|^2_2 + \int_{\Omega} \mathbf{S} : \mathbb{D} \, dx + \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}_\tau \, dS = 0
\]

For the power-law fluids

\[
\mathbf{S} = |\mathbb{D}|^{r-2} \mathbb{D} \iff \mathbb{D} = |\mathbf{S}|^{r'-2} \mathbf{S} \quad r' = r/(r - 1)
\]

\[
\mathbf{S} : \mathbb{D} = \left( \frac{1}{r} + \frac{1}{r'} \right) \mathbf{S} : \mathbb{D} = \frac{1}{r} |\mathbb{D}|^r + \frac{1}{r'} |\mathbf{S}|^{r'}
\]

For Navier’s slip

\[
\mathbf{S} = \gamma^*_r \mathbf{v}_\tau \iff \mathbf{v}_\tau = \frac{1}{\gamma^*_r} \mathbf{S}
\]

\[
\mathbf{S} \cdot \mathbf{v}_\tau = (\frac{1}{2} + \frac{1}{2}) \mathbf{S} \cdot \mathbf{v}_\tau = \frac{\gamma^*_r}{2} |\mathbf{v}_\tau|^2 + \frac{1}{2\gamma^*_r} |\mathbf{S}|^2
\]
Implicit constitutive equations in bulk - maximal monotone $r$-graph setting

Define

$$\begin{align*}
(S, D) \in A & \iff G(S, D) = 0
\end{align*}$$

Assumptions - $A$ is a maximal monotone $r$-graph, $r \in (1, +\infty)$

(A1) $(\emptyset, \emptyset) \in A$

(A2) **Monotone graph:** For any $(S_1, D_1), (S_2, D_2) \in A$

$$\quad (S_1 - S_2) \cdot (D_1 - D_2) \geq 0$$

(A3) **Maximal monotone graph:** Let $(S_*, D_*) \in \mathbb{R}^{d \times d}_{\text{sym}} \times \mathbb{R}^{d \times d}_{\text{sym}}$.

If $(S_* - S) \cdot (D_* - D) \geq 0 \quad \forall (S, D) \in A$ then $(S_*, D_*) \in A$

(A4) **$r$-graph:** There are $\alpha_* > 0$ and $c_* \geq 0$ so that for any $(S, D) \in A$

$$S \cdot D \geq \alpha_* \left(|D|' + |S|'\right) - c_*$$
Implicit formulation of BCs - maximal monotone $q$-graph setting

Define

\[(s, v_\tau) \in \mathcal{B} \iff g(s, v_\tau) = 0\]

**B1** $\mathcal{B}$ contains the origin. $(0, 0) \in \mathcal{B}$.

**B2** $\mathcal{B}$ is a monotone graph.

\[
(s_1 - s_2) \cdot (v_\tau^1 - v_\tau^2) \geq 0 \quad \text{for all } (s_1, v_\tau^1), (s_2, v_\tau^2) \in \mathcal{B}.
\]

**B3** $\mathcal{B}$ is a maximal monotone graph. Let for some $(s, u)$ holds:

\[
(s - \bar{s}) \cdot (v_\tau - u) \geq 0 \quad \text{for all } (s, v_\tau) \in \mathcal{B} \quad \text{then } (s, u) \in \mathcal{B}.
\]

**B4** $\mathcal{B}$ is a $q$-graph. For any $q \in (1, \infty)$ fixed there are $\beta_* > 0$ and $d_* \geq 0$ such that

\[
s \cdot v_\tau \geq \beta_*(|v_\tau|^q + |s|^{q/(q-1)}) - d_* \quad \text{for all } (s, v_\tau) \in \mathcal{B}.
\]

- No-slip boundary condition is excluded by **B4**
- For all our examples $q = 2$
Basic estimates

A PRIORI ESTIMATES REVISITED

Recall

\[ \frac{1}{2} \frac{d}{dt} \| \mathbf{v} \|^2 + \int_\Omega \mathbf{S} : \mathbb{D} \, dx + \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}_\tau \, dS = 0 \]

Using (A4) and (B4) and integrating the result from 0 to any \( t \in (0, T] \):

\[ \frac{1}{2} \| \mathbf{v}(t) \|^2 + \alpha_* \int_0^t \| \mathbf{S} \|_{\mathbb{S}'} + \| \mathbb{D} \|_r + \beta_* \int_0^t \| \mathbf{s} \|^2_{2, \partial \Omega} + \| \mathbf{v}_\tau \|^2_{2, \partial \Omega} \]

\[ \leq \frac{1}{2} \| \mathbf{v}_0 \|^2 + c_* | Q_T | + d_* | \Sigma_T | \]

Consequently,

\[ (\mathbf{v}, \mathbf{s}, \mathbb{S}) \in FS \]

Any reasonable (numerical) approximations should fulfil uniform estimates in FS
Function spaces - Stick-slip versus No-slip

\[ W_{n}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}, \]
\[ W_{n, \text{div}}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \text{div } \mathbf{v} = 0; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}, \]
versus

\[ W_{0}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathbf{v} = 0 \text{ on } \partial \Omega \}, \]
\[ W_{0, \text{div}}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \text{div } \mathbf{v} = 0; \mathbf{v} = 0 \text{ on } \partial \Omega \}, \]
\[ \mathbf{v} \in L^\infty(0, T; L^2) \cap L^r(0, T; W^{1,r}_{n,\text{div}}) \cap L^{\frac{5r}{3}}(0, T; L^{\frac{5r}{3}}(\Omega)^d) \]
\[ \mathcal{S} \in L^{r'}(0, T; L^{r'}(\Omega)^{d \times d}) \]
\[ \mathbf{s} \in L^2(0, T; L^2(\partial\Omega)^d) \]
\[ \partial_t \mathbf{v} \in \left( L^{r}(0, T; W^{1,r}_{n,\text{div}}) \cap L^{\frac{5r}{6}}(0, T; W^{1,\frac{5r}{6}}_{n,\text{div}}) \right)^* \]
\[ = \begin{cases} 
L^{r'}(0, T; W^{-1,r'}_{n,\text{div}}) & \text{if } r \geq \frac{11}{5} \\
L^{\frac{5r}{5r-6}}(0, T; W^{-1,\frac{5r}{5r-6}}_{n,\text{div}}) & \text{if } r < \frac{11}{5}
\end{cases} \]

- **FS** compactly embedded into \( L^2(0, T; L^2(\Omega)) \) if \( r > 6/5 \)
- **FS** compactly embedded into \( L^2(0, T; L^2(\partial\Omega)) \) if \( r > 8/5 \)
Part #3

Weak stability of Problem
Weak stability of Problem

Assume that

- for each \( n \in \mathbb{N} \): \((v^n, s^n, S^n)\) solves Problem
- \((v^n, s^n, S^n)\) converges weakly to \((v, s, S)\) in FS

Is \((v, s, S)\) also solution of Problem?
Balance of linear momentum - equation of motion

For all \( \tilde{\mathbf{w}} \in (W^{1,r}(\Omega) \cap C^1(\Omega))^3 \) with \( \text{div} \, \tilde{\mathbf{w}} = 0 \) in \( \Omega \) and \( \tilde{\mathbf{w}} \cdot n = 0 \) on \( \partial \Omega \):

\[
\int_0^T \left\{ \langle \partial_t \mathbf{v}^n, \tilde{\mathbf{w}} \rangle + (\mathbb{S}^n, \mathbb{D}\tilde{\mathbf{w}})_\Omega + (s^n, \tilde{\mathbf{w}}_\tau)_{\partial \Omega} - (\mathbf{v}^n \otimes \mathbf{v}^n, \nabla \tilde{\mathbf{w}})_\Omega \right\} \, dt = 0
\]

converges to

\[
\int_0^T \left\{ \langle \partial_t \mathbf{v}, \tilde{\mathbf{w}} \rangle + (\mathbb{S}, \mathbb{D}\tilde{\mathbf{w}})_\Omega + (s, \tilde{\mathbf{w}}_\tau)_{\partial \Omega} - (\mathbf{v} \otimes \mathbf{v}, \nabla \tilde{\mathbf{w}})_\Omega \right\} \, dt = 0
\]

provided that \( W^{1,r}(\Omega) \) is compactly embedded into \( L^2(\Omega) \), which holds if

\[ r > 6/5. \]

It remains to show that

\[ (\mathbb{S}, \mathbb{D}\mathbf{v}) \in \mathcal{A} \quad \text{and} \quad (s, \mathbf{v}_\tau) \in \mathcal{B}. \]
Convergence lemma

Lemma

Let $U \subset Q_T$ be arbitrary (measurable) and $r \in (1, \infty)$. Assume that

- $A$ is a maximal monotone graph (satisfying (A2)–(A3))
- $\{S^n\}_{n=1}^\infty$ and $\{D^n\}_{n=1}^\infty$ satisfy
  \[
  (S^n, D^n) \in A
  \]
  for a.a. $(t, x) \in U$
  weakly in $L^r(U)^{d \times d}$
  weakly in $L^{r'}(U)^{d \times d}$

\[
\limsup_{n \to \infty} \int_U S^n \cdot D^n \, dx \, dt \leq \int_U S \cdot D \, dx \, dt.
\]

Then

\[
(S, D) \in A \quad \text{almost everywhere in } U.
\]

- **Local version**
- Last assumption suggests to use energy (entropy) inequality
Step 1. \( S^n \cdot D^n \rightarrow S \cdot D \) weakly in \( L^1(U) \)

From (A2)

\[
0 \leq (S^n - S^m) \cdot (D^n - D^m) \quad \text{a.e. in } U
\]

Hence, by the assumptions,

\[
\lim_{n \to \infty} \lim_{m \to \infty} \| (S^n - S^m) \cdot (D^n - D^m) \|_1 \leq 0
\]

which implies

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_U (S^n - S^m) \cdot (D^n - D^m) \varphi = 0 \quad \forall \varphi \in L^\infty(U)
\]

Setting \( L := \lim_{\ell \to \infty} \int_U (S^\ell \cdot D^\ell) \varphi \) we conclude that

\[
0 = \lim_{n \to \infty} \lim_{m \to \infty} \left[ \int_U S^n \cdot D^n \varphi - \int_U S^n \cdot D^m \varphi - \int_U S^m \cdot D^n \varphi + \int_U S^m \cdot D^m \varphi \right]
\]

\[
= 2 \left( L - \int_U S \cdot D \varphi \right)
\]
Step 2. \((S, D) \in \mathcal{A}\) a.e. in \(U\)

Take arbitrarily 
\((S^*, D^*) \in \mathcal{A}\) and a nonnegative \(\varphi \in L^\infty(U)\)

Then from (A2) and Step 1

\[
0 \leq \lim_{n \to \infty} \int_U (S^n - S^*) \cdot (D^n - D^*) \varphi = \int_U (S - S^*) \cdot (D - D^*) \varphi.
\]

Since \(\varphi \geq 0\) arbitrary we get

\[
0 \leq (S - S^*) \cdot (D - D^*) \quad \text{a.e. in } U
\]

Since \((S^*, D^*) \in \mathcal{A}\) is arbitrary, the maximality of the graph implies

\((S, D) \in \mathcal{A}\) a.e. in \(U\)
Identification of the limit for boundary terms

Assume that
\[ s^n \rightharpoonup s \quad \text{weakly in } L^2(0, T; L^2(\partial\Omega)^3), \]
\[ v^n \rightharpoonup v \quad \text{weakly in } L^2(0, T; L^2(\partial\Omega)^3) \]
and \((s^n, v^n) \in \mathcal{B}\)

- it is enough to show that

\[ \limsup_{n \to \infty} \int_{\partial\Omega} s^n \cdot v^n \leq \int_{\partial\Omega} s \cdot v \]

- however we also have

\[ v^n \rightharpoonup v \quad \text{strongly in } L^1(0, T; L^1(\partial\Omega)^3) \]

By Egorov theorem, for any \(\varepsilon > 0\) there exists \(U_\varepsilon \subset \Sigma_T\) such that \(|\Sigma_T \setminus U_\varepsilon| \leq \varepsilon\) and

\[ v^n \rightharpoonup v \quad \text{strongly in } L^\infty(U_\varepsilon)^3 \]

\[ \limsup_{n \to \infty} \int_{U_\varepsilon} s^n \cdot v^n \leq \int_{U_\varepsilon} s \cdot v \]

and \((s, v) \in \mathcal{B}\) a.e. in \(U_\varepsilon\). But \(\varepsilon\) is arbitrary and \((s, v) \in \mathcal{B}\) a.e. on \(\Sigma_T\)
Weak stability of Problem Identification \((\mathcal{S}, \mathcal{D} \mathbf{v}) \in \mathcal{A}\) - the convective term neglected

Take \(\mathbf{v}^n\) as a test function in weak formulation of BLM for Problem\((n)\):

\[
\frac{1}{2} \| \mathbf{v}^n(T) \|^2_2 + \int_{Q_T} \mathbf{S}^n : \mathbf{D}^n + \int_{\Sigma_T} \mathbf{s}^n \cdot \mathbf{v}^n = \frac{1}{2} \| \mathbf{v}_0 \|^2_2 \tag{1}
\]

Take \(\mathbf{v}\) as a test function in weak formulation of BLM for Problem:

\[
\frac{1}{2} \| \mathbf{v}(T) \|^2_2 + \int_{Q_T} \mathbf{S} : \mathbf{D} + \int_{\Sigma_T} \mathbf{s} \cdot \mathbf{v} = \frac{1}{2} \| \mathbf{v}_0 \|^2_2 \tag{2}
\]

Letting \(n \to \infty\) in (1) and comparing the result with (2) we observe that

\[
\limsup_{n \to \infty} \int_{Q_T} \mathbf{S}^n : \mathbf{D}^n \leq \int_{Q_T} \mathbf{S} : \mathbf{D}
\]

which is the fourth assumption of Convergence lemma. Therefore

\((\mathcal{S}, \mathcal{D}) \in \mathcal{A}\)
Weak stability of Problem

Identification \((S, Dv) \in A\) - with the convective term

Since

\[
\int_{\Omega} v^n_k \frac{\partial v^n}{\partial x_k} \cdot v^n = \int_{\Omega} v^n_k \frac{1}{2} \frac{\partial |v^n|^2}{\partial x_k} = \int_{\Omega} \frac{1}{2} \text{div}(|v|^2 v) = 0
\]

and similarly

\[
\int_{\Omega} v_k \frac{\partial v}{\partial x_k} \cdot v = 0
\]

the above stated proof remains unchanged if

\[
v_k \frac{\partial v}{\partial x_k} \cdot v \in L^1(Q_T)
\]

(3)

Since \(v \in L^{\frac{5r}{3}}(Q_T)\), (3) holds if \(r \geq \frac{11}{5}\).

Weak stability of Problem is proved. The result include Rigid/shear-thickening fluids, activated NS fluids, and Euler/shear-thickening fluids if \(r \geq 11/5\)

Q: What about the Euler/NS fluid or Bingham fluids when \(r = 2\)?
Part #4

Bingham fluids with threshold slip - existence of unsteady flows for large data
Bingham fluids with threshold slip

\[ G(\mathbf{S}, \mathbf{D}) := \mathbf{D} - \frac{(|\mathbf{S}| - \tau^*_\ast) + \mathbf{S}}{|\mathbf{S}|} \quad \text{Bingham fluid} \]

\[ g(s, v) := v - \frac{(|s| - \sigma^*_\ast) + s}{|s|} \quad \text{Threshold slip} \]

\textbf{Theorem}

Let \( \Omega \subset \mathbb{R}^d \) be a \( C^{1,1} \) domain. Then for any \( v_0 \in L^2_{0,\text{div}} \) there exists

\[ v \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; W^{1,2}_{n,\text{div}}) \]

\[ \mathbf{S} \in L^2(Q)^{d \times d}_{\text{sym}}, \quad s \in L^2(0, T; L^2(\partial\Omega)^d) \]

\[ p_1 \in L^2(Q), \quad p_2 \in L^{\frac{d+2}{d+1}}(0, T; W^{1,\frac{d+2}{d+1}}(\Omega)) \]

solving for almost all time \( t \in (0, T) \) and for all \( w \in W^{1,\infty}_n \)

\[ \langle \partial_t v, w \rangle - \int_\Omega (v \otimes v) \cdot \nabla w + \int_\Omega \mathbf{S} : \mathbf{D}(w) + \int_{\partial\Omega} s \cdot w = \int_\Omega (p_1 + p_2) \text{div} w \]

and fulfilling

\[ G(\mathbf{S}, \mathbf{D}v) = 0 \quad \text{a.e. in } Q_T \quad \text{and} \quad g(s, v_T) = 0 \quad \text{a.e. in } \Sigma_T \]
Function spaces - Stick-slip versus Slip

\[ W_{n,q}^{1} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}, \]

\[ W_{0,\text{div}}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \text{div } \mathbf{v} = 0; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}, \]

versus

\[ W_{0,q}^{1} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathbf{v} = 0 \text{ on } \partial \Omega \}, \]

\[ W_{0,\text{div}}^{1,q} := \{ \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \text{div } \mathbf{v} = 0; \mathbf{v} = 0 \text{ on } \partial \Omega \}, \]

By the Helmholtz decomposition, for \( q \in (1, \infty) \):

\[ W_{n,q}^{1} = W_{n,\text{div}}^{1,q} \oplus \{ \nabla \varphi; \varphi \in W^{2,q}, \nabla \varphi \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}. \]

Similar decomposition for \( W_{0,q}^{1}(\Omega)^d \) is open.

- Essential difference in the weak formulation
- \( \sigma_\ast \) can be artificial (big enough) so that it is never active
  - in analysis if \( \mathbf{v} \in L^\infty(0, T; C(\overline{\Omega})) \)
  - in computer simulations
Proof - \(n\)-approximations

Consider

\[
G^n(S, D) := D - \left( \frac{(|S| - \tau_\ast)_+}{|S|} + \frac{1}{n} \right) S
\]

Bingham fluid, \( (Bn) \)

\[
g^n(s, v) := v - \left( \frac{(|s| - \sigma_\ast)_+}{|s|} + \frac{1}{n} \right) s
\]

threshold slip \( (Tn) \)

and smooth \( G_n, |G'_n| \leq \frac{1}{n} \)

\[
G_n(s) := 1 \text{ for } s \leq n, \quad G_n(s) = 0 \text{ for } s > 2n.
\]

Take approximation

\[
\partial_t v^n + \text{div}(v^n \otimes v^n) G_n(|v^n|) - \text{div} S^n = -\nabla p^n
\]

with constitutive equations \( (Bn) \) and \( (Tn) \). Since \( (Bn) \) and \( (Tn) \) imply

\[
S = S^*_n(D), \quad s = s^*_n(v)
\]

with \( S^*_n \) and \( s^*_n \) being continuous monotone with linear growth (at infinity), the existence follows from monotone operator theory (due to the presence of \( G_n \))
Pressure for $n$ fixed

$$\langle \partial_t \mathbf{v}^n, \mathbf{w} \rangle + (\mathbb{S}^n, \mathbb{D}(\mathbf{w})) + (\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G(|\mathbf{v}^n|), \mathbf{w}) + (s^n, \mathbf{w}_\tau)_{\partial \Omega}$$

$$- \langle \mathbf{b}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W_{n,\text{div}}^1 \text{ and a.a. } t \in (0, T)$$

Define $p^n$ as the solution of the following problem

$$\nabla p^n + \nabla \mathbf{z} + (\mathbb{S}^n, \nabla^{(2)} \mathbf{z}) + (\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G(|\mathbf{v}^n|), \nabla \mathbf{z})$$

$$+ (s^n, \nabla \mathbf{z})_{\partial \Omega} = 0$$

for all $\nabla \mathbf{z} \in W^{2,2}(\Omega)$ with $\nabla \mathbf{z} \cdot \mathbf{n} = 0$ on $\partial \Omega$ and a.a. $t \in (0, T)$

$$w = \mathbf{w} + \nabla \mathbf{z}$$

$$= \langle \partial_t \mathbf{v}^n, \mathbf{w} \rangle + (\mathbb{S}^n, \mathbb{D}(\mathbf{w})) + (\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G(|\mathbf{v}^n|), \mathbf{w}) + (s^n, \mathbf{w}_\tau)_{\partial \Omega} - \langle \mathbf{b}, \mathbf{w} \rangle$$

$$= \langle \partial_t \mathbf{v}^n, \mathbf{w} \rangle + (\nabla p, \nabla \mathbf{z})$$

$$= \langle \partial_t \mathbf{v}^n, \mathbf{w} \rangle + \nabla \mathbf{z} + (\nabla p, \mathbf{w})$$

which finally leads to:

$$\langle \partial_t \mathbf{v}^n, \mathbf{w} \rangle + (\mathbb{S}^n, \mathbb{D}(\mathbf{w})) + (\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G(|\mathbf{v}^n|), \mathbf{w}) + (s^n, \mathbf{w}_\tau)_{\partial \Omega}$$

$$= (p^n, \text{div } \mathbf{w}) + \langle \mathbf{b}, \mathbf{w} \rangle \text{ for all } \mathbf{w} \in W_{n,\text{div}}^1 \text{ and a.a. } t \in (0, T)$$
Apriori estimates

I. Test by \( \mathbf{v}^n \) (convective term) vanishes to get

\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{v}^n \|_2^2 + \int_\Omega \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^n) + \int_{\partial \Omega} \mathbf{s}^n \cdot \mathbf{v}^n = 0
\]

\[
\sup_{t \in (0, T)} \| \mathbf{v}^n(t) \|_2^2 + \int_{Q_T} |\mathbf{S}^n|^2 + |\nabla \mathbf{v}^n|^2 + |\mathbf{v}^n|^2 \frac{2(d+2)}{d} + \int_{(0, T) \times \partial \Omega} |\mathbf{s}^n|^2 + |\mathbf{v}^n|^2 \leq C(\mathbf{v}_0)
\]

II. Find \( p^n_2 \) with zero mean value solving at each time level

\[
\int_\Omega \nabla p^n_2 \cdot \nabla \varphi = - \int_\Omega \text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G_n(|\mathbf{v}^n|) \cdot \varphi
\]

But

\[
\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G_n(|\mathbf{v}^n|) = v^k_n \frac{\partial \mathbf{v}^n}{\partial x_k} G_n(|\mathbf{v}^n|)
\]

\[
\int_{Q_T} |\text{div}(\mathbf{v}^n \otimes \mathbf{v}^n) G_n(|\mathbf{v}^n|)| \frac{d+2}{d+1} \leq C \implies \int_0^T \| p^n_2 \|_{1, \frac{d+2}{d+1}} \leq C
\]

Define \( p^n_1 := p^n - p^n_2 \).
### III. For $p^n_1 := p^n - p^n_2$ find $\varphi$ with zero mean value such that $\nabla \varphi \cdot n = 0$ on $\partial \Omega$ solving

$$\Delta \varphi = p^n_1 \implies \int_{Q_T} |\nabla^2 \varphi|^2 + \int_{(0,T) \times \partial \Omega} |\nabla \varphi|^2 \leq \int_{Q_T} |p^n_1|^2$$

Test by $\nabla \varphi$ and integrate over $Q_T$

$$\int_{Q_T} |p^n_1|^2 = -\int_{Q_T} \nabla p^n_1 \cdot \nabla \varphi = \int_{Q_T} (\nabla p^n_2 - \text{div}(v^n \otimes v^n) G_n(|v^n|)) \cdot \nabla \varphi$$

$$+ \int_{Q_T} S^n \cdot \nabla^2 \varphi + \int_{(0,T) \times \partial \Omega} s^n \cdot \nabla \varphi$$

$$= \int_{Q_T} S^n \cdot \nabla^2 \varphi + \int_{(0,T) \times \partial \Omega} s^n \cdot \nabla \varphi$$

$$\leq C \left( \int_{Q_T} |p^n_1|^2 \right)^{\frac{1}{2}}$$

### IV. $\|\partial_t v^n\|_{(L^2(0,T; W^{1,2}_n) \cap L^{d+2}(Q_T))^*} \leq C$
Convergences

Aubin-Lions and apriori estimates:

\[ \mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0, T; W^{1,2}_n), \]
\[ \mathcal{S}^n \rightharpoonup \mathcal{S} \quad \text{weakly in } L^2(Q)^{d \times d}, \]
\[ \mathbf{s}^n \rightharpoonup \mathbf{s} \quad \text{weakly in } L^2(0, T; L^2(\partial \Omega)), \]
\[ \mathbf{v}^n \rightarrow \mathbf{v} \quad \text{strongly in } L^2(Q), \]
\[ \mathbf{v}^n \rightarrow \mathbf{v} \quad \text{strongly in } L^2(0, T; L^2(\partial \Omega)), \]
\[ \rho_n^1 \rightharpoonup \rho_1 \quad \text{weakly in } L^2(Q), \]
\[ \rho_n^2 \rightharpoonup \rho_2 \quad \text{weakly in } L^{\frac{d+2}{d+1}}(0, T; W^{1, \frac{d+2}{d+1}}(\Omega)), \]
\[ \partial_t \mathbf{v}^n \rightharpoonup \partial_t \mathbf{v} \quad \text{weakly in } (L^2(0, T; W^{1,2}_n) \cap L^{d+2}(Q_T))^* \]

solving the original problem, and also \( g(s, \mathbf{v}_\tau) = 0 \)

It remains to show the validity of \( G(\mathcal{S}, \mathcal{D}\mathbf{v}) = 0 \).
Convergence III

Assume that \( \{k^n\}_{n=1}^\infty \) is such that \( 0 < A \leq k^n \leq B < \infty \). Test the \( n \)-th approximation by

\[
 w^n := T_k^n(v^n - v) := (v^n - v) \min \left\{ 1, \frac{k^n}{|v^n - v|} \right\}
\]

Note \( T_k(u) = u \) if \( |u| \leq k \).

Taking \( w^n \) as a test function

\[
 \limsup_{n \to \infty} \int_{Q_T} \mathbb{S}^n \cdot \mathbb{D}(w^n) - p_1^n \div w^n \\
= \limsup_{n \to \infty} \int_{Q_T} -\langle \partial_t v^n, w^n \rangle - (\div (v^n \otimes v^n) G_n(|v^n|)) + \nabla p_2^n \cdot w^n \\
+ \int_{\Sigma_T} s^n \cdot w^n \leq 0
\]

J. Málek

Incompressible Fluids with activation
Find $\bar{S} \in L^2(Q)$ fulfilling

\[
\mathcal{D}(v) = \frac{(\bar{S} - \tau) + \bar{S}}{\bar{S}}
\]

Then

\[
\limsup_{n \to \infty} \int_{Q_T} (S^n - \bar{S}) \cdot \mathcal{D}(w^n) \leq \limsup_{n \to \infty} \int_{|v^n - v| \geq k^n} \frac{k^n}{|v^n - v|} p_1^n (|\nabla v^n| + |\nabla v|)
\]

Considering

\[
I^n := C_\ast (|p_1^n|^2 + |\nabla v^n|^2 + |\nabla v|^2 + |\bar{S}|^2 + |\bar{S}^n|)
\]

\[
\sup_n \int_{Q_T} I^n < \infty
\]

we observe that

\[
\limsup_{n \to \infty} \int_{|v^n - v| < k^n} (S^n - \bar{S}) \cdot \mathcal{D}(v^n - v) \leq \limsup_{n \to \infty} \int_{|v^n - v| \geq k^n} \frac{k^n}{|v^n - v|} I^n
\]

**AIM:** RHS should tend to zero by making a proper choice for $A$, $B$ and $k^n$. 

---

**J. Málek**

*Incompressible Fluids with activation*
For $N \in \mathbb{N}$ arbitrary, fix $A := N$ and $B := N^{N+1}$ and define

$$Q^n_i := \{ (t, x) \in Q_T; N^i \leq |\mathbf{v}^n - \mathbf{v}| \leq N^{i+1} \} \quad i = 1, \ldots, N.$$  

Since

$$\sum_{i=1}^{N} \int_{Q^n_i} I^n \leq C_*,$$

there is, for each $n \in \mathbb{N}$, an index $i_n \in \{1, \ldots, N\}$ such that

$$\int_{Q^n_{i_n}} I^n < \frac{C_*}{N}.$$  

Setting $k^n := N^{i_n+1}$, RHS is estimated in the following way:

$$\int_{|\mathbf{v}^n - \mathbf{v}| \geq N^{i_n+1}} \frac{k^n}{|\mathbf{v}^n - \mathbf{v}|} I^n = \int_{N^{i_n+2} \geq |\mathbf{v}^n - \mathbf{v}| \geq N^{i_n+1}} \cdots + \int_{|\mathbf{v}^n - \mathbf{v}| \geq N^{i_n+2}} \cdots$$

$$= \int_{Q^n_{i_n}} I^n \cdots + \int_{|\mathbf{v}^n - \mathbf{v}| \geq N^{i_n+2}} I^n \leq \frac{C_*}{N}. \quad (4)$$

Next, using the constitutive equation for $\mathbb{D}\mathbf{v}^n$ and $\mathbb{D}\mathbf{v}$ we conclude that

$$\limsup_{n \to \infty} \int_{|\mathbf{v}^n - \mathbf{v}| \leq k^n} (S^n - \overline{S}) \cdot \left( \left( \frac{(S^n - \tau^*)}{|S^n|} + \frac{1}{n} \right) S^n - \frac{(\overline{S} - \tau^*)}{|\overline{S}|} \overline{S} \right) \leq \frac{C_*}{N},$$
Thus, for $Z^n := (\mathcal{S}^n - \bar{S}) \cdot \left( \frac{(\mathcal{S}^n - \tau^*) + \mathcal{S}^n - (\bar{S} - \tau^*) + \bar{S}}{|\mathcal{S}^n|} \right) \geq 0$

\[
\limsup_{n \to \infty} \int_{|\mathbf{v}^n - \mathbf{v}| \leq k^n} Z^n \leq \frac{C^*}{N} + \limsup_{n \to \infty} \frac{1}{n} \int_{Q_T} |\mathcal{S}^n||\mathcal{S}^n - \bar{S}| \leq \frac{C^*}{N}
\]

Since $A = N$ and $k^n \geq N$

\[
\limsup_{n \to \infty} \int_{|\mathbf{v}^n - \mathbf{v}| \leq N} Z^n \leq \frac{C^*}{N}
\]

Splitting $Q_T$ into a union of $\{ |\mathbf{v}^n - \mathbf{v}| \leq N \}$ and $\{ |\mathbf{v}^n - \mathbf{v}| > N \}$ one concludes

\[
\limsup_{n \to \infty} \int_{Q_T} \sqrt{Z^n} \leq \frac{C^*}{N} \implies Z^n \to 0 \quad \text{a.e. in } Q_T
\]

Applying then the biting lemma, one then concludes that

\[
Z^n \to 0 \quad \text{strongly in } L^1(Q_T \setminus E_j) \quad E_j \subset Q_T : \lim_{j \to \infty} |E_j| = 0
\]

\[
\implies \limsup_{n \to \infty} \int_{Q_T \setminus E_j} \mathcal{S}^n \cdot (\nabla \mathbf{v}^n - \frac{1}{n} \mathcal{S}^n) = \int_{Q_T \setminus E_j} \mathcal{S} \cdot \nabla \mathbf{v}.
\]

Convergence lemma and the properties of $E_j$: $\implies (\mathcal{S}, \nabla \mathbf{v}) \in \mathcal{A}$ a.e. in $Q_T$. 
Part #5

Implicitly constituted fluids described by maximal monotone \( \psi \)-graph - existence of unsteady flows subject to Navier’s slip for large data
Definition of weak solution to the Problem with Navier’s slip bcs

**Definition**

We say \((p, \mathbf{v}, \mathcal{S})\) is weak solution to *Problem* with Navier’s slip

\[
\begin{align*}
p &\in L^1(Q_T) \\
\mathbf{v} &\in C_{\text{weak}}(0, T; L_n^2, \text{div}) \cap L^q(0, T; W_n^{1,1}) \text{ with } \mathbb{D}(\mathbf{v}) \in L^{\psi}(Q_T) \\
\mathcal{S} &\in L^{\psi^*}(Q_T) \\
\lim_{t \to 0^+} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 &= 0 \\
\langle \mathbf{v}', \mathbf{w} \rangle + \langle \mathcal{S}, \mathbb{D}(\mathbf{w}) \rangle - (\mathbf{v} \otimes \mathbf{v}, \mathbb{D}(\mathbf{w})) + \alpha^*(\mathbf{v}_\tau, \mathbf{w}_\tau)_{\partial \Omega} &= \langle \mathbf{b}, \mathbf{w} \rangle + (p, \text{div } \mathbf{w}) \\
\text{for all } \mathbf{w} &\in W_n^{1,1} \text{ such that } \mathbb{D}(\mathbf{w}) \in L^\infty(\Omega)^{d \times d} \text{ and a.a. } t \in (0, T), \\
(\mathbb{D}(\mathbf{v}(t, x)), \mathcal{S}(t, x)) &\in \mathcal{A} \text{ for a.a. } (t, x) \in Q_T.
\end{align*}
\]
Theorem

Let \( \Omega \subset \mathbb{R}^3 \) and \( A \) satisfy the assumptions (A1)–(A4) with \( \psi \) fulfilling

\[
\begin{align*}
    c_1 s^r - c_2 &\leq \psi(s) \leq c_3 s^r + c_4 \quad \text{with} \quad r > \frac{2d}{d+2}
\end{align*}
\]

Then for any \( \Omega \in C^{1,1} \) and \( T \in (0, \infty) \) and for arbitrary

\[
    v_0 \in L^2_{n, \text{div}}, \quad b \in L^2(0, T; L^2(\Omega)^d) \quad \text{and} \quad \gamma_* \geq 0,
\]

there exists weak solution to Problem.

Novel tools:
(i) Structural assumptions (A1)–(A4) on \( G(S, D) = 0 \)
(ii) Convergence lemma
(iii) Understanding the interplay between the chosen boundary conditions and global integrability of \( p \)
(iv) Lipschitz approximations of Sobolev-Orlicz and Bochner functions

Methods

- subcritical case
  - Minty’s method
  - energy equality - $\mathbf{v}$ is an admissible test function
- supercritical case
  - Generalized Minty’s method - Convergence lemma
    - Lipschitz approximation in Orlicz-Sobolev spaces
    - $L^\infty$-truncation of Sobolev functions
No-slip versus Threshold slip (Stick-slip)

- Homogeneous Dirichlet boundary conditions are considered as the simplest for many PDEs.
- In incompressible fluid dynamical problems, it is however, in general, open whether $p \in L^1(Q_T)$ for no-slip boundary conditions.
- Exceptions are the cases when we control $\partial_t \mathbf{v}$ is an integrable function, e.g.,
  - Navier-Stokes model (linearity, Solonnikov)
  - Ladyzhenskaya model for $r \geq \frac{12}{5}$ in 3D setting provided that data are smooth (potentiality of $S$, test by time derivative, Ladyzhenskaya)
  - All models above with uniform monotonicity (but no assumption on having potential), whenever we can test by $\mathbf{v}$ (bootstrap in non-integer time derivatives, Bulíček, Etwein, Kaplický, Pražák) $r > \frac{11}{5}$
Concluding Remarks

- implicitly constitutive theory seems to suitable approach to include various activation criteria both in the bulk and on the boundary
- threshold slip is the way how to overcome the troubles connected with the analysis of unsteady flows subject to homogeneous Dirichlet boundary conditions (no-slip) - fits nicely to the framework of implicitly constituted materials
- for implicitly constituted fluids characterized by (A1)-(A4) and \( r > 6/5 \), we define the solution and show its large data existence - object to be studied numerically and computationally.
- new options how to numerically discretize the problems - some give interesting results (second order vs. first order PDEs) - J.Hron, P. Minakowski, G.Tierra.
Thank you