Growth and Singularity in 2D Fluids

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Euler equations in 2D

The (incompressible) Euler equations are

\[ u_t + (u \cdot \nabla)u + \nabla p = 0 \]
\[ \nabla \cdot u = 0 \]

on \( D \times (0, T) \) for some domain \( D \subseteq \mathbb{R}^d \) and time \( T \leq \infty \), with

\[ u \cdot n = 0 \]

on \( \partial D \times (0, T) \) (no-flow boundary condition) and given \( u(\cdot, 0) \).

In 2D, their vorticity form is the active scalar equation

\[ \omega_t + u \cdot \nabla \omega = 0 \]

with vorticity \( \omega := \nabla \times u = -(u_1)_x^2 + (u_2)_x \in \mathbb{R} \) and

\[ u = \nabla^\perp \Delta^{-1} \omega \]

Here \( \Delta \) is the Dirichlet Laplacian (no-flow boundary condition).
Growth of solutions to the 2D Euler equations

Solutions of any transport equation

$$\omega_t + u \cdot \nabla \omega = 0$$

are uniformly bounded, so blow-up might only be possible in the derivatives of $\omega$ (loss of regularity).

- Wolibner (1933) and Hölder (1933) showed that solutions remain regular, with the double-exponential bound

$$\|\nabla \omega(\cdot, t)\|_{L^\infty} \leq Ce^{ct}$$

- Z. (2015) proved existence of at least exponential growth for $\omega(\cdot, 0) \in C^{1,\frac{1}{2}}(\mathbb{T}^2) \cap C^\infty(\mathbb{T}^2 \setminus \{0\})$ (hence $\partial D = \emptyset$). Double-exponential growth on $\mathbb{R}^2$ and $\mathbb{T}^2$ is still open.
- Kiselev-Z. (2015) showed finite time blow-up on a domain with (two) singular points.
Double-exponential (i.e., fast) growth for the 2D Euler equations suggests that they could be critical in the sense that finite time blow-up could happen for more singular models. Particularly interesting is the surface quasi-geostrophic (SQG) equation

\[ \omega_t + u \cdot \nabla \omega = 0 \]
\[ u = -\nabla^\perp (-\Delta)^{-1/2} \omega \]

It is used in atmospheric science models and was first rigorously studied by Constantin-Majda-Tabak (1994).

2D Euler and SQG are extremal members of the natural family

\[ \omega_t + u \cdot \nabla \omega = 0 \]
\[ u = -\nabla^\perp (-\Delta)^{-1+\alpha} \omega \]

of modified SQG (m-SQG) equations, with parameter \( \alpha \in [0, \frac{1}{2}] \). The regularity/blow-up question remains open for all \( \alpha > 0 \).
I will talk about the corresponding patch problem (Bertozzi, Chemin, Constantin, Córdoba, Denissov, Depauw, Gancedo, Rodrigo, Yudovich,...) on the half-plane \( D = \mathbb{R} \times \mathbb{R}^+ \). Here

\[
\omega(\cdot, t) = \sum_{n=1}^{N} \theta_n \chi_{\Omega_n(t)}
\]

with \( \theta_n \in \mathbb{R} \setminus \{0\} \), and each patch \( \Omega_n(t) \subseteq D \) is a bounded open set advected by \( u = -\nabla^\perp (-\Delta)^{-1} + \alpha \omega \) (see later). For the half-plane \( D \), this is (with \( \bar{y} = (y_1, -y_2) \) and some \( c_\alpha > 0 \))

\[
u(x, t) = -c_\alpha \int_D \left( \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} - \frac{(x - \bar{y})^\perp}{|x - \bar{y}|^{2+2\alpha}} \right) \omega(y, t) dy
\]

We require patch-like initial data with some regularity:

- Patches do not touch each other or themselves:
  - \( \Omega_n(0) \cap \Omega_m(0) = \emptyset \) for \( n \neq m \)
  - each \( \partial \Omega_n(0) \) is a simple closed curve

- All \( \partial \Omega_n(0) \) have certain prescribed regularity.

Blow-up happens if one of these fails at some time \( t > 0 \).
Global regularity of $C^{1,\gamma}$ Euler patches on $\mathbb{R} \times \mathbb{R}^+$

<table>
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<th>Theorem (Kiselev-Ryzhik-Yao-Z., 2015)</th>
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<td>Let $\alpha = 0$ and $\gamma \in (0, 1]$. Then for each $C^{1,\gamma}$ patch-like initial data $\omega(\cdot, 0)$, there exists a unique global $C^{1,\gamma}$ patch solution $\omega$.</td>
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- The same whole-plane result for a single patch was proved by Chemin (1993). Our proof is motivated by an alternative approach by Bertozzi-Constantin (1993).
- Specifically, each patch boundary is the zero-level set of a function which is advected by $u$. The rates of change of their $C^{1,\gamma}$ norms, of their gradients on their zero-level sets, and of the distances of their zero-level sets are controlled.
- Previously Depauw (1999) proved local regularity on the half-plane (and global if patches do not touch $\partial D$ initially).
- A result of Dutrifoy (2003) implies global existence in $C^{1,s}$ for some $s < \gamma$. 
Blow-up of $H^3$ patches on $\mathbb{R} \times \mathbb{R}^+$ for small $\alpha > 0$

Theorem (Kiselev-Yao-Z., 2015)

Let $\alpha \in (0, \frac{1}{24})$. Then for each $H^3$ patch-like initial data $\omega(\cdot, 0)$, there exists a unique local $H^3$ patch solution $\omega$. Moreover, if the maximal time $T_\omega$ of existence of $\omega$ is finite, then at $T_\omega$ either two patches touch, or a patch boundary touches itself, or a patch boundary loses $H^3$ regularity (i.e., blow-up).

Local existence on the whole plane was proved for $\alpha \in (0, \frac{1}{2})$ by Gancedo (2008). We can prove uniqueness and the last claim.

Theorem (Kiselev-Ryzhik-Yao-Z., 2015)

Let $\alpha \in (0, \frac{1}{24})$. Then there are $H^3$ patch-like initial data $\omega(\cdot, 0)$ for which the solution $\omega$ blows up in finite time (i.e., $T_\omega < \infty$).

To the best of our knowledge, this is the first rigorous result proving finite time blow-up in this type of fluid dynamics models.
Definition of patch solutions

In the Euler case one usually requires that $\Phi_t : \bar{D} \to \bar{D}$ given by

$$\frac{d}{dt} \Phi_t(x) = u(\Phi_t(x), t) \quad \text{and} \quad \Phi_0(x) = x$$

preserves each patch: $\Phi_t(\Omega_n(0)) = \Omega_n(t)$ for each $t \in (0, T)$. However, the map $\Phi_t$ need not be uniquely defined for $\alpha > 0$.

Definition

A patch-like (i.e., no touches of patches at any $t \in [0, T]$ plus continuity of each $\partial \Omega_n(t)$ in time w.r.t Hausdorff distance)

$$\omega(\cdot, t) = \sum_{n=1}^{N} \theta_n \chi_{\Omega_n(t)}$$

is a patch solution to m-SQG on $[0, T)$ if for each $t, n$ we have

$$\lim_{h \to 0} \frac{d_H \left( \partial \Omega_n(t + h), X^h_{u(\cdot, t)}[\partial \Omega_n(t)] \right)}{h} = 0,$$

with $d_H$ Hausdorff distance and $X^h_u[A] = \{x + hu(x) \mid x \in A\}$. 
Properties of patch solutions

Denote $\Omega(t) = \bigcup_n \Omega_n(t)$. The definition shows that:

- $\partial \Omega(t)$ is moving with velocity $u(x, t)$ at $x \in \partial \Omega(t)$.
- Patch solutions to m-SQG are also weak solutions (and weak solutions with $C^1$ boundaries which move with some continuous velocity are patch solutions).
- In the Euler case it is equivalent to the definition via $\Phi$.
- It is also essentially equivalent to the definition via $\Phi$ in the case of $H^3$ patch solutions to m-SQG with $\alpha < \frac{1}{4}$ [KYZ].
- In fact, $\Phi_t(x)$ is uniquely defined for $x \in \overline{D} \setminus \partial \Omega(0)$, and

$$
\Phi_t : \Omega_n(0) \rightarrow \Omega_n(t) \quad \text{and} \quad \Phi_t : \left[ \overline{D} \setminus \overline{\Omega(0)} \right] \rightarrow \left[ \overline{D} \setminus \overline{\Omega(t)} \right].
$$

Also, these maps are measure preserving bijections and we have $\Phi_t(\partial \Omega_n(0)) = \partial \Omega_n(t)$ in an appropriate sense.

- This uses that the normal component of $u$ (w.r.t. $\partial \Omega(t)$) is Lipschitz in the normal direction if $\alpha < \frac{1}{4}$. 

Local $H^3$ regularity: The contour equation

For simplicity assume a single patch. Parametrize $\partial \Omega(t)$ by $z(\cdot, t) \in H^3(\mathbb{T})$. Then for any $x = z(\xi, t) \in \partial \Omega(t)$ we obtain

$$u(x, t) = \frac{c_\alpha \theta}{2\alpha} \sum_{i=1}^{2} \int_{\mathbb{T}} \frac{-\partial_\xi z^i(\xi - \eta, t)}{|z(\xi, t) - z^i(\xi - \eta, t)|^{2\alpha}} d\eta$$

with

$$z^1(\xi, t) := z(\xi, t) \quad \text{and} \quad z^2(\xi, t) := \bar{z}(\xi, t)$$

Next add a multiple of the tangent vector $\partial_\xi z(\xi, t)$ so that the integrand becomes more regular, and get the contour equation

$$\partial_t z(\xi, t) = \frac{c_\alpha \theta}{2\alpha} \sum_{i=1}^{2} \int_{\mathbb{T}} \frac{\partial_\xi z(\xi, t) - \partial_\xi z^i(\xi - \eta, t)}{|z(\xi, t) - z^i(\xi - \eta, t)|^{2\alpha}} d\eta$$

Gancedo proves local regularity for the contour equation in $\mathbb{R}^2$ (which has only $i = 1$, and also a single patch) for any $\alpha < \frac{1}{2}$. 
Local $H^3$ regularity: Existence of a patch solution

We prove local regularity on $D = \mathbb{R} \times \mathbb{R}^+$ for $\alpha < \frac{1}{24}$, via

$$\frac{d}{dt} \|z(\cdot, t)\| \leq C(\alpha) \theta \|z(\cdot, t)\|^8$$

where $\|\cdot\| = \|z(\cdot, t)\|_{H^3} + \text{inverse Lipschitz norm of } z(\cdot, t)$ (+ distance of patches when $N \geq 2$). Quite a bit more involved...

- The method does not seem to work for Hölder norms.

Limitation on $\alpha$ is essentially due to insufficient bounds on the tangential velocity. Where a patch departs $x_1$-axis, tangential velocity generated by its reflection might deform it excessively.

- Most of the proof works for $\alpha < \frac{1}{4}$.

This local contour solution $z$ then yields a patch solution $\omega$. 
Local $H^3$ regularity: Independence of parametrization

Proving uniqueness via some version of Gronwall difficult:

$$|u(x) - \tilde{u}(x)| \lesssim d_H(\partial \Omega, \partial \tilde{\Omega})^{1-2\alpha}.$$ 

- Gronwall does apply to $\|z - \tilde{z}\|_{L^2}$ but $z, \tilde{z}$ might not exist.

First step towards uniqueness is showing independence of the “contour” patch from parametrization of $\partial \Omega(0)$.

- Regularize:

$$u^\varepsilon(x, t) = -c_\alpha \int_D \left( \frac{(x - y)^\perp}{(|x - y|^2 + \varepsilon^2)^{1+\alpha}} - \frac{(x - \tilde{y})^\perp}{(|x - \tilde{y}|^2 + \varepsilon^2)^{1+\alpha}} \right) \omega(y, t) dy$$

- Show uniqueness of patch solution $\omega^\varepsilon$ (e.g., via Gronwall). Then any contour solutions $z^\varepsilon, \tilde{z}^\varepsilon$ which parametrize the same initial patch must yield the same $\omega^\varepsilon$.

- Show $z^\varepsilon \to z$ if they have the same initial parametrization. Similarly $\tilde{z}^\varepsilon \to \tilde{z}$, hence $z, \tilde{z}$ must yield the same $\omega$. 
Let $\omega$ be any patch solution and $\omega^s$ the “contour” patch solution with $\omega^s(\cdot, s) = \omega(\cdot, s)$ ($\omega^s$ is unique). For small $T > 0$ and $J \in \mathbb{N}$:

Successive estimation of the rates of change of $d_H(\partial \Omega, \partial \tilde{\Omega})$ and $\|z - \tilde{z}\|_{L^2}$ and telescoping give $|\Omega(T) \triangle \Omega^0(T)| \lesssim J^{1-1/2\alpha}$. Then take $J \to \infty$ and get $\Omega = \Omega^0$ on $[0, T]$. 
Finite time blow-up in $H^3$: Initial data and symmetry

Our initial data will be made of two patches and odd in $x_1$.

Then local uniqueness shows that before blow-up we have

$$
\omega(\cdot, t) = \chi_{\Omega(t)} - \chi_{\tilde{\Omega}(t)}
$$

with $\Omega(t) \subseteq D^+ = (\mathbb{R}^+)^2$ and $\tilde{y} = (-y_1, y_2)$. Then (let $c_\alpha = 1$)

$$
u(x, t) = -\int_{\Omega(t)} H(x, y)dy
$$

$$
H(x, y) = \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} - \frac{(x - \tilde{y})^\perp}{|x - \tilde{y}|^{2+2\alpha}} - \frac{(x - \tilde{y})^\perp}{|x - \tilde{y}|^{2+2\alpha}} + \frac{(x + y)^\perp}{|x + y|^{2+2\alpha}}
$$
Finite time blow-up in $H^3$: A barrier argument

Goal: show that if $\Omega(0) \supseteq [\varepsilon, 3] \times [0, 3]$ and $\varepsilon > 0$ is small, then

$$\Omega(t) \supseteq K(t) = \{X(t) < x_1 < 2\} \cap \{0 < x_2 < x_1\}$$

until blow-up, where $X(0) = \varepsilon$ and $X'(t) = -\frac{1}{100\alpha} X(t)^{1-2\alpha}$.

This gives blow-up because $X(50\varepsilon^{2\alpha}) = 0$.

If $t < 50\varepsilon^{2\alpha}$ is the first time with $D^+ \setminus \Omega(t) \cap K(t) \neq \emptyset$, then by

$$\|u\|_{L^\infty} \leq C_1\|\omega(\cdot, 0)\|_{L^\infty} + C_2\|\omega(\cdot, 0)\|_{L^1} \leq C$$

the touch can only be on $I_1 \cup I_2$ (since $\Omega(t) \supseteq \Omega_\alpha$ by $\varepsilon$ small). Also uses that the patch cannot separate from the $x_1$-axis...
Finite time blow-up in $H^3$: Estimates on the flow

We have $u_1(x, t) = -\int_{\Omega(t)} H_1(x, y) dy$, where

$$H_1(x, y) = \frac{y_2 - x_2}{|x - y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x - \tilde{y}|^{2+2\alpha}} + \frac{y_2 + x_2}{|x - \bar{y}|^{2+2\alpha}} - \frac{y_2 + x_2}{|x + y|^{2+2\alpha}}$$

Then $|x - \tilde{y}| \leq |x + y|$ on $\Omega(t) \subseteq D^+$ gives

$$u_1(x, t) \leq -\int_{\Omega(t)} \left( \frac{y_2 - x_2}{|x - y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x - \tilde{y}|^{2+2\alpha}} \right) G(x, y) dy$$

From $K(t) \subseteq \Omega(t)$ we have for $x \in K(t) \cap \{x_1 \leq 1\}$

$$u_1(x, t) \leq \int_{\mathbb{R} \times (0, x_2)} |G(x, y)| dy - \int_{A(x)} G(x, y) dy$$

because $\text{sgn}(G(x, y)) = \text{sgn}(y_2 - x_2)$.

Small $\alpha$ is crucial for $A(x)$ to compensate limited control near $x$. Blow-up may be easier to prove in slightly super-critical models.
A computation and cancellations yield for \( x_2 \leq x_1 \leq \delta_\alpha \) (> 0)

\[
\int_{\mathbb{R} \times (0, x_2)} |G(x, y)| \, dy \leq \frac{1}{\alpha} \left( \frac{1}{1 - 2\alpha} - 2^{-\alpha} \right) x_1^{1-2\alpha}
\]

\[
- \int_{A(x)} G(x, y) \, dy \leq - \frac{1}{\alpha} \left( \frac{1}{6 \cdot 20^\alpha} \right) x_1^{1-2\alpha}
\]

and we get for small \( \alpha \) and \( x \in I_1 \cup I_2 \) (using \( x_1 \geq X(t) \))

\[
u_1(x, t) \leq - \frac{1}{50^\alpha} x_1^{1-2\alpha} < - \frac{1}{100^\alpha} X(t)^{1-2\alpha} = X'(t)
\]

So touch cannot happen on \( I_1 \).

Similarly, for small \( \alpha \) and \( x \in I_2 \)

\[
u_2(x, t) \geq \frac{1}{50^\alpha} x_2^{1-2\alpha} > 0
\]

so touch cannot happen on \( I_2 \).