Nečas Center for Mathematical Modeling

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MINISTRY OF EDUCATION, YOUTH AND SPORTS

Activated fluids: continuum description, analysis and computational results

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Fluid and Solid Mechanics

Continuum Thermodynamics - concept of continuum

- balance equations
  - conservation of mass, energy
  - principles of classical Newtonian mechanics applied to subsets of the body: $\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}$ with $\boldsymbol{v} = \frac{d\boldsymbol{\chi}}{dt}$
  - principles of classical thermodynamics applied to subsets of the body assumed to be at local equilibrium
- boundary conditions
- initial conditions

Insufficient to describe mechanical and thermal processes inside the body

Initial and boundary value problems

- balance equations

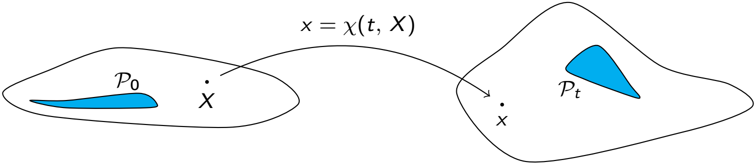
$$
\begin{aligned}
\dot{\rho} &= \rho \operatorname{div} \boldsymbol{v} \\
\rho \dot{\boldsymbol{v}} &= \operatorname{div} \mathbb{T} \\
\rho \dot{E} &= \operatorname{div}(\mathbb{T}\boldsymbol{v} - \boldsymbol{j}_e) \qquad E := e + \tfrac{1}{2}|\boldsymbol{v}|^2
\end{aligned}
$$

  - the density ρ
  - the velocity $\boldsymbol{v} = (v_1, v_2, v_3)$
  - the internal energy e
  - the Cauchy stress tensor $\mathbb{T} = (T_{11}, T_{12}, T_{13}, T_{22}, T_{23}, T_{33})$
  - the energy flux $\boldsymbol{j}_e = (j_{e1}, j_{e2}, j_{e3})$
- boundary conditions
- initial conditions

Insufficient to predict mechanical processes inside the body
Closure - constitutive (material) equations involving $\mathbb{T}$ and $\boldsymbol{j}_e$

Section 1

Balance equations and stress power



Balance equation for z

$$\frac{d}{dt}\int_{\mathcal{P}_t} z(t,x)\,dx = \int_{\partial\mathcal{P}_t} \boldsymbol{j}_z(t,x)\cdot\boldsymbol{n}(t,x)\,dS + \int_{\mathcal{P}_t} s_z(t,x)\,dx$$

Incompressibility:

$$\frac{d}{dt}\mathrm{Vol}(\mathcal{P}_t) = 0 \quad\Longleftrightarrow\quad \frac{d}{dt}\int_{\mathcal{P}_t} dx = 0$$

General form of the balance equations

For all $\mathcal{P}_t \subset \Omega$: $\dot{z} := \frac{\partial z}{\partial t} + \boldsymbol{v} \cdot \nabla z \qquad \boldsymbol{v} := \frac{\partial \chi}{\partial t}$

$$\boxed{\begin{aligned} \int_{\mathcal{P}_t} \{\dot{z} + z \operatorname{div} \boldsymbol{v} - \operatorname{div} \boldsymbol{j}_z + s_z\} \, dx = 0 \\ \dot{z} + z \operatorname{div} \boldsymbol{v} - \operatorname{div} \boldsymbol{j}_z + s_z = 0 \end{aligned}}$$

For mass density ρ: $\dot{\rho} = -\rho \operatorname{div} \boldsymbol{v}$

For linear momentum $\rho\boldsymbol{v}$: $\dot{\overline{\rho\boldsymbol{v}}} + \rho\boldsymbol{v} \operatorname{div} \boldsymbol{v} = \rho\dot{\boldsymbol{v}} = \operatorname{div} \mathbb{T}$

For total energy ρE: $\dot{\overline{\rho E}} + \varrho E \operatorname{div} \boldsymbol{v} = \rho \dot{E} = \operatorname{div}(\mathbb{T}\boldsymbol{v} - \boldsymbol{j}_e)$

Incompressibility: $\operatorname{div} \boldsymbol{v} = 0 \implies$

$$\dot{\rho} = 0 \iff \rho(t, \chi(t, X)) = \rho_0(X)$$

Stress power

Multiplying $\rho\dot{\boldsymbol{v}} = \operatorname{div}\mathbb{T}$ scalarly by $\boldsymbol{v}$:

$$\boxed{\mathbb{D} = \tfrac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)}$$

$$\boxed{\tfrac{1}{2}\rho\dot{\overline{|\boldsymbol{v}|^2}} = \operatorname{div}(\mathbb{T}\boldsymbol{v}) - \mathbb{T} : \mathbb{D}}$$

Subtracting this from $\rho\dot{E} = \operatorname{div}(\mathbb{T}\boldsymbol{v} - \boldsymbol{j}_e)$

$$\boxed{\rho\dot{e} = \operatorname{div}\boldsymbol{j}_e + \mathbb{T} : \mathbb{D}}$$

Stress power - source s_e

$$\boxed{\mathbb{T} : \mathbb{D} = \mathbb{S} : \mathbb{D}_\delta + m\operatorname{div}\boldsymbol{v}} = \mathbb{S} : \mathbb{D}_\delta \text{ if } \operatorname{div}\boldsymbol{v} = 0$$

where

$$\mathbb{A}_\delta := \mathbb{A} - \frac{1}{3}(\operatorname{Tr}\mathbb{A})\mathbb{I}$$

and

$$\boxed{\mathbb{T} = \mathbb{S} + m\mathbb{I}} \qquad \mathbb{S} := \mathbb{T}_\delta \text{ and } m := \tfrac{1}{3}\operatorname{Tr}\mathbb{T}$$

Stress power and the 2nd law of thermodynamics

So far, continuum thermodynamics entered only through the conservation of energy (First law of thermodynamics). For classical compressible fluids the rate of entropy production takes the form (Second law of thermodynamics)

$$\boxed{\theta\xi = \mathbb{S} : \mathbb{D}_\delta + (m + p_{\mathrm{th}}) \operatorname{div} \boldsymbol{v} - \boldsymbol{j}_e \cdot \frac{\nabla\theta}{\theta} \quad \text{and } \xi \geq 0} \qquad (1)$$

Remarks

- $\mathbb{S} : \mathbb{D}_\delta + (m + p_{\mathrm{th}}) \operatorname{div} \boldsymbol{v} \neq \mathbb{S} : \mathbb{D}_\delta + m \operatorname{div} \boldsymbol{v}$
- For incompressible fluids and isothermal processes: $\boxed{\xi = \mathbb{S} : \mathbb{D} \geq 0}$ represents gain/loss for internal/kinetic energy
- A purely mechanical systems (isothermal processes) are merely approximation
- Classification of incompressible fluids based on stress power - towards model with activation (mixing)
- Constitutive theory for $\mathbb{T}$ and $\boldsymbol{j}$ stemming from (1) - towards geo-physical models

Section 2

Classification of incompressible fluids

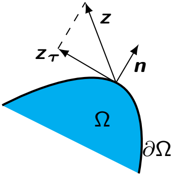
Josiah Williard Gibbs (1839-1903): One of the principal objects of theoretical reserach in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity.

Incompressible fluids with constant density ρ_*

$$\boxed{\begin{array}{ll} \operatorname{div} \boldsymbol{v} = 0 & \\ \rho_* \left(\dfrac{\partial \boldsymbol{v}}{\partial t} + v_k \dfrac{\partial \boldsymbol{v}}{\partial x_k} \right) = \nabla m + \operatorname{div} \mathbb{S} & \text{in } (0,T) \times \Omega \end{array}}$$

Internal flows

$$\boxed{\boldsymbol{v} \cdot \boldsymbol{n} = 0 \qquad \text{on } (0,T) \times \partial\Omega}$$



Balance of energy $\boldsymbol{s} := (-\mathbb{S}\boldsymbol{n})_{\boldsymbol{\tau}}$ $\qquad \boldsymbol{z}_{\boldsymbol{\tau}} := \boldsymbol{z} - (\boldsymbol{z} \cdot \boldsymbol{n})\boldsymbol{n}$

$$\boxed{\frac{d}{dt} \int_\Omega \rho_* \frac{|\boldsymbol{v}|^2}{2}\, dx + \int_\Omega \mathbb{S} : \mathbb{D}\, dx + \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}\, dS = 0}$$

Two dissipative mechanisms

$\boxed{\mathbb{S} : \mathbb{D}}$ mechanical energy due to friction between layers of the fluid in the bulk and due to further microstructural changes, transformed into heat: growth of the internal energy

- $\mathbb{D}$ the symmetric part of the velocity gradient
- $\mathbb{S}$ the traceless part of the Cauchy stress

$\boxed{\boldsymbol{s} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}}$ mechanical energy due to mutual interaction of the fluid in bulk and the solid that forms the boundary; transformed into the heat: growth of internal energy

- $\boldsymbol{v}_{\boldsymbol{\tau}}$ tangential part of the velocity on $\partial\Omega$
- $\boldsymbol{s}$ projection of the normal traction to the tangent plane

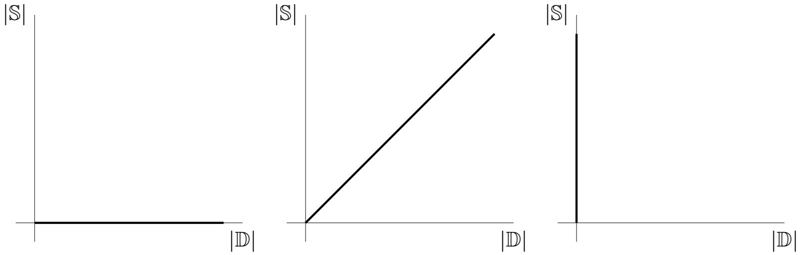
Requirements

$$\boxed{\mathbb{S} : \mathbb{D} \geq 0} \qquad \text{and} \; \boxed{\boldsymbol{s} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} \geq 0}$$

We formulate the whole cascade of models in bulk (i.e. constitutive equations relating $\mathbb{S}$ and $\mathbb{D}$) and the whole cascade of boundary conditions, for internal flows (i.e. constitutive equations relating $\boldsymbol{s}$ and $\boldsymbol{v}_{\boldsymbol{\tau}}$)

From Euler through NS fluid to rigid body motion

$\boxed{\mathbb{S} : \mathbb{D} \geq 0}$



A linear relation $\boxed{\mathbb{S} = 2\nu_* \mathbb{D}}$ $\iff$ $\boxed{\mathbb{S} = \alpha_* \mathbb{D}}$ Navier-Stokes fluid

- $\nu_* > 0$ is the shear viscosity
- $\alpha_* > 0$ is the fluidity $\alpha_* = \frac{1}{2\nu_*}$

Two remarkable trivial situations

- $\boxed{\mathbb{S} = \mathbb{O}}$ $\iff$ $\mathbb{T} = m\mathbb{I}$ Euler fluid
- $\boxed{\mathbb{D} = \mathbb{O}}$ $\iff$ $\boldsymbol{v}(t, x) = \boldsymbol{a}(t) \times x + \boldsymbol{b}(t)$ rigid body motion

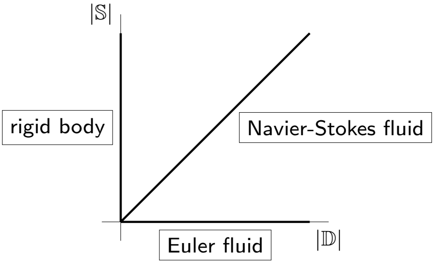


Figure: Response of Euler fluid, Navier-Stokes fluid, and rigid body.

Implicit constitutive relations

$$\boxed{\mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbb{O}}$$

$$\boxed{\mathbb{S} = |\mathbb{D}|^{r-2}\mathbb{D}} \qquad \Longleftrightarrow \qquad \boxed{\mathbb{D} = |\mathbb{S}|^{r'-2}\mathbb{S}}$$

Meaningful for $\boxed{r > 1}$ $\qquad$ $\boxed{r' = r/(r-1)}$

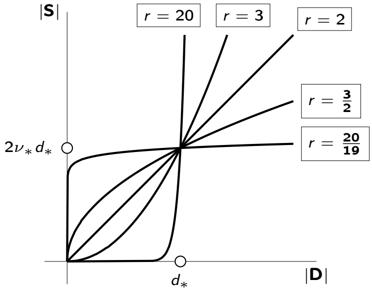
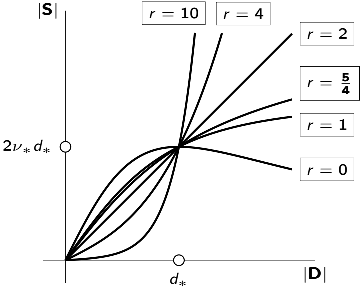


Figure: Response of the power-law model for various values of r.

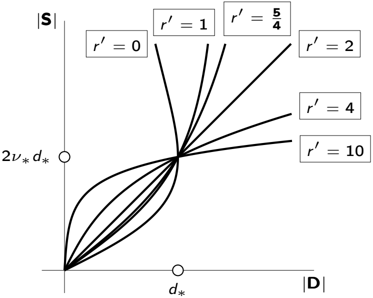
A generalization of power-law fluid

$$\mathbb{S} = 2\nu_* \left(\frac{1}{2} + \frac{1}{2} \frac{|\mathbb{D}|^2}{d_*^2} \right)^{\frac{r-2}{2}} \mathbb{D}$$

$$\mathbb{D} = \frac{1}{2\nu_*} \left(\frac{1}{2} + \frac{1}{2} \frac{|\mathbb{S}|^2}{(2\nu_* d_*)^2} \right)^{\frac{r'-2}{2}} \mathbb{S}$$



$r \in \mathbb{R}$

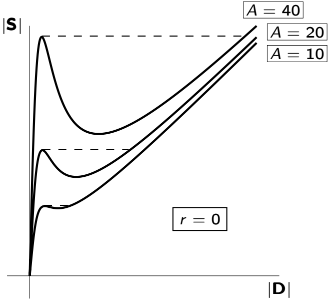


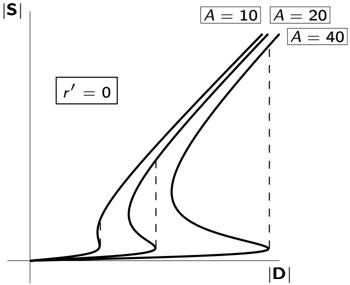
$r' \in \mathbb{R}$

A generalization of power-law fluid

$$\mathbb{S} = \left(1 + A\left(1 + |\mathbb{D}|^2\right)^{\frac{r-2}{2}}\right)\mathbb{D}$$

$$\mathbb{D} = \left(1 + A\left(1 + |\mathbb{S}|^2\right)^{\frac{r'-2}{2}}\right)\mathbb{S}$$





Both models can be simplified by making the response *monotone* (dashed line). Note that only on the left $\mathbb{S}$ is a function of $\mathbb{D}$; on the right, $\mathbb{D}$ is a function of $\mathbb{S}$.

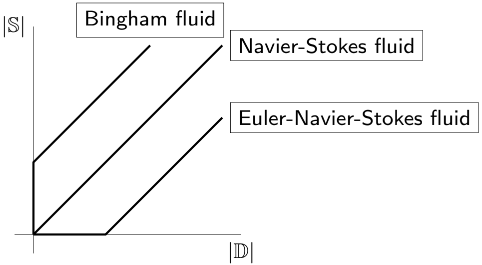


Figure: Response of the Bingham fluid, the Navier-Stokes fluid, and activated Euler-Navier-Stokes fluid.

Bingham fluid

- mixes rigid body behaviour with fluid behaviour
- a key model of viscoplasticity
- a special issue of IJNonNFM (2015)

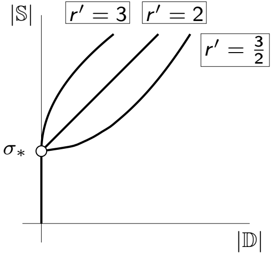
Euler/Navier-Stokes fluid

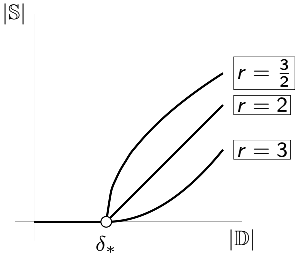
- connects behavior of fluids where shear effects are neglegible in parts of the fluid domain
- a possible model in boundary layer theory
- superfluids

Activated power-law fluids

$$\mathbb{D} = \mathbb{O} \iff |\mathbb{S}| \leq \sigma_*$$
$$\mathbb{D} \neq \mathbb{O} \iff \mathbb{S} = \sigma_* \frac{\mathbb{D}}{|\mathbb{D}|} + 2\nu_{\mathrm{g}}\left(|\mathbb{D}|^2\right)\mathbb{D}$$

$$\mathbb{S} = \mathbb{O} \iff |\mathbb{D}| \leq \delta_*$$
$$\mathbb{S} \neq \mathbb{O} \iff \mathbb{D} = \delta_* \frac{\mathbb{S}}{|\mathbb{S}|} + \frac{1}{2\nu_{\mathrm{g}}\left(|\mathbb{D}|^2\right)}\mathbb{S}$$





$$\mathbb{D} = \frac{1}{2\nu_{\mathrm{g}}\left(|\mathbb{D}|^2\right)} \frac{\left(|\mathbb{S}| - \sigma_*\right)^+}{|\mathbb{S}|}\mathbb{S}$$

$$\mathbb{S} = 2\nu_{\mathrm{g}}\left(|\mathbb{D}|^2\right) \frac{\left(|\mathbb{D}| - \delta_*\right)^+}{|\mathbb{D}|}\mathbb{D}$$

Workshop Viscoplastic fluids: from theory to application (2013) (Xavier Chateau, Antony Wachs)

- The realistic and accurate modeling of viscoplastic and thixotropic materials still remains an unsolved question in the field
- Efforts in designing new numerical approaches with enhanced accuracy and fast convergence have seemed to slow down and the workshop was an occasion to acknowledge that this research should be revived

A novel approach

$$\boxed{\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{O}}$$

Continuous curve over the Cartesian product $\mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ (replaces viewpoints through "multivalued" or "discontinuus" functions, or variational inequalities)

Fluids with limiting shear-stress/shear-rate

$$\boxed{\mathbb{S} = 2\nu_* \left(1 + \frac{|\mathbb{D}|^2}{d_*^2}\right)^{-\frac{1}{2}} \mathbb{D}}$$

$$\boxed{\mathbb{D} = \frac{1}{2\nu_*} \left(1 + \frac{|\mathbb{S}|^2}{d_*^2}\right)^{-\frac{1}{2}} \mathbb{S}}$$

Fluids with limiting shear-stress/shear-rate

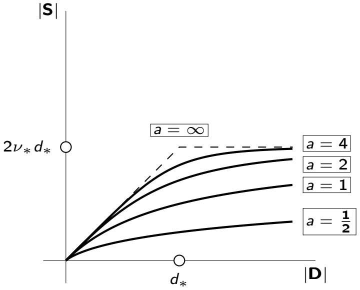
$$\mathbb{S} = 2\nu_* \left(1 + \frac{|\mathbb{D}|^{a}}{d_*^{a}}\right)^{-\frac{1}{a}} \mathbb{D}$$

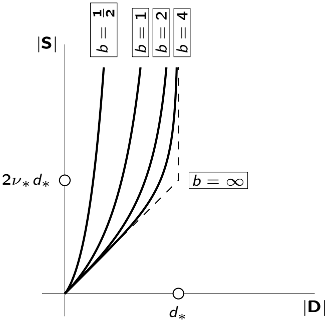
$$\mathbb{D} = \frac{1}{2\nu_*} \left(1 + \frac{|\mathbb{S}|^{b}}{d_*^{b}}\right)^{-\frac{1}{b}} \mathbb{S}$$

Fluids with limiting shear-stress/shear-rate

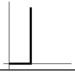
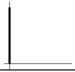
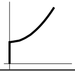
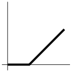
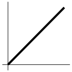
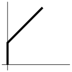
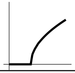
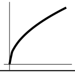
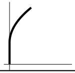
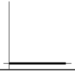
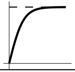
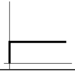
$$\mathbb{S} = 2\nu_* \left(1 + \frac{|\mathbb{D}|^{a}}{d_*^{a}}\right)^{-\frac{1}{a}} \mathbb{D}$$

$$\mathbb{D} = \frac{1}{2\nu_*} \left(1 + \frac{|\mathbb{S}|^{b}}{d_*^{b}}\right)^{-\frac{1}{b}} \mathbb{S}$$



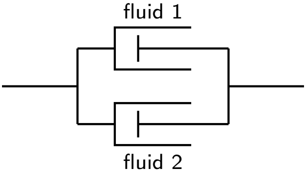


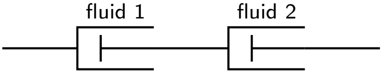
Summary

Euler/limiting shear-rate 	limiting shear-rate 	rigid body 
Euler/shear-thickening 	shear-thickening 	rigid/shear-thickening 
Euler/Navier-Stokes 	Navier-Stokes 	Bingham = rigid/Navier-Stokes 
Euler/shear-thinning 	shear-thinning 	rigid/shear-thinning 
Euler 	limiting shear stress 	perfect plastic 
$\lvert\mathbb{D}\rvert \leq \delta_* \iff \mathbb{S} = \mathbb{O}$	no activation	$\lvert\mathbb{S}\rvert \leq \sigma_* \iff \mathbb{D} = \mathbb{O}$

Summary of systematic classification of fluid-like responses with corresponding $\lvert\mathbb{S}\rvert$ vs $\lvert\mathbb{D}\rvert$ diagrams.

“Mixing” two of the above given fluids





$$\boxed{\mathbb{S} = \mathbb{S}_1 + \mathbb{S}_2}$$

$$\boxed{\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2}$$

From slip through Navier's slip to no-slip

$$\boxed{\boldsymbol{s} \cdot \boldsymbol{v_\tau} \geq 0}$$

A linear relation $\boxed{\boldsymbol{s} = \gamma_* \boldsymbol{v_\tau}}$ $\Longleftrightarrow$ $\boxed{\boldsymbol{v_\tau} = \frac{1}{\gamma_*}\boldsymbol{s}}$ Navier's slip

Two remarkable trivial situations

- $\boxed{\boldsymbol{s} = \boldsymbol{0}}$ slip
- $\boxed{\boldsymbol{v_\tau} = \boldsymbol{0}}$ no-slip

$$\boldsymbol{v_\tau} = \boldsymbol{0} \iff |\boldsymbol{s}| \leq s_*$$

$$\boldsymbol{v_\tau} \neq \boldsymbol{0} \iff \boldsymbol{s} = s_* \frac{\boldsymbol{v_\tau}}{|\boldsymbol{v_\tau}|} + \gamma_* \boldsymbol{v_\tau}$$

$$\boxed{\boldsymbol{v_\tau} = \frac{1}{\gamma_*}\frac{(|\boldsymbol{s}| - s_*)^+}{|\boldsymbol{s}|}\boldsymbol{s}}$$

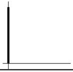
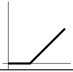
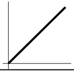
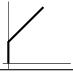
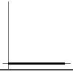
stick-slip

$$\boldsymbol{s} = \boldsymbol{0} \iff |\boldsymbol{v_\tau}| \leq v_*$$

$$\boldsymbol{s} \neq \boldsymbol{0} \iff \boldsymbol{v_\tau} = v_* \frac{\boldsymbol{s}}{|\boldsymbol{s}|} + \frac{1}{\gamma_*}\boldsymbol{v_\tau}$$

$$\boxed{\boldsymbol{s} = \gamma_* \frac{(|\boldsymbol{v_\tau}| - v_*)^+}{|\boldsymbol{v_\tau}|}\boldsymbol{v_\tau}}$$

slip/Navier's slip

		no-slip 
slip/Navier's slip 	Navier's slip 	stick-slip 
slip 		
$\lvert \boldsymbol{v}_{\boldsymbol{\tau}} \rvert \leq \delta_* \iff \boldsymbol{s} = \mathbf{0}$	no activation	$\lvert \boldsymbol{s} \rvert \leq \sigma_* \iff \boldsymbol{v}_{\boldsymbol{\tau}} = \mathbf{0}$

Summary of systematic classification of boundary conditions
with corresponding $|\boldsymbol{s}|$ vs $|\boldsymbol{v}_{\boldsymbol{\tau}}|$ diagrams.

Section 3

Is the developed framework useful?

Importance of and $\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{O}$ and $\mathbb{G}(\mathbb{T}, \mathbb{L}) = \mathbb{O}$

NAVIER-STOKES FLUID can not describe several phenomena that have been observed and documented experimentally:

- **shear thinning, shear thickening** - ν_g depends on $|\mathbb{D}|^2$ and/or $|\mathbb{S}|^2$
- **pressure thickening** - ν_g depends on p
- **the presence of activation or deactivation criteria** - "jump" singularities
- **the presence of the normal stress differences at simple shear flows**
- stress relaxation
- non-linear creep
- responses of anisotropic fluids
- thixotropy

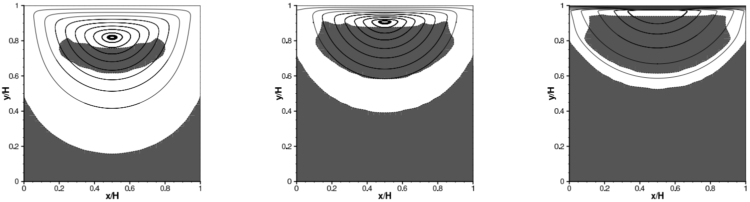
$\mathbb{G}(\mathbb{T}, \mathbb{L}) = \mathbb{O}$ has potential to describe four of them - rich structure.

Models connected with names like Ostwald (1925), de Waele (1923), Carreau (1972), Yasuda (1979), Eyring (1958), Cross (1965), Sisko (1958), Matsuhisa and Bird (1965), Glen (1955), Blatter (1995), Barus (1893), Bingham (1922) etc.

Lid driven cavity with Bingham fluid (J. Hron, J. Málek J. Stebel, K. Touška) (2016)

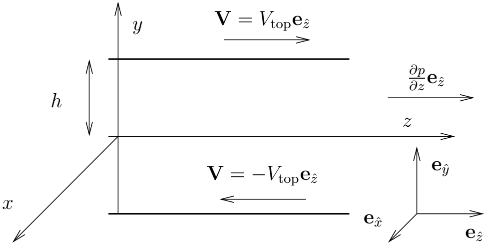
- Unknowns $(\boldsymbol{v}, p, \mathbb{S})$:

$$-\operatorname{div}\mathbb{S} = -\nabla p + \mathbf{b}$$
$$\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{O}$$
$$\mathbb{D}(\boldsymbol{v}) = \mathbb{D} \qquad \text{improves convergence for larger } \tau_*$$

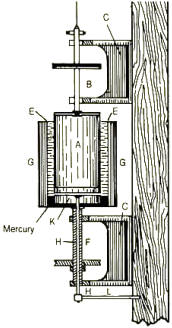


$\tau^* = 5$ $\tau^* = 50$ $\tau^* = 500$

D. Vola, L. Boscardin, J.C. Latché: Laminar unsteady flows of Bingham fluids: a numerical strategy and some benchmark results, 2003.



$\mathbf{V} = V_{\text{top}}\mathbf{e}_{\hat{z}}$
y
h
$\frac{\partial p}{\partial z}\mathbf{e}_{\hat{z}}$
z
$\mathbf{e}_{\hat{y}}$
$\mathbf{V} = -V_{\text{top}}\mathbf{e}_{\hat{z}}$
x
$\mathbf{e}_{\hat{x}}$
$\mathbf{e}_{\hat{z}}$



C
B
E
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G
C
Mercury
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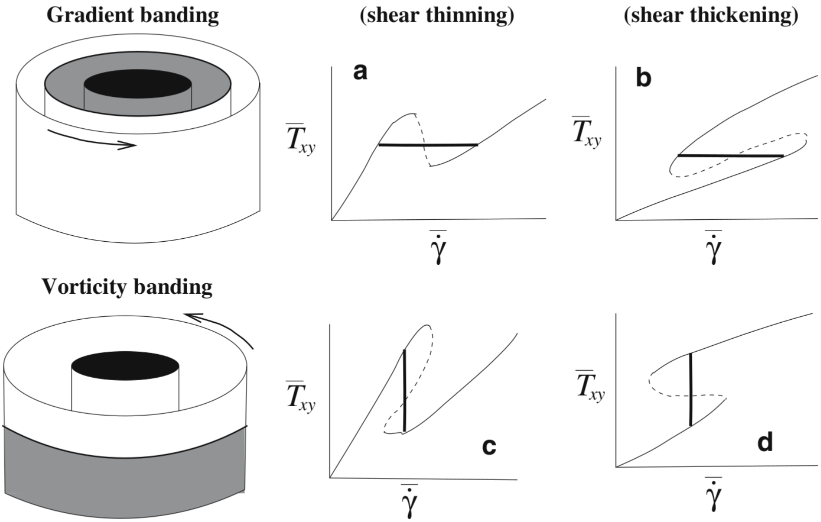
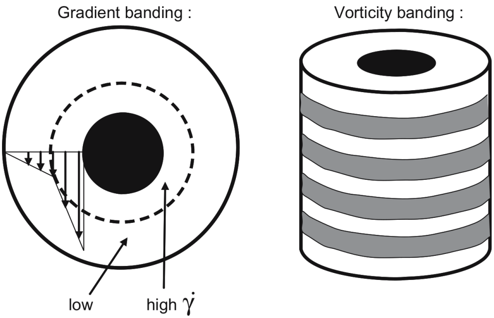
Nonmonotone response



P.D. Olmstedt: Perspectives on shear banding in complex fluids, *Rheol. Acta*, Vol. 47, pp. 283–300 (2008)

Nonmonotone response – gradient and vorticity banding

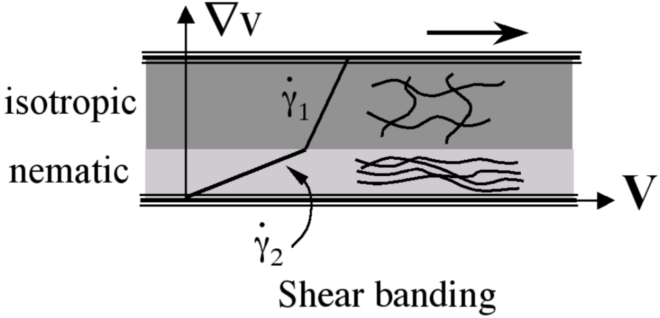


J.F. Berret: Rheology of wormlike micelles - Perspectives on shear banding in complex fluids, In R.G. Weiss and P.Terech (eds.) *Molecular gels*, pp. 567–720 (2006)

J.K.G. Dhont and W. J. Briels: Gradient and vorticity banding, *Rheol. Acta*, Vol. 47, pp. 257–281 (2008)

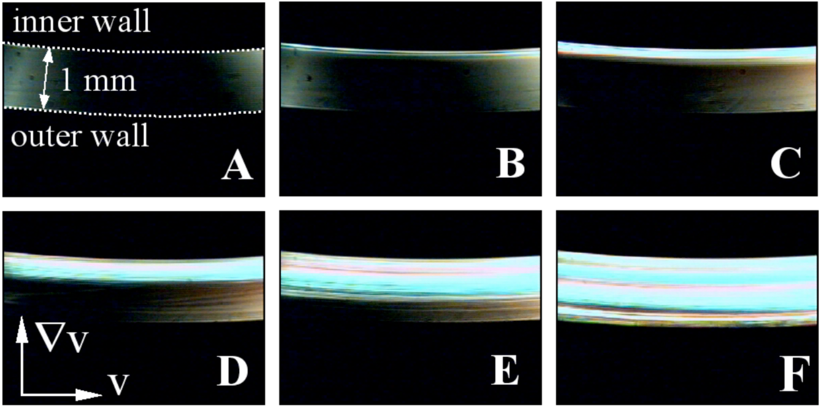
Nonmonotone response – gradient and vorticity banding

Equillibrium properties and shear banding transitions



J.F. Berret: Rheology of wormlike micelles - Perspectives on shear banding in complex fluids, In R.G. Weiss and P.Terech (eds.) *Molecular gels*, pp. 567–720 (2006)

Nonmonotone response – gradient and vorticity banding

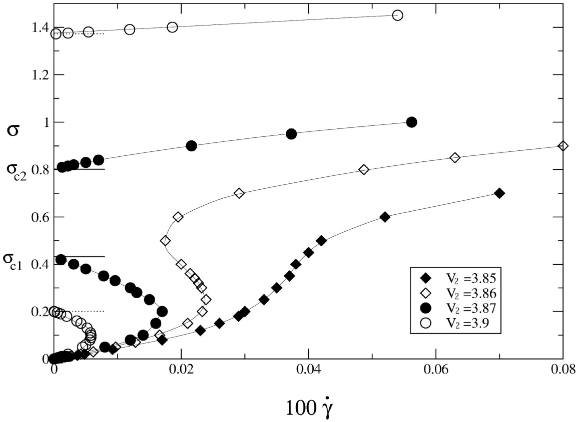


J.F. Berret: Rheology of wormlike micelles - Perspectives on shear banding in complex fluids, In R.G. Weiss and P.Terech (eds.) *Molecular gels*, pp. 567–720 (2006)

J. Málek, V. Pruša, G. Tierra: Numerical scheme for simulation of transient flows of non-Newtonian fluids characterized by a non-monotone relation between $\mathbb{D}$ and $\mathbb{S}$, in preparation (2016)

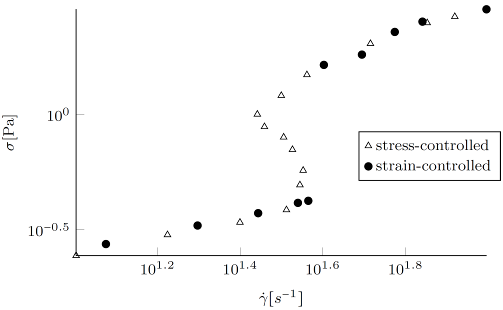
Experimental data for colloidal suspensions

Can one describe such non-monotone response of fluid-like materials?



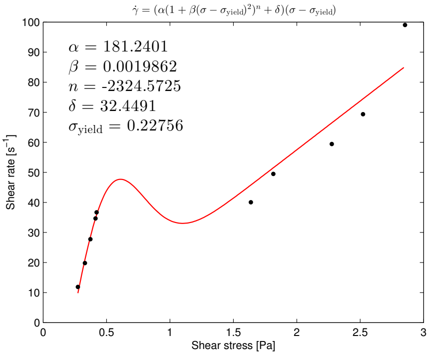
C. B. Holmes, M. E. Cates, M. Fuchs, P. Sollich: Glass transitions and shear thickening suspension rheology, *J. Rheology*, Vol. 49, pp. 237–269 (2005)

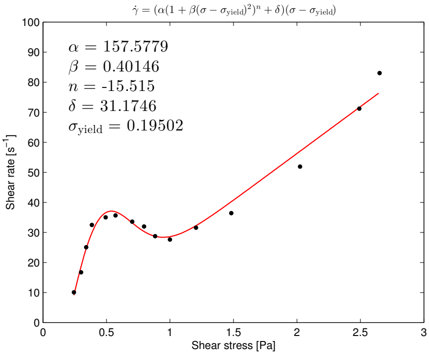
Stress-controlled and strain-controlled data



Tris (2-hydroxyethyl) ammonium acetate (TTAA) surfactant dissolved in water with addition of sodium salicylate (NaSal)

P. Boltenhagen, Y. Hu, E.F. Mathys, D.J. Pine: Observation of bulk phase separation and coexistence in asheared micellar solution , *Phys. Rev. Lett.*, Vol. 79, pp. 2359–2362 (1997)





T. Perlácová, V. Průša: Tensorial implicit constitutive relations in mechanics of incompressible non-Newtonian fluids , *J. Non-Newton. Fluid Mech.*, Vol. 216, pp. 13–21 (2015)

A.Janečka, V. Průša: Perspectives on using implicit type constitutive relations in the modelling of the behavior of non-newtonian fluids, *AIP Conference Proceedings*, Vol. 1662 (2015)

Summary

1. Some fluids exhibit new qualitative phenomena (shear banding, vorticity banding).
2. Experimental data can be explained by nonmonotone shear stress/shear rate relation.
3. The framework of implicit constitutive relations seems suitable to described fluids with activation
4. A new way to look at the problems from perspective of PDE analysis and numerical simulations

Activated fluids: continuum description, analysis and computational results
II. A continuum thermodynamic approach

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and

Mathematical Institute of Charles University in Prague

Faculty of Mathematics and Physics

May 23, 2016, University of Maryland

A Modern Thermodynamic Approach to Constitutive Theory

Classical equilibrium thermodynamics

- $E = E(S, V)$
- $T =_{\mathrm{def}} \frac{\partial E}{\partial S}$, $P =_{\mathrm{def}} -\frac{\partial E}{\partial V}$
- $\mathrm{d}S \geq \frac{\mathrm{d}Q}{T}$ or $\mathrm{d}S = \frac{\mathrm{d}Q}{T}$ for reversible processes

Continuum mechanics equilibrium thermodynamics

- $e = e(\eta, \rho)$
- $\theta =_{\mathrm{def}} \frac{\partial e}{\partial \eta}$, $p_{\mathrm{th}} =_{\mathrm{def}} -\frac{\partial e}{\partial \left(\frac{1}{\rho}\right)} = \rho^2 \frac{\partial e}{\partial \rho}$
- $\rho \dot{\eta} + \operatorname{div}\left(\frac{\mathbf{j}_\eta}{\theta}\right) \geq 0$

$$\boxed{\rho \xi =_{\mathrm{def}} \rho \dot{\eta} + \operatorname{div}\left(\frac{\mathbf{j}_\eta}{\theta}\right) \geq 0 \quad \text{and} \quad \xi \geq 0}$$

Navier–Stokes–Fourier Fluid

- $e = e(\eta, \rho) \Longrightarrow \rho \dot{e} = \rho \underbrace{\frac{\partial e}{\partial \eta}}_{\theta} \dot{\eta} + \rho \underbrace{\frac{\partial e}{\partial \rho}}_{\frac{p_{\mathrm{th}}}{\rho}} \dot{\rho}$
- Use the balance equations

$$\begin{aligned} \rho\theta\dot{\eta} &= \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_e + p_{\mathrm{th}} \operatorname{div} \mathbf{v} \\ &= \mathbb{S} : \mathbb{D}_\delta - \operatorname{div} \mathbf{j}_e + (m + p_{\mathrm{th}}) \operatorname{div} \mathbf{v} \end{aligned}$$

$$\boxed{\rho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{j}_e}{\theta}\right) = \frac{1}{\theta}\left[\mathbb{S} : \mathbb{D}_\delta + (m + p_{\mathrm{th}}) \operatorname{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right]}$$

$$\rho\theta\xi =_{\mathrm{def}} \left[\mathbb{S} : \mathbb{D}_\delta + (m + p_{\mathrm{th}}) \operatorname{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right] > 0$$

Navier–Stokes–Fourier Fluid

$$\boxed{\rho\theta\xi =_{\text{def}} \left[\mathbb{S} : \mathbb{D}_\delta + (m + p_{\text{th}})\,\text{div}\,\mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right] > 0}$$

$$\mathbb{S} = 2\nu\mathbb{D}_\delta, \qquad \nu > 0 \qquad \text{(a)}$$

$$m + p_{\text{th}} = \tilde{\lambda}\,\text{div}\,\mathbf{v}, \qquad \tilde{\lambda} > 0 \qquad \text{(b)}$$

$$\mathbf{j}_e = -k\nabla\theta, \qquad k > 0 \qquad \text{(c)}$$

From (a) and (b) follows:

$$\begin{aligned}\mathbb{T} = \mathbb{S} + m\mathbb{I} &= 2\nu\mathbb{D}_\delta + \left(\tilde{\lambda}\,\text{div}\,\mathbf{v} - p_{\text{th}}\right)\mathbb{I} \\ &= 2\nu\mathbb{D} - p_{\text{th}}\mathbb{I} + \left(\tilde{\lambda} - \frac{2\nu}{3}\right)(\text{div}\,\mathbf{v})\mathbb{I} \\ &= -p_{\text{th}}\mathbb{I} + 2\nu\mathbb{D} + \lambda(\text{div}\,\mathbf{v})\mathbb{I}\end{aligned}$$

$\lambda =_{\text{def}} \tilde{\lambda} - \frac{2\nu}{3} \iff \tilde{\lambda} = \frac{2\nu+3\lambda}{3}$

General Thermodynamic Framework

1. $\boxed{e = e(\eta, y_1, \dots, y_n)}$ is increasing function w.r.t. η
2. $$\boxed{\rho\dot{e} = \rho\frac{\partial e}{\partial \eta}\dot{\eta} + \rho\sum_j \frac{\partial e}{\partial y_j}\dot{y}_j}$$
   We need to know $\dot{y}_j$ from balance equations or kinematics
3. $\theta = \frac{\partial e}{\partial \eta} > 0$
4. $\rho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{j}_\eta}{\theta}\right) = s_\eta$, where $s_\eta = \frac{1}{\theta}\sum_\alpha J_\alpha A_\alpha$
   each $J_\alpha A_\alpha$ represents independent dissipative mechanism
5. Identify s_η with $\rho\xi$

$$\boxed{\rho\xi =_{\text{def}} \frac{1}{\theta}\sum_\alpha J_\alpha A_\alpha \quad \text{and} \quad \xi \geq 0} \tag{1}$$

6.
   1. Linear non-equilibrium thermodynamics: $J_\alpha = \gamma_\alpha A_\alpha$, $\gamma_\alpha > 0$
   2. Non-linear non-equilibrium thermodynamics: specification of constitutive equation for ξ and its maximization with the constraint (1)

K. R. Rajagopal, A. R. Srinivasa: On thermomechanical restrictions of continua, *Proc. R. Soc. Lond. A* Vol. 460, pp. 631–651 (2004).

J. Málek, V. Průša: Derivation of equations for continuum mechanics and thermodynamics of fluids, *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids (eds. Y. Giga, A. Novotný)*, submitted (2015).

Korteweg–NSF Fluid

- Korteweg (1901)

$$\mathbb{T} = -p\mathbb{I} + 2\nu(\rho)\mathbb{D} + \Big(\lambda(\rho)\,\mathrm{div}\,\mathbf{v} + \alpha(\rho)\,|\nabla\rho|^2 + \beta(\rho)\Delta\rho\Big)\,\mathbb{I} + \gamma(\rho)\nabla\rho\otimes\nabla\rho$$

- **Q:** Is this model compatible with 2nd law of thermodynamics?
- **Q:** How to extend this model to include thermal processes?

$$\boxed{e = e_{\mathrm{NSF}}(\eta,\rho) + \frac{\sigma}{2\rho}\,|\nabla\rho|^2} \qquad \dot{\overline{\nabla\rho}} = -\nabla\left(\rho\,\mathrm{div}\,\mathbf{v}\right) + [\nabla\mathbf{v}]^{\top}\,\nabla\rho$$

$$\begin{aligned}\rho\xi =& \frac{1}{\theta}\Big[\left(\mathbb{T}_\delta - \sigma\left(\nabla\rho\otimes\nabla\rho\right)_\delta\right) : \mathbb{D}_\delta \\ &+ \left(m + p_{\mathrm{th}}^{\mathrm{K}} - \frac{\sigma}{3}\,|\nabla\rho|^2 - \sigma\rho\Delta\rho + (1-\delta)\sigma\rho\left(\nabla\rho\right)\cdot\frac{\nabla\theta}{\theta}\right)\mathrm{div}\,\mathbf{v} \\ &- \left(\mathbf{j}_e - \delta\sigma\rho(\mathrm{div}\,\mathbf{v})\nabla\rho\right)\cdot\frac{\nabla\theta}{\theta}\Big]\end{aligned}$$

$p_{\mathrm{th}}^{\mathrm{K}} = p_{\mathrm{th}}^{\mathrm{NSF}} - \frac{\sigma}{2}\,|\nabla\rho|^2$, $\delta\in[0,1]$

Rate Type Fluid Models

- Popular class of phenomenological models in visco-elasticity
- Broad applications of visco-elasticity:
  - Bio-mechanics (soft tissues, bio-fluids)
  - Polymer industry, glass technology
  - Food industry
  - Geo-mechanics (Earth's mantle, tectonic plates, glacier, soil)
- Derivation of complete 3D models that are consistent with the second law of thermodynamics is very recent

K. R. Rajagopal, A. R. Srinivasa: A thermodynamic frame work for rate type fluid models, *J. Non-Newton. Fluid*, Vol. 88, pp. 207–227 (2000).

J. Málek, K. R. Rajagopal, K. Tůma: On a variant of the Maxwell and Oldroyd-B models within the context of a thermodynamic basis, *Int. J. Non-Linear Mech.*, Vol. 76, pp. 42–47 (2015).

J. Málek, V. Průša: Derivation of equations for continuum mechanics and thermodynamics of fluids, *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids (eds. Y. Giga, A. Novotný)*, submitted (2015).

Asphalt binders

- Widely used
- Microstructure and chemistry are not well understood $\Longrightarrow$ macroscopic decription is the only possible choice
- An example of a complex material with complicated microstructure exhibiting—with clear evidence—visco-elastic phenomena (stress relaxation, non-linear creep, normal stress differences) $\Longrightarrow$ their response cannot be described by standard models
- Good access to available experimental data

J. M. Krishnan, K. R. Rajagopal: On the mechanical behavior of asphalt, *Mech. Mater.*, Vol. 37, pp. 1085–1100 (2005).

Asphalt binder

- Glue in the asphalt concrete (very sticky)
- Almost incompressible (compared to asphalt concrete)
- Mixture of a large number of hydrocarbons
- Exhibits viscoelastic behavior

Solid- and Fluid-Like Materials

Year	Event
1930	Plug trimmed off
1938	1st drop
1954	3rd drop
1970	5th drop
1988	7th drop
2000	8th drop
2014	9th drop

Incompressible Rate-Type Fluid Models

- Balance equations for compressible fluids

$$\boxed{\begin{aligned} \dot{\rho} &= -\rho \,\mathrm{div}\, \mathbf{v} \\ \rho\dot{\mathbf{v}} &= \mathrm{div}\, \mathbb{T}, \quad \mathbb{T} = \mathbb{T}^\top \end{aligned}}$$

- Balance equations for incompressible fluids

$$\boxed{\begin{aligned} &\mathrm{div}\, \mathbf{v} = 0 \\ &\rho_* \dot{\mathbf{v}} = \mathrm{div}\, \mathbb{T}_\delta + \nabla m, \quad \mathbb{T}_\delta = \mathbb{T}_\delta^\top \end{aligned}}$$

- **Goal:** To find an additional evolution equation for a part of the stress

Standard Viscoelastic Rate-Type Fluid Models

Cauchy stress $\mathbb{T} = -p\mathbb{I} + \mathbb{S}$, $\boxed{\overset{\triangledown}{\mathbb{S}} =_{\text{def}} \frac{\mathrm{d}\mathbb{S}}{\mathrm{d}t} - \mathbb{L}\mathbb{S} - \mathbb{S}\mathbb{L}^{\top}}$, $\mathbb{L} =_{\text{def}} \nabla \mathbf{v}$

- Maxwell (1867)

$$\boxed{\mathbb{S} + \lambda\overset{\triangledown}{\mathbb{S}} = 2\mu\mathbb{D}}$$

- Oldroyd-B (1950)

$$\boxed{\mathbb{S} + \lambda\overset{\triangledown}{\mathbb{S}} = 2\eta_1\mathbb{D} + 2\eta_2\overset{\triangledown}{\mathbb{D}}}$$

- Burgers (1939)

$$\boxed{\mathbb{S} + \lambda_1\overset{\triangledown}{\mathbb{S}} + \lambda_2\overset{\triangledown\triangledown}{\mathbb{S}} = 2\eta_1\mathbb{D} + 2\eta_2\overset{\triangledown}{\mathbb{D}}}$$

- Giesekus (1982)

$$\boxed{\mathbb{S} + \lambda_1\overset{\triangledown}{\mathbb{S}} - \frac{\alpha\lambda_2}{\mu}\mathbb{S}^2 = -2\mu\mathbb{D}}$$

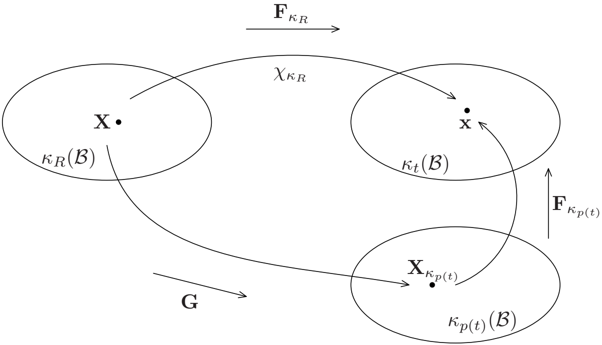
- Models due to Phan-Thien–Tanner (1977), Johnson–Segelman (1977), White–Metzer (1977), etc.

Basic Questions

- **Q:** Are these models compatible with 2nd law of thermodynamics?
- **Q:** How to extend these models to include thermal processes?

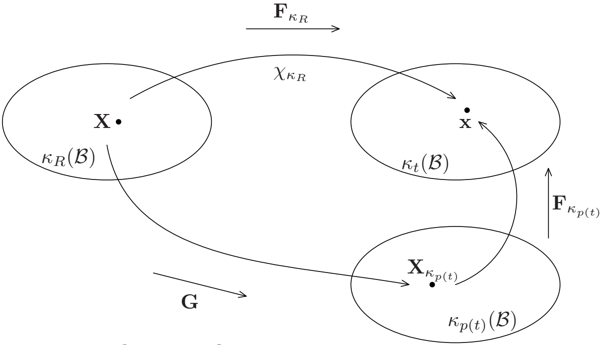
Natural Configuration

- Deformation gradient $\mathbb{F} =_{\text{def}} \mathbb{F}_{\kappa_R}$ is split into the **elastic** and the **dissipative** part: $\mathbb{F}_{\kappa_{p(t)}}$ and $\mathbb{G}$



- $\boxed{\mathbb{F} = \mathbb{F}_{\kappa_{p(t)}} \mathbb{G}}$

Natural Configuration Kinematics



- Left and right Cauchy–Green tensors:

$$\mathbb{B}_{\kappa_p(t)} =_{\text{def}} \mathbb{F}_{\kappa_p(t)}\mathbb{F}^{\top}_{\kappa_p(t)}, \quad \mathbb{C}_{\kappa_p(t)} =_{\text{def}} \mathbb{F}^{\top}_{\kappa_p(t)}\mathbb{F}_{\kappa_p(t)}$$

- $\mathbb{L}_{\kappa_{p(t)}} =_{\text{def}} \dot{\mathbb{G}}\mathbb{G}^{-1}, \mathbb{D}_{\kappa_{p(t)}} =_{\text{def}} \frac{1}{2}\left(\mathbb{L}_{\kappa_{p(t)}} + \mathbb{L}^{\top}_{\kappa_{p(t)}}\right)$

$$\dot{\mathbb{B}}_{\kappa_{p(t)}} = \mathbb{L}\mathbb{B}_{\kappa_{p(t)}} + \mathbb{B}_{\kappa_{p(t)}}\mathbb{L}^{\top} - 2\mathbb{F}_{\kappa_{p(t)}}\mathbb{D}_{\kappa_{p(t)}}\mathbb{F}^{\top}_{\kappa_{p(t)}} \implies$$

$$\overset{\triangledown}{\mathbb{B}}_{\kappa_{p(t)}} = -2\mathbb{F}_{\kappa_{p(t)}}\mathbb{D}_{\kappa_{p(t)}}\mathbb{F}^{\top}_{\kappa_{p(t)}}$$

Constitutive equations

- **Internal energy e for compressible neo-Hookean solid**

$$e = e_{\mathrm{NSF}}(\eta, \rho) + \frac{\mu}{2\rho}\left(\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 3 - \log\det\mathbb{B}_{\kappa_{p(t)}}\right)$$

$$\rho\xi = \frac{1}{\theta}\Bigg[\left(\mathbb{T} - \mu\mathbb{B}_{\kappa_{p(t)}}\right)_\delta : \mathbb{D}_\delta + \mu\left(\mathbb{C}_{\kappa_p(t)} - \mathbb{I}\right) : \mathbb{D}_{\kappa_{p(t)}} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}$$
$$+ \left(m + p_{\mathrm{th}}^{\mathrm{M}} - \mu\left(\frac{1}{3}\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 1\right)\right)\operatorname{div}\mathbf{v}\Bigg]$$

$$p_{\mathrm{th}}^{\mathrm{M}} =_{\mathrm{def}} p_{\mathrm{th}}^{\mathrm{NSF}} - \tfrac{\mu}{2}\left(\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 3 - \log\det\mathbb{B}_{\kappa_{p(t)}}\right)$$

- **Linearity**

$$\left(\mathbb{T} - \mu\mathbb{B}_{\kappa_{p(t)}}\right)_\delta = 2\nu\mathbb{D}_\delta, \qquad \nu > 0$$

$$m + p_{\mathrm{th}}^{\mathrm{M}} - \mu\left(\frac{1}{3}\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 1\right) = \frac{2\nu + 3\lambda}{3}\operatorname{div}\mathbf{v}, \qquad 2\nu + 3\lambda > 0$$

$$\mu\left(\mathbb{C}_{\kappa_p(t)} - \mathbb{I}\right) = 2\nu_1\mathbb{D}_{\kappa_{p(t)}}, \qquad \nu_1 > 0$$

$$\mathbf{j}_e = -k\nabla\theta, \qquad k > 0$$

Compressible Giesekus Fluid

❶

$$\left(\mathbb{T} - \mu \mathbb{B}_{\kappa_{p(t)}}\right)_\delta = 2\nu \mathbb{D}_\delta$$

$$m + p_{\mathrm{th}}^{\mathrm{M}} - \mu \left(\frac{1}{3} \operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}} - 1\right) = \frac{2\nu + 3\lambda}{3} \operatorname{div} \mathbf{v}$$

imply

$$\boxed{\mathbb{T} = \mathbb{T}_\delta + m\mathbb{I} = -p_{\mathrm{th}}^{\mathrm{M}} \mathbb{I} + 2\nu \mathbb{D} + \lambda (\operatorname{div} \mathbf{v})\, \mathbb{I} + \mu \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right)}$$

❷ $\mu \left(\mathbb{C}_{\kappa_p(t)} - \mathbb{I}\right) = 2\nu_1 \mathbb{D}_{\kappa_{p(t)}}$ and $\overset{\triangledown}{\mathbb{B}}_{\kappa_{p(t)}} = -2\mathbb{F}_{\kappa_{p(t)}} \mathbb{D}_{\kappa_{p(t)}} \mathbb{F}_{\kappa_{p(t)}}^\top$
imply

$$\boxed{\mu \mathbb{B}_{\kappa_{p(t)}}^2 - \mu \mathbb{B}_{\kappa_{p(t)}} = \nu_1 \overset{\triangledown}{\mathbb{B}}_{\kappa_{p(t)}}}$$

Incompressible Giesekus Fluid

- **Internal energy e for compressible neo-Hookean solid**

$$e = e_{\mathrm{NSF}}(\eta, \rho) + \frac{\mu}{2\rho}\left(\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 3 - \log\det\mathbb{B}_{\kappa_{p(t)}}\right)$$

$$\rho\xi = \frac{1}{\theta}\left[\left(\mathbb{T} - \mu\mathbb{B}_{\kappa_{p(t)}}\right)_\delta : \mathbb{D}_\delta + \mu\left(\mathbb{C}_{\kappa_p(t)} - \mathbb{I}\right) : \mathbb{D}_{\kappa_{p(t)}} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right]$$

- **Linearity** implies

$$\mathbb{T} = \mathbb{T}_\delta + m\mathbb{I} = m\mathbb{I} + 2\nu\mathbb{D} + \mu\left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right)$$

- $\mu\left(\mathbb{C}_{\kappa_p(t)} - \mathbb{I}\right) = 2\nu_1\mathbb{D}_{\kappa_{p(t)}}$ and $\overset{\triangledown}{\mathbb{B}}_{\kappa_{p(t)}} = -2\mathbb{F}_{\kappa_{p(t)}}\mathbb{D}_{\kappa_{p(t)}}\mathbb{F}^{\top}_{\kappa_{p(t)}}$ imply

$$\mu\mathbb{B}^2_{\kappa_{p(t)}} - \mu\mathbb{B}_{\kappa_{p(t)}} = \nu_1\overset{\triangledown}{\mathbb{B}}_{\kappa_{p(t)}}$$

Compressible Maxwell and Oldroyd-B Fluid

- **Compressible Maxwell fluid**
  - **Internal energy e**

$$e = e_{\mathrm{NSF}}(\eta, \rho) + \frac{\mu}{2\rho}\left(\mathrm{Tr}\,\mathbb{B}_{\kappa_{p(t)}} - 3 - \log\det\mathbb{B}_{\kappa_{p(t)}}\right)$$

  - **Rate of entropy production ξ:**

$$\xi = 2\mu_1 \mathbb{D}_{\kappa_p(t)} : \mathbb{C}_{\kappa_p(t)} \mathbb{D}_{\kappa_p(t)} \geq 0$$

- **Compressible Oldroyd-B fluid**
  - **Internal energy e**

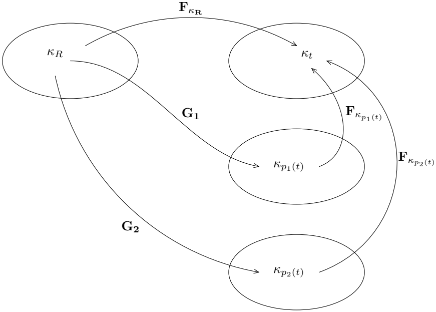
$$e = e_{\mathrm{NSF}}(\eta, \rho) + \frac{\mu}{2\rho}\left(\mathrm{Tr}\,\mathbb{B}_{\kappa_{p(t)}} - 3 - \log\det\mathbb{B}_{\kappa_{p(t)}}\right)$$

  - **Rate of entropy production ξ:**

$$\xi = 2\mu_1 \mathbb{D}_{\kappa_p(t)} : \mathbb{C}_{\kappa_p(t)} \mathbb{D}_{\kappa_p(t)} + 2\mu_2 \mathbb{D} : \mathbb{D} \geq 0$$

Burgers Fluid

$$\boxed{\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \lambda_2 \overset{\triangledown\triangledown}{\mathbb{S}} = 2\eta_1 \mathbb{D} + 2\eta_2 \overset{\triangledown}{\mathbb{D}}}$$



Summary

- An important step towards analysis of initial and boundary value problems (a priori estimates) – specifying the object for relevant computer simulations
- Material coefficients may, in general, depend on state variables
- Compressible and incompressible Navier–Stokes–Fourier (NSF) fluids, Korteweg NSF fluids, Rate type fluids

K. R. Rajagopal, A. R. Srinivasa: A thermodynamic frame work for rate type fluid models, *J. Non-Newton. Fluid*, Vol. 88, pp. 207–227 (2000).

M. Heida, J. Málek: On compressible Korteweg fluid-like materials, *Int. J. Eng. Sci.*, Vol. 48, pp. 1313–1324 (2010).

J. Málek, V. Průša: Derivation of equations for continuum mechanics and thermodynamics of fluids, *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids (eds. Y. Giga, A. Novotný)*, submitted (2015).

Summary

- Cahn–Hilliard NSF fluids

M. Heida, J. Málek, K. R. Rajagopal: On the development and generalizations of Cahn–Hilliard equations within a thermodynamic framework, *Z. Angew. Math. Phys.*, Vol. 63, pp. 145–169 (2012).

- Allen–Cahn NSF fluids

M. Heida, J. Málek, K. R. Rajagopal: On the development and generalizations of Allen–Cahn and Stefan equations within a thermodynamic framework, *Z. Angew. Math. Phys.*, Vol. 63, pp. 759–776 (2012).

- Binary mixtures with and without chemical reactions

O. Souček, V. Průša, J. Málek, K. R. Rajagopal: On the natural structure of thermodynamic potentials and fluxes in the theory of chemically non-reacting binary mixtures, *Acta Mech.*, Vol. 225, pp. 3157–3186 (2014).

Activated fluids: continuum description, analysis and computational results

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Faculty of Mathematics and Physics

May 24, 2016

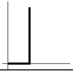
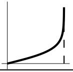
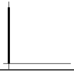
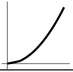
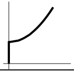
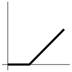
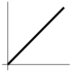
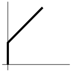
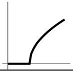
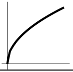
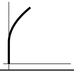
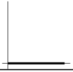
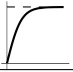
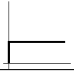
MINISTRY OF EDUCATION, YOUTH AND SPORTS

ERC-CZ project LL1202 - MORE
Implicitly constituted material models: from theory through model reduction to efficient numerical methods
http://more.karlin.mff.cuni.cz/

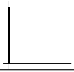
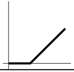
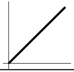
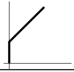
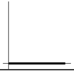
Contents

Part #1

Incompressible fluids and boundary conditions with activation

Euler/limiting shear-rate 	limiting shear-rate 	rigid body 
Euler/shear-thickening 	shear-thickening 	rigid/shear-thickening 
Euler/Navier-Stokes 	Navier-Stokes 	Bingham = rigid/Navier-Stokes 
Euler/shear-thinning 	shear-thinning 	rigid/shear-thinning 
Euler 	limiting shear stress 	perfect plastic 
$\lvert\mathbb{D}\rvert \leq \delta_* \iff \mathbb{S} = \mathbb{O}$	no activation	$\lvert\mathbb{S}\rvert \leq \sigma_* \iff \mathbb{D} = \mathbb{O}$

Summary of systematic classification of fluid-like responses with corresponding $\lvert\mathbb{S}\rvert$ vs $\lvert\mathbb{D}\rvert$ diagrams.

		no-slip 
slip/Navier's slip 	Navier's slip 	stick-slip 
slip 		
$\lvert \boldsymbol{v_\tau} \rvert \le \delta_* \iff \boldsymbol{s} = \boldsymbol{0}$	no activation	$\lvert \boldsymbol{s} \rvert \le \sigma_* \iff \boldsymbol{v_\tau} = \boldsymbol{0}$

Summary of systematic classification of boundary conditions with corresponding $\lvert \boldsymbol{s} \rvert$ vs $\lvert \boldsymbol{v_\tau} \rvert$ diagrams.

Formulation of the problem

PROBLEM

$$
\left.\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \\
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbb{S} &= -\nabla p + \boldsymbol{b} \\
\mathbb{G}(\mathbb{S}, \mathbb{D}) &= \mathbb{O}
\end{aligned}\right\} \text{ in } Q_T
$$

$$
\left.\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\
\boldsymbol{s} := -(\mathbb{S}\boldsymbol{n})_{\boldsymbol{\tau}} \qquad \boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_{\boldsymbol{\tau}}) &= \boldsymbol{0}
\end{aligned}\right\} \text{ on } \Sigma_T
$$

$$
\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \qquad \text{in } \Omega
$$

DATA

- $\Omega \subset \mathbb{R}^d$ bounded, open set with $\partial\Omega \in \mathcal{C}^{1,1}$ and $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^d$
- $T > 0$ and $Q_T := (0, T) \times \Omega$, $\Sigma_T := (0, T) \times \partial\Omega$
- $\boldsymbol{v}_0$, $\boldsymbol{b}$
- $\mathbb{G}$ and $\boldsymbol{g}$ - constitutive functions in the bulk and on the boundary

Main questions addressed

UNKNOWNS triplet $(\boldsymbol{v}, p, \mathbb{S})$ defined on Q_T and $\boldsymbol{s}$ defined on Σ_T

$$\left.\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbb{S} &= -\nabla p + \boldsymbol{b} \\ \mathbb{G}(\mathbb{S}, \mathbb{D}) &= \mathbb{O} \end{aligned}\right\} \text{ in } Q_T$$

$$\left.\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\ \boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_{\boldsymbol{\tau}}) &= \boldsymbol{0} \end{aligned}\right\} \text{ on } \Sigma_T$$

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \qquad \text{in } \Omega$$

AIM

- To establish large data existence of solution for any set of data (Ω, T, $\boldsymbol{v}_0$, $\boldsymbol{b}$) and for robust class of constitutive equations described by $\mathbb{G}$ and $\boldsymbol{g}$
- To develop a theory with $p \in L^1(Q_T)$ - important
  - heat-conducting incompressible fluids (M. Bulíček, E. Feireisl - G. Schimperna)
  - one/two equation turbulence model (M. Bulíček, R. Lewandowski)
  - incompressible fluids with pressure and shear-rate dependent viscosity (J. Nečas, KR Rajagopal, M. Bulíček, M. Majdoub, A. Hirn, J. Stebel, M. Lanzendörfer, ...)
  - corresponding numerical methods and their analysis

Theoretical results

- Existence of WS to NSEs in 2d and 3d (Leray (1929-1934), Oseen (1922))
- Existence of WS to NSEs in bounded domains, its 2d uniqueness and 3d *conditional* uniqueness and existence (Hopf (1952), Kiselev & Ladyzhenkaya (1959), Prodi (1959), Serrin (1963))
- Existence of WS to $\mathbb{S} = 2(\nu_0 + \nu_1|\mathbb{D}|^{r-2})\mathbb{D}$ for $r \geq \frac{11}{5}$ and its uniqueness if $r \geq \frac{5}{2}$ (Ladyzhenskaya (1967-1972), J.-L. Lions (1969))
  - Nečas, Bellout, Bloom, Málek, Růžička (1993-2000): $r \geq \frac{9}{5}$
  - DalMaso, Murat (1996), Frehse, Málek, Steinhauer, Růžička (1996-2000), Bulíček, Málek, Rajagopal (2007), Wolf (2009): $r \geq \frac{8}{5}$
  - Diening, Růžička, Wolf (2010), Breit, Diening, Schwarzacher (2015): $r > \frac{6}{5}$
  - Bulíček, Ettwein, Kaplický, Pražák (2010): uniqueness for $r > \frac{11}{5}$
- Existence of WS to monotone (rather than strictly monotone) response, Orlicz function-type response (Bulíček, Gwiazda, Málek, Świerczewska-Gwiazda (2012): $r > \frac{6}{5}$)
- Existence of WS to activated fluids with activated boundary conditions (Bulíček, Málek (2016): $r > \frac{6}{5}$)

Part #2

Structure of implicit relations

Basic information

A PRIORI ESTIMATES

Multiplying the 2nd Eq. by $\boldsymbol{v}$ ($\boldsymbol{b} \equiv 0$)

$$\frac{1}{2}\frac{\partial|\boldsymbol{v}|^2}{\partial t} + \operatorname{div}(\tfrac{1}{2}|\boldsymbol{v}|^2\boldsymbol{v}) - \operatorname{div}(\mathbb{S}\boldsymbol{v}) + \mathbb{S}\cdot\mathbb{D} = -\operatorname{div}(p\boldsymbol{v})$$

Since $\boldsymbol{v}\cdot\boldsymbol{n} = 0$, integrating it over Ω leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_2^2 + \int_\Omega \mathbb{S} : \mathbb{D}\, dx + \int_{\partial\Omega} \boldsymbol{s}\cdot\boldsymbol{v}_{\boldsymbol{\tau}}\, dS = 0$$

For the power-law fluids $\boxed{\mathbb{S} = |\mathbb{D}|^{r-2}\mathbb{D} \iff \mathbb{D} = |\mathbb{S}|^{r'-2}\mathbb{S} \qquad r' = r/(r-1)}$:

$$\mathbb{S} : \mathbb{D} = \left(\frac{1}{r} + \frac{1}{r'}\right)\mathbb{S} : \mathbb{D} = \frac{1}{r}|\mathbb{D}|^r + \frac{1}{r'}|\mathbb{S}|^{r'}$$

For Navier's slip $\boxed{\boldsymbol{s} = \gamma_*\boldsymbol{v}_{\boldsymbol{\tau}} \iff \boldsymbol{v}_{\boldsymbol{\tau}} = \frac{1}{\gamma_*}\boldsymbol{s}}$:

$$\boldsymbol{s}\cdot\boldsymbol{v}_{\boldsymbol{\tau}} = (\tfrac{1}{2} + \tfrac{1}{2})\boldsymbol{s}\cdot\boldsymbol{v}_{\boldsymbol{\tau}} = \frac{\gamma_*}{2}|\boldsymbol{v}_{\boldsymbol{\tau}}|^2 + \frac{1}{2\gamma_*}|\boldsymbol{s}|^2$$

Implicit constitutive equations in bulk - maximal monotone r-graph setting

Define

$$\boxed{(\mathbb{S},\mathbb{D})\in\mathcal{A}\quad\Longleftrightarrow\quad\mathbb{G}(\mathbb{S},\mathbb{D})=\mathbb{O}}$$

Assumptions - $\mathcal{A}$ is a maximal monotone r-graph, $\boxed{r\in(1,+\infty)}$

(A1) $(\mathbb{O},\mathbb{O})\in\mathcal{A}$

(A2) **Monotone graph:** For any $(\mathbb{S}_1,\mathbb{D}_1),(\mathbb{S}_2,\mathbb{D}_2)\in\mathcal{A}$

$$(\mathbb{S}_1-\mathbb{S}_2)\cdot(\mathbb{D}_1-\mathbb{D}_2)\geq 0$$

(A3) **Maximal monotone graph:** Let $(\mathbb{S}_*,\mathbb{D}_*)\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

$$\text{If}\quad(\mathbb{S}_*-\mathbb{S})\cdot(\mathbb{D}_*-\mathbb{D})\geq 0\quad\forall\,(\mathbb{S},\mathbb{D})\in\mathcal{A}\quad\text{then}\quad(\mathbb{S}_*,\mathbb{D}_*)\in\mathcal{A}$$

(A4) ***r*-graph:** There are $\alpha_*>0$ and $c_*\geq 0$ so that for any $(\mathbb{S},\mathbb{D})\in\mathcal{A}$

$$\mathbb{S}\cdot\mathbb{D}\geq\alpha_*\left(|\mathbb{D}|^r+|\mathbb{S}|^{r'}\right)-c_*$$

Implicit formulation of BCs - maximal monotone q-graph setting

Define

$$\boxed{(\boldsymbol{s}, \boldsymbol{v}_\tau) \in \mathcal{B} \iff \boldsymbol{g}(\boldsymbol{s}, \boldsymbol{v}_{\boldsymbol{\tau}}) = \boldsymbol{0}}$$

(B1) $\mathcal{B}$ **contains the origin.** $(\boldsymbol{0}, \boldsymbol{0}) \in \mathcal{B}$.

(B2) $\mathcal{B}$ **is a monotone graph.**

$$(\boldsymbol{s}_1 - \boldsymbol{s}_2) \cdot (\boldsymbol{v}_{\boldsymbol{\tau}}^1 - \boldsymbol{v}_{\boldsymbol{\tau}}^2) \geq 0 \qquad \text{for all } (\boldsymbol{s}_1, \boldsymbol{v}_{\boldsymbol{\tau}}^1), (\boldsymbol{s}_2, \boldsymbol{v}_{\boldsymbol{\tau}}^2) \in \mathcal{B}.$$

(B3) $\mathcal{B}$ **is a maximal monotone graph.** Let for some $(\boldsymbol{s}, \boldsymbol{u})$ holds:

$$\text{If} \quad (\bar{\boldsymbol{s}} - \boldsymbol{s}) \cdot (\bar{\boldsymbol{v}}_{\boldsymbol{\tau}} - \boldsymbol{u}) \geq 0 \quad \text{for all } (\bar{\boldsymbol{s}}, \bar{\boldsymbol{v}}_{\boldsymbol{\tau}}) \in \mathcal{B} \quad \text{then } (\boldsymbol{s}, \boldsymbol{u}) \in \mathcal{B}.$$

(B4) $\mathcal{B}$ **is a q-graph.** For any $q \in (1, \infty)$ fixed there are $\beta_* > 0$ and $d_* \geq 0$ such that

$$\boldsymbol{s} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} \geq \beta_*(|\boldsymbol{v}_{\boldsymbol{\tau}}|^q + |\boldsymbol{s}|^{q/(q-1)}) - d_* \qquad \text{for all } (\boldsymbol{s}, \boldsymbol{v}_{\boldsymbol{\tau}}) \in \mathcal{B}\,.$$

- No-slip boundary condition is excluded by **(B4)**
- For all our examples $q = 2$

Basic estimates

A PRIORI ESTIMATES REVISITED

Recall

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_2^2 + \int_\Omega \mathbb{S} : \mathbb{D}\, dx + \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}\, dS = 0$$

Using **(A4)** and **(B4)** and integrating the result from 0 to any $t \in (0, T]$:

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \alpha_* \int_0^t \|\mathbb{S}\|_{r'}^{r'} + \|\mathbb{D}\|_r^r + \beta_* \int_0^t \|\boldsymbol{s}\|_{2,\partial\Omega}^2 + \|\boldsymbol{v}_{\boldsymbol{\tau}}\|_{2,\partial\Omega}^2 \leq \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + c_*|Q_T| + d_*|\Sigma_T|$$

Consequently,

$$(\boldsymbol{v}, \boldsymbol{s}, \mathbb{S}) \in \mathtt{FS}$$

Any reasonable (numerical) approximations should fulfil uniform estimates in $\mathtt{FS}$

Function spaces - Stick-slip versus No-slip

$$W^{1,q}_{\boldsymbol{n}} := \{\boldsymbol{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$W^{1,q}_{\boldsymbol{n},\mathrm{div}} := \{\boldsymbol{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathrm{div}\, \boldsymbol{v} = 0; \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

versus

$$W^{1,q}_{0} := \{\boldsymbol{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$
$$W^{1,q}_{0,\mathrm{div}} := \{\boldsymbol{v} \in W^{1,q}(\Omega; \mathbb{R}^d); \mathrm{div}\, \boldsymbol{v} = 0; \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$

FS

$$
\begin{aligned}
\boldsymbol{v} &\in L^{\infty}(0,T;L^2)\cap L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})\cap L^{\frac{5r}{3}}(0,T;L^{\frac{5r}{3}}(\Omega)^d)\\
\mathbb{S} &\in L^{r'}(0,T;L^{r'}(\Omega)^{d\times d})\\
\boldsymbol{s} &\in L^2(0,T;L^2(\partial\Omega)^d)\\
\partial_t\boldsymbol{v} &\in \left(L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})\cap L^{\frac{5r}{6}}(0,T;W^{1,\frac{5r}{6}}_{\boldsymbol{n},\mathrm{div}})\right)^*\\
&= \begin{cases} L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n},\mathrm{div}}) & \text{if } r\ge \frac{11}{5}\\ L^{\frac{5r}{5r-6}}(0,T;W^{-1,\frac{5r}{5r-6}}_{\boldsymbol{n},\mathrm{div}}) & \text{if } r< \frac{11}{5}\end{cases}
\end{aligned}
$$

- FS compactly embedded into $L^2(0,T;L^2(\Omega))$ if $r>6/5$
- FS compactly embedded into $L^2(0,T;L^2(\partial\Omega))$ if $r>8/5$

Part #3

Weak stability of Problem

Weak stability of Problem

Assume that

- for each $n \in \mathbb{N}$: $(\boldsymbol{v}^n, \boldsymbol{s}^n, \mathbb{S}^n)$ solves Problem
- $(\boldsymbol{v}^n, \boldsymbol{s}^n, \mathbb{S}^n)$ converges weakly to $(\boldsymbol{v}, \boldsymbol{s}, \mathbb{S})$ in $\mathtt{FS}$

Is $(\boldsymbol{v}, \boldsymbol{s}, \mathbb{S})$ also solution of Problem?

Balance of linear momentum - equation of motion

For all $\tilde{\boldsymbol{w}} \in (W^{1,r}(\Omega) \cap \mathcal{C}^1(\Omega))^3$ with div $\tilde{\boldsymbol{w}} = 0$ in Ω and $\tilde{\boldsymbol{w}} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$:

$$\int_0^T \{\langle \partial_t \boldsymbol{v}^n, \tilde{\boldsymbol{w}} \rangle + (\mathbb{S}^n, \mathbb{D}\tilde{\boldsymbol{w}})_\Omega + (\boldsymbol{s}^n, \tilde{\boldsymbol{w}}_{\boldsymbol{\tau}})_{\partial\Omega} - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n, \nabla \tilde{\boldsymbol{w}})_\Omega\} \, dt = 0$$

converges to

$$\int_0^T \{\langle \partial_t \boldsymbol{v}, \tilde{\boldsymbol{w}} \rangle + (\mathbb{S}, \mathbb{D}\tilde{\boldsymbol{w}})_\Omega + (\boldsymbol{s}, \tilde{\boldsymbol{w}}_{\boldsymbol{\tau}})_{\partial\Omega} - (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla \tilde{\boldsymbol{w}})_\Omega\} \, dt = 0$$

provided that $W^{1,r}(\Omega)$ is compactly embedded into $L^2(\Omega)$, which holds if

$$r > 6/5.$$

It remains to show that

$$(\mathbb{S}, \mathbb{D}\boldsymbol{v}) \in \mathcal{A} \quad \text{and} \quad (\boldsymbol{s}, \boldsymbol{v}_{\boldsymbol{\tau}}) \in \mathcal{B}.$$

Convergence lemma

Lemma

Let $U \subset Q_T$ be arbitrary (measurable) and $r \in (1, \infty)$. Assume that

- *$\mathcal{A}$ is a maximal monotone graph (satisfying **(A2)–(A3)**)*
- *$\{\mathbb{S}^n\}_{n=1}^{\infty}$ and $\{\mathbb{D}^n\}_{n=1}^{\infty}$ satisfy*

$$(\mathbb{S}^n, \mathbb{D}^n) \in \mathcal{A} \qquad \text{for a.a. } (t,x) \in U$$

$$\mathbb{D}^n \rightharpoonup \mathbb{D} \qquad \text{weakly in } L^r(U)^{d \times d}$$

$$\mathbb{S}^n \rightharpoonup \mathbb{S} \qquad \text{weakly in } L^{r'}(U)^{d \times d}$$

$$\limsup_{n \to \infty} \int_U \mathbb{S}^n \cdot \mathbb{D}^n \, dx \, dt \le \int_U \mathbb{S} \cdot \mathbb{D} \, dx \, dt.$$

Then

$$(\mathbb{S}, \mathbb{D}) \in \mathcal{A} \qquad \text{almost everywhere in } U.$$

- *Local* version
- Last assumption suggests to use energy (entropy) inequality

Step 1. $\boxed{\mathbb{S}^n \cdot \mathbb{D}^n \rightharpoonup \mathbb{S} \cdot \mathbb{D} \text{ weakly in } L^1(U)}$

From **(A2)**

$$0 \leq (\mathbb{S}^n - \mathbb{S}^m) \cdot (\mathbb{D}^n - \mathbb{D}^m) \qquad \text{a.e. in } U$$

Hence, by the assumptions,

$$\lim_{n\to\infty} \lim_{m\to\infty} \|(\mathbb{S}^n - \mathbb{S}^m) \cdot (\mathbb{D}^n - \mathbb{D}^m)\|_1 \leq 0$$

which implies

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_U (\mathbb{S}^n - \mathbb{S}^m) \cdot (\mathbb{D}^n - \mathbb{D}^m)\varphi = 0 \qquad \forall \varphi \in L^\infty(U)$$

Setting $L := \lim_{\ell\to\infty} \int_U (\mathbb{S}^\ell \cdot \mathbb{D}^\ell)\varphi$ we conclude that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \lim_{m\to\infty} \left[\int_U \mathbb{S}^n \cdot \mathbb{D}^n \varphi - \int_U \mathbb{S}^n \cdot \mathbb{D}^m \varphi - \int_U \mathbb{S}^m \cdot \mathbb{D}^n \varphi + \int_U \mathbb{S}^m \cdot \mathbb{D}^m \varphi \right] \\ &= 2\left(L - \int_U \mathbb{S} \cdot \mathbb{D}\,\varphi \right) \end{aligned}$$

Step 2. $\boxed{(\mathbb{S},\mathbb{D}) \in \mathcal{A} \text{ a.e. in } U}$

Take arbitrarily

$$(\mathbb{S}^*,\mathbb{D}^*) \in \mathcal{A} \quad \text{and a nonnegative } \varphi \in L^\infty(U)$$

Then from **(A2)** and Step 1

$$0 \leq \lim_{n\to\infty} \int_U (\mathbb{S}^n - \mathbb{S}^*) \cdot (\mathbb{D}^n - \mathbb{D}^*)\varphi = \int_U (\mathbb{S} - \mathbb{S}^*) \cdot (\mathbb{D} - \mathbb{D}^*)\varphi.$$

Since $\varphi \geq 0$ arbitrary we get

$$0 \leq (\mathbb{S} - \mathbb{S}^*) \cdot (\mathbb{D} - \mathbb{D}^*) \qquad \text{a.e. in } U$$

Since $(\mathbb{S}^*,\mathbb{D}^*) \in \mathcal{A}$ is arbitrary, the maximality of the graph implies

$$(\mathbb{S},\mathbb{D}) \in \mathcal{A} \qquad \text{a.e. in } U$$

□

Identification of the limit for boundary terms

Assume that

$$\boldsymbol{s}^n \rightharpoonup \boldsymbol{s} \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)^3),$$
$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)^3)$$

and $(\boldsymbol{s}^n, \boldsymbol{v}^n) \in \mathcal{B}$

- it is enough to show that

$$\limsup_{n\to\infty} \int_{\partial\Omega} \boldsymbol{s}^n \cdot \boldsymbol{v}^n \le \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{v}$$

- however we also have

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^1(0,T;L^1(\partial\Omega)^3)$$

By Egorov theorem, for any $\varepsilon > 0$ there exists $U_\varepsilon \subset \Sigma_T$ such that $|\Sigma_T \setminus U_\varepsilon| \le \varepsilon$ and

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^\infty(U_\varepsilon)^3$$
$$\implies \limsup_{n\to\infty} \int_{U_\varepsilon} \boldsymbol{s}^n \cdot \boldsymbol{v}^n \le \int_{U_\varepsilon} \boldsymbol{s} \cdot \boldsymbol{v}$$

and $(\boldsymbol{s}, \boldsymbol{v}) \in \mathcal{B}$ a.e. in U_ε. But ε is arbitrary and $(\boldsymbol{s}, \boldsymbol{v}) \in \mathcal{B}$ a.e. on Σ_T

Identification $(\mathbb{S}, \mathbb{D}\boldsymbol{v}) \in \mathcal{A}$ - the convective term neglected

Take $\boldsymbol{v}^n$ as a test function in weak formulation of BLM for Problem(n):

$$\frac{1}{2}\|\boldsymbol{v}^n(T)\|_2^2 + \int_{Q_T} \mathbb{S}^n : \mathbb{D}^n + \int_{\Sigma_T} \boldsymbol{s}^n \cdot \boldsymbol{v}^n = \frac{1}{2}\|\boldsymbol{v}_0^n\|_2^2 \tag{1}$$

Take $\boldsymbol{v}$ as a test function in weak formulation of BLM for Problem:

$$\frac{1}{2}\|\boldsymbol{v}(T)\|_2^2 + \int_{Q_T} \mathbb{S} : \mathbb{D} + \int_{\Sigma_T} \boldsymbol{s} \cdot \boldsymbol{v} = \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 \tag{2}$$

Letting $n \to \infty$ in (1) and comparing the result with (2) we observe that

$$\limsup_{n\to\infty} \int_{Q_T} \mathbb{S}^n : \mathbb{D}^n \le \int_{Q_T} \mathbb{S} : \mathbb{D}$$

which is the fourth assumption of Convergence lemma. Therefore

$$(\mathbb{S}, \mathbb{D}) \in \mathcal{A}$$

Identification $(\mathbb{S}, \mathbb{D}\boldsymbol{v}) \in \mathcal{A}$ - with the convective term

Since

$$\int_\Omega v_k^n \frac{\partial \boldsymbol{v}^n}{\partial x_k} \cdot \boldsymbol{v}^n = \int_\Omega v_k^n \frac{1}{2} \frac{\partial |\boldsymbol{v}^n|^2}{\partial x_k} = \int_\Omega \frac{1}{2} \operatorname{div}(|\boldsymbol{v}|^2 \boldsymbol{v}) = 0$$

and similarly

$$\int_\Omega v_k \frac{\partial \boldsymbol{v}}{\partial x_k} \cdot \boldsymbol{v} = 0$$

the above stated proof remains unchanged if

$$v_k \frac{\partial \boldsymbol{v}}{\partial x_k} \cdot \boldsymbol{v} \in L^1(Q_T) \tag{3}$$

- Since $\boldsymbol{v} \in L^{\frac{5r}{3}}(Q_T)$, (3) holds if $r \geq \frac{11}{5}$.

- Weak stability of Problem is proved. The result include Rigid/shear-thickening fluids, activated NS fluids, and Euler/shear-thickening fluids if $r \geq 11/5$

Q: What about the Euler/NS fluid or Bingham fluids when $r = 2$?

Part #4

Bingham fluids with threshold slip - existence of unsteady flows for large data

$$\mathbb{G}(\mathbb{S},\mathbb{D}) := \mathbb{D} - \frac{(|\mathbb{S}| - \tau_*)_+}{|\mathbb{S}|}\mathbb{S} \qquad \text{Bingham fluid}$$

$$\boldsymbol{g}(\boldsymbol{s},\boldsymbol{v}) := \boldsymbol{v} - \frac{(|\boldsymbol{s}| - \sigma_*)_+}{|\boldsymbol{s}|}\boldsymbol{s} \qquad \text{Threshold slip}$$

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^{1,1}$ domain. Then for any $\boldsymbol{v}_0 \in L^2_{0,\mathrm{div}}$ there exists

$$\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}})$$

$$\mathbb{S} \in L^2(Q)^{d\times d}_{sym}, \quad \boldsymbol{s} \in L^2(0,T;L^2(\partial\Omega)^d)$$

$$p_1 \in L^2(Q), \quad p_2 \in L^{\frac{d+2}{d+1}}(0,T;W^{1,\frac{d+2}{d+1}}(\Omega))$$

solving for almost all time $t \in (0,T)$ and for all $\boldsymbol{w} \in W^{1,\infty}_{\boldsymbol{n}}$

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})\cdot\nabla\boldsymbol{w} + \int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{w}) + \int_{\partial\Omega} \boldsymbol{s}\cdot\boldsymbol{w} = \int_\Omega (p_1+p_2)\,\mathrm{div}\,\boldsymbol{w}$$

and fulfilling

$$\mathbb{G}(\mathbb{S},\mathbb{D}\boldsymbol{v}) = \mathbb{O} \ \textit{a.e. in } Q_T \quad \textit{and} \quad \boldsymbol{g}(\boldsymbol{s},\boldsymbol{v}_{\boldsymbol{\tau}}) = \boldsymbol{0} \ \textit{a.e. in } \Sigma_T$$

M. Bulíček, J. Málek: On unsteady internal flows of Bingham fluids subject to threshold slip on the impermeable boundary, in Recent Developments of Mathematical Fluid Mechanics (eds. H. Amann et al.), pp. 135-156

Function spaces - Stick-slip versus Slip

$$W^{1,q}_{\boldsymbol{n}} := \{\boldsymbol{v} \in W^{1,q}(\Omega;\mathbb{R}^d); \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$W^{1,q}_{\boldsymbol{n},\mathrm{div}} := \{\boldsymbol{v} \in W^{1,q}(\Omega;\mathbb{R}^d); \mathrm{div}\, \boldsymbol{v} = 0; \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

versus

$$W^{1,q}_{0} := \{\boldsymbol{v} \in W^{1,q}(\Omega;\mathbb{R}^d); \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$
$$W^{1,q}_{0,\mathrm{div}} := \{\boldsymbol{v} \in W^{1,q}(\Omega;\mathbb{R}^d); \mathrm{div}\, \boldsymbol{v} = 0; \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$

By the Helmholtz decomposition, for $q \in (1,\infty)$:

$$W^{1,q}_{\boldsymbol{n}} = W^{1,q}_{\boldsymbol{n},\mathrm{div}} \oplus \{\nabla\varphi; \varphi \in W^{2,q}, \nabla\varphi\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

Similar decomposition for $W^{1,q}_0(\Omega)^d$ is open.

- Essential difference in the weak formulation
- σ_* can be artificial (big enough) so that it is never active
  - in analysis if $\boldsymbol{v} \in L^\infty(0,T;C(\overline{\Omega}))$
  - in computer simulations

Proof - n-approximations

Consider

$$\mathbb{G}^n(\mathbb{S},\mathbb{D}) := \mathbb{D} - \left(\frac{(|\mathbb{S}| - \tau_*)_+}{|\mathbb{S}|} + \frac{1}{n} \right) \mathbb{S} \qquad \text{Bingham fluid,} \qquad \text{(Bn)}$$

$$\boldsymbol{g}^n(\boldsymbol{s},\boldsymbol{v}) := \boldsymbol{v} - \left(\frac{(|\boldsymbol{s}| - \sigma_*)_+}{|\boldsymbol{s}|} + \frac{1}{n} \right) \boldsymbol{s} \qquad \text{threshold slip} \qquad \text{(Tn)}$$

and smooth G_n, $|G_n'| \leq \frac{1}{n}$

$$G_n(s) := 1 \text{ for } s \leq n, \quad G_n(s) = 0 \text{ for } s > 2n.$$

Take approximation

$$\partial_t \boldsymbol{v}^n + \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|) - \operatorname{div} \mathbb{S}^n = -\nabla p^n$$

with constitutive equations (Bn) and (Tn). Since (Bn) and (Tn) imply

$$\mathbb{S} = \mathbb{S}^*_n(\mathbb{D}), \qquad \boldsymbol{s} = \boldsymbol{s}^*_n(\boldsymbol{v})$$

with $\mathbb{S}^*_n$ and $\boldsymbol{s}^*_n$ being continuous monotone with linear growth (at infinity), the existence follows from monotone operator theory (due to the presence of G_n)

Pressure for n fixed

$$\langle \partial_t \boldsymbol{v}^n, \tilde{\boldsymbol{w}} \rangle + (\mathbb{S}^n, \mathbb{D}(\tilde{\boldsymbol{w}})) + (\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G(|\boldsymbol{v}^n|), \tilde{\boldsymbol{w}}) + (\boldsymbol{s}^n, \tilde{\boldsymbol{w}}_{\boldsymbol{\tau}})_{\partial\Omega} - \langle \boldsymbol{b}, \tilde{\boldsymbol{w}} \rangle = \boldsymbol{0} \qquad \text{for all } \tilde{\boldsymbol{w}} \in W^{1,2}_{\boldsymbol{n},\mathrm{div}} \text{ and a.a. } t \in (0, T)$$

Define p^n as the solution of the following problem

$$(\nabla p^n, \nabla z) + (\mathbb{S}^n, \nabla^{(2)} z) + (\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G(|\boldsymbol{v}^n|), \nabla z) + (\boldsymbol{s}^n, (\nabla z)_{\boldsymbol{\tau}})_{\partial\Omega} - \langle \boldsymbol{b}, \nabla z \rangle = 0$$

$$\text{for all } z \in W^{2,2}(\Omega) \text{ with } \nabla z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and a.a. } t \in (0, T)$$

$$\boxed{\boldsymbol{w} = \tilde{\boldsymbol{w}} + \nabla z}$$

$$\begin{aligned}(\mathbb{S}^n, \mathbb{D}(\boldsymbol{w})) + (\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G(|\boldsymbol{v}^n|), \boldsymbol{w}) + (\boldsymbol{s}^n, \boldsymbol{w}_{\boldsymbol{\tau}})_{\partial\Omega} - \langle \boldsymbol{b}, \boldsymbol{w} \rangle \\ = \langle \partial_t \boldsymbol{v}^n, \tilde{\boldsymbol{w}} \rangle + (\nabla p, \nabla z) \\ = \langle \partial_t \boldsymbol{v}^n, \tilde{\boldsymbol{w}} + \nabla z \rangle + (\nabla p, \tilde{\boldsymbol{w}} + \nabla z)\end{aligned}$$

which finally leads to:

$$\langle \partial_t \boldsymbol{v}^n, \boldsymbol{w} \rangle + (\mathbb{S}^n, \mathbb{D}(\boldsymbol{w})) + (\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G(|\boldsymbol{v}^n|), \boldsymbol{w}) + (\boldsymbol{s}^n, \boldsymbol{w}_{\boldsymbol{\tau}})_{\partial\Omega} = (p^n, \operatorname{div} \boldsymbol{w}) + \langle \boldsymbol{b}, \boldsymbol{w} \rangle \qquad \text{for all } \boldsymbol{w} \in W^{1,2}_{\boldsymbol{n}} \text{ and a.a. } t \in (0, T)$$

Apriori estimates

I. Test by $\boldsymbol{v}^n$ (convective term) vanishes to get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n\|_2^2 + \int_\Omega \mathbb{S}^n \cdot \mathbb{D}(\boldsymbol{v}^n) + \int_{\partial\Omega} \boldsymbol{s}^n \cdot \boldsymbol{v}^n = 0$$

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_2^2 + \int_{Q_T} |\mathbb{S}^n|^2 + |\nabla \boldsymbol{v}^n|^2 + |\boldsymbol{v}^n|^{\frac{2(d+2)}{d}} + \int_{(0,T)\times\partial\Omega} |\boldsymbol{s}^n|^2 + |\boldsymbol{v}^n|^2 \leq C(\boldsymbol{v}_0)$$

II. Find p_2^n with zero mean value solving at each time level

$$\int_\Omega \nabla p_2^n \cdot \nabla \varphi = -\int_\Omega \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|) \cdot \varphi$$

But

$$\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|) = v_k^n \frac{\partial \boldsymbol{v}^n}{\partial x_k} G_n(|\boldsymbol{v}^n|)$$

$$\int_{Q_T} |\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|)|^{\frac{d+2}{d+1}} \leq C \implies \int_0^T \|p_2^n\|_{1,\frac{d+2}{d+1}}^{\frac{d+2}{d+1}} \leq C$$

Define $p_1^n := p^n - p_2^n$.

Apriori estimates - continuation

III. For $p_1^n := p^n - p_2^n$ find φ with zero mean value such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ solving

$$\Delta\varphi = p_1^n \implies \int_{Q_T} |\nabla^2\varphi|^2 + \int_{(0,T)\times\partial\Omega} |\nabla\varphi|^2 \leq \int_{Q_T} |p_1^n|^2$$

Test by $\nabla\varphi$ and integrate over Q_T

$$\begin{aligned}
\int_{Q_T} |p_1^n|^2 &= -\int_{Q_T} \nabla p_1^n \cdot \nabla\varphi = \int_{Q_T} (\nabla p_2^n - \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|)) \cdot \nabla\varphi \\
&\quad + \int_{Q_T} \mathbb{S}^n \cdot \nabla^2\varphi + \int_{(0,T)\times\partial\Omega} \boldsymbol{s}^n \cdot \nabla\varphi \\
&= \int_{Q_T} \mathbb{S}^n \cdot \nabla^2\varphi + \int_{(0,T)\times\partial\Omega} \boldsymbol{s}^n \cdot \nabla\varphi \\
&\leq C \left(\int_{Q_T} |p_1^n|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

IV. $$\|\partial_t \boldsymbol{v}^n\|_{(L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap L^{d+2}(Q_T))^*} \leq C$$

Convergences

Aubin-Lions and apriori estimates:

$$
\begin{aligned}
\boldsymbol{v}^n &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}),\\
\mathbb{S}^n &\rightharpoonup \mathbb{S} && \text{weakly in } L^2(Q)^{d\times d},\\
\boldsymbol{s}^n &\rightharpoonup \boldsymbol{s} && \text{weakly in } L^2(0,T;L^2(\partial\Omega)),\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^2(Q),\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)),\\
p_1^n &\rightharpoonup p_1 && \text{weakly in } L^2(Q),\\
p_2^n &\rightharpoonup p_2 && \text{weakly in } L^{\frac{d+2}{d+1}}(0,T;W^{1,\frac{d+2}{d+1}}(\Omega)),\\
\partial_t\boldsymbol{v}^n &\rightharpoonup \partial_t\boldsymbol{v} && \text{weakly in } (L^2(0,T;W^{1,2}_{\boldsymbol{n}})\cap L^{d+2}(Q_T))^*
\end{aligned}
$$

solving the original problem, and also $\boldsymbol{g}(\boldsymbol{s},\boldsymbol{v}_{\boldsymbol{\tau}})=0$

It remains to show the validity of $\mathbb{G}(\mathbb{S},\mathbb{D}\boldsymbol{v})=\mathbb{O}$.

Convergence III

Assume that $\{k^n\}_{n=1}^{\infty}$ is such that $0 < A \le k^n \le B < \infty$. Test the n-th approximation by

$$\boldsymbol{w}^n := T_{k^n}(\boldsymbol{v}^n - \boldsymbol{v}) := (\boldsymbol{v}^n - \boldsymbol{v}) \min\left\{1, \frac{k^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}\right\}$$

Note $T_k(\boldsymbol{u}) = \boldsymbol{u}$ if $|\boldsymbol{u}| \le k$.

Taking $\boldsymbol{w}^n$ as a test function

$$\begin{aligned}\limsup_{n\to\infty} \int_{Q_T} & \mathbb{S}^n \cdot \mathbb{D}(\boldsymbol{w}^n) - p_1^n \operatorname{div} \boldsymbol{w}^n \\ &= \limsup_{n\to\infty} \int_{Q_T} -\langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}^n\rangle - (\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|)) + \nabla p_2^n) \cdot \boldsymbol{w}^n \\ &\quad + \int_{\Sigma_T} \boldsymbol{s}^n \cdot \boldsymbol{w}^n \le 0\end{aligned}$$

Find $\overline{\mathbb{S}} \in L^2(Q)$ fulfilling

$$\boxed{\mathbb{D}(\boldsymbol{v}) = \frac{(\overline{\mathbb{S}}| - \tau)_+}{|\overline{\mathbb{S}}|}\overline{\mathbb{S}}}$$

Then

$$\limsup_{n\to\infty} \int_{Q_T} (\mathbb{S}^n - \overline{\mathbb{S}}) \cdot \mathbb{D}(\boldsymbol{w}^n) \leq \limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \geq k^n} \frac{k^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} |p_1^n| (|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|)$$

Considering

$$I^n := C_*(|p_1^n|^2 + |\nabla \boldsymbol{v}^n|^2 + |\nabla \boldsymbol{v}|^2 + |\overline{\mathbb{S}}|^2 + |\overline{\mathbb{S}}^n|) \qquad \boxed{\sup_n \int_{Q_T} I^n < \infty}$$

we observe that

$$\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| < k^n} (\mathbb{S}^n - \overline{\mathbb{S}}) \cdot \mathbb{D}(\boldsymbol{v}^n - \boldsymbol{v}) \leq \limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \geq k^n} \frac{k^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n$$

AIM: RHS should tend to zero by making a proper choice for A, B and k^n.

For $N \in \mathbb{N}$ arbitrary, fix $A := N$ and $B := N^{N+1}$ and define

$$Q_i^n := \{(t,x) \in Q_T; N^i \le |\boldsymbol{v}^n - \boldsymbol{v}| \le N^{i+1}\} \qquad i = 1, \dots, N.$$

Since

$$\sum_{i=1}^{N} \int_{Q_i^n} I^n \le C_*,$$

there is, for each $n \in \mathbb{N}$, an index $i_n \in \{1, \dots, N\}$ such that

$$\int_{Q_{i_n}^n} I^n < \frac{C_*}{N}$$

Setting $k^n := N^{i_n+1}$, RHS is estimated in the following way:

$$\begin{aligned} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \ge N^{i_n+1}} \frac{k^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n &= \int_{N^{i_n+2} \ge |\boldsymbol{v}^n - \boldsymbol{v}| \ge N^{i_n+1}} \cdots + \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \ge N^{i_n+2}} \cdots \\ &= \int_{Q_{i_n}^n} \cdots + \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \ge N^{i_n+2}} I^n \le \frac{C_*}{N}. \end{aligned} \tag{4}$$

Next, using the constitutive equation for $\mathbb{D}\boldsymbol{v}^n$ and $\mathbb{D}\boldsymbol{v}$ we conclude that

$$\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \le k^n} (\mathbb{S}^n - \overline{\mathbb{S}}) \cdot \left(\left(\frac{(\mathbb{S}^n - \tau_*)_+}{|\mathbb{S}^n|} + \frac{1}{n} \right) \mathbb{S}^n - \frac{(\overline{\mathbb{S}}| - \tau)_+}{|\overline{\mathbb{S}}|} \overline{\mathbb{S}} \right) \le \frac{C_*}{N},$$

Thus, for $\boxed{Z^n := (\mathbb{S}^n - \overline{\mathbb{S}}) \cdot \left(\frac{(\mathbb{S}^n - \tau_*)_+}{|\mathbb{S}^n|}\mathbb{S}^n - \frac{(\overline{\mathbb{S}}| - \tau_*)_+}{|\overline{\mathbb{S}}|}\overline{\mathbb{S}} \right) \geq 0.}$

$$\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \leq k^n} Z^n \leq \frac{C_*}{N} + \limsup \frac{1}{n} \int_{Q_T} |\mathbb{S}^n||\mathbb{S}^n - \overline{\mathbb{S}}| \leq \frac{C_*}{N}$$

Since $A = N$ and $k^n \geq N$

$$\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \leq N} Z^n \leq \frac{C_*}{N}$$

Splitting Q_T into a union of $\{|\boldsymbol{v}^n - \boldsymbol{v}| \leq N\}$ and $\{|\boldsymbol{v}^n - \boldsymbol{v}| > N\}$ one concludes

$$\limsup_{n\to\infty} \int_{Q_T} \sqrt{Z^n} \leq \frac{C_*}{N} \implies Z^n \to 0 \quad \text{a.e. in } Q_T$$

Applying then the biting lemma, one then concludes that

$$Z^n \to 0 \quad \text{strongly in } L^1(Q_T \setminus E_j) \qquad E_j \subset Q_T : \quad \lim_{j\to\infty} |E_j| = 0$$

$$\implies \limsup_{n\to\infty} \int_{Q_T \setminus E_j} \mathbb{S}^n \cdot (\mathbb{D}\boldsymbol{v}^n - \frac{1}{n}\mathbb{S}^n) = \int_{Q_T \setminus E_j} \mathbb{S} \cdot \mathbb{D}\boldsymbol{v}\,.$$

Convergence lemma and the properties of E_j: $\implies (\mathbb{S}, \mathbb{D}\boldsymbol{v}) \in \mathcal{A}$ a.e. in Q_T.

Part #5

Implicitly constituted fluids described by maximal monotone ψ-graph - existence of unsteady flows subject to Navier's slip for large data

Definition of weak solution to the Problem with Navier's slip bcs

Definition

We say $(p, \boldsymbol{v}, \mathbb{S})$ is weak solution to *Problem* with Navier's slip

$$p \in L^1(Q_T)$$

$$\boldsymbol{v} \in C_{\mathrm{weak}}(0,T; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^q(0,T; W^{1,1}_{\mathbf{n},\mathrm{div}}) \text{ with } \mathbb{D}(\boldsymbol{v}) \in L^{\psi}(Q_T)$$

$$\mathbb{S} \in L^{\psi^*}(Q_T)$$

$$\lim_{t \to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0$$

$$\langle \boldsymbol{v}', \boldsymbol{w} \rangle + (\mathbb{S}, \mathbb{D}(\boldsymbol{w})) - (\boldsymbol{v} \otimes \boldsymbol{v}, \mathbb{D}(\boldsymbol{w})) + \alpha_*(\boldsymbol{v}_{\boldsymbol{\tau}}, \boldsymbol{w}_{\boldsymbol{\tau}})_{\partial\Omega} = \langle \boldsymbol{b}, \boldsymbol{w} \rangle, + (p, \operatorname{div} \boldsymbol{w}),$$

$$\text{for all } \boldsymbol{w} \in W^{1,1}_{\boldsymbol{n}} \text{ such that } \mathbb{D}(\boldsymbol{w}) \in L^{\infty}(\Omega)^{d \times d} \text{ and a.a. } t \in (0,T),$$

$$(\mathbb{D}(\boldsymbol{v}(t,x)), \mathbb{S}(t,x)) \in \mathcal{A} \text{ for a.a. } (t,x) \in Q_T.$$

Theorem

Theorem

Let $\Omega \subset \mathbb{R}^3$ and $\mathcal{A}$ satisfy the assumptions **(A1)–(A4)** *with ψ fulfilling*

$$\boxed{c_1 s^r - c_2 \le \psi(s) \le c_3 s^{\tilde{r}} + c_4 \quad \textit{with } r > \frac{2d}{d+2}}$$

Then for any $\Omega \in \mathcal{C}^{1,1}$ and $T \in (0, \infty)$ and for arbitrary

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}, \quad \boldsymbol{b} \in L^2(0,T; L^2(\Omega)^d) \ \textit{and}\ \gamma_* \ge 0\,, \tag{5}$$

there exists weak solution to Problem.

Novel tools:
(i) Structural assumptions **(A1)–(A4)** on $\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{O}$
(ii) Convergence lemma
(iii) Understanding the interplay between the chosen boundary conditions and global integrability of p
(iv) Lipschitz approximations of Sobolev-Orlicz and Bochner functions

M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda: On Unsteady Flows of Implicitly Constituted Incompressible Fluids, *SIAM J. Math. Anal.*, Vol. 44, No. 4, pp. 2756–2801 (2012)

Methods

- subcritical case
  - Minty's method
  - energy equality - $\boldsymbol{v}$ is an admissible test function
- supercritical case
  - Generalized Minty's method - Convergence lemma
    - Lipschitz approximation in Orlicz-Sobolev spaces
    - L^∞-truncation of Sobolev functions

No-slip versus Threshold slip (Stick-slip)

- Homogeneous Dirichlet boundary conditions are considered as the simplest for many PDEs
- In incompressible fluid dynamical problems, it is however, in general, open whether $p \in L^1(Q_T)$ for no-slip boundary conditions
- Exceptions are the cases when we control $\partial_t \boldsymbol{v}$ is an integrable function, e.g.,
  - Navier-Stokes model (linearity, Solonnikov)
  - Ladyzhenskaya model for $r \geq \frac{12}{5}$ in 3D setting provided that data are smooth (potentiality of $\mathbb{S}$, test by time derivative, Ladyzhenskaya)
  - All models above with uniform monotonicity (but no assumption on having potential), whenever we can test by $\boldsymbol{v}$ (bootstrap in non-integer time derivatives, Bulíček, Etwein, Kaplický, Pražák) $r > \frac{11}{5}$

Concluding Remarks

- implicitly constitutive theory seems to suitable approach to include various activation criteria both in the bulk and on the boundary
- threshold slip is the way how to overcome the troubles connected with the analysis of unsteady flows subject to homogeneous Dirichlet boundary conditions (no-slip) - fits nicely to the framework of implicitly constituted materials
- for implicitly constituted fluids characterized by **(A1)**-**(A4)** and $r > 6/5$, we define the solution and show its large data existence - object to be studied numerically and computationally.
- new options how to numerically discretize the problems - some give interesting results (second order vs. first order PDEs) - J.Hron, P. Minakowski, G.Tierra.

Thank you