Imaging through random media by speckle intensity correlations

Josselin Garnier (Ecole Polytechnique)
http://www.josselin-garnier.org/

Sensor array imaging and (some of) its main limitations

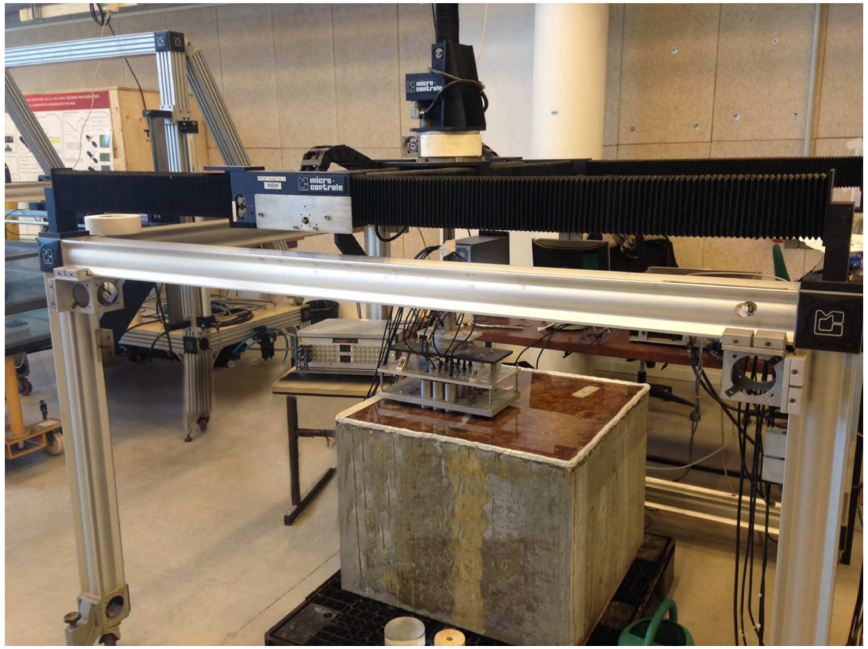
• Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, etc) has two steps:
- data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver array.
- data processing: the recorded signals are processed to identify the quantities of interest (source locations, reflector locations, etc).

• Example:
Ultrasound echography

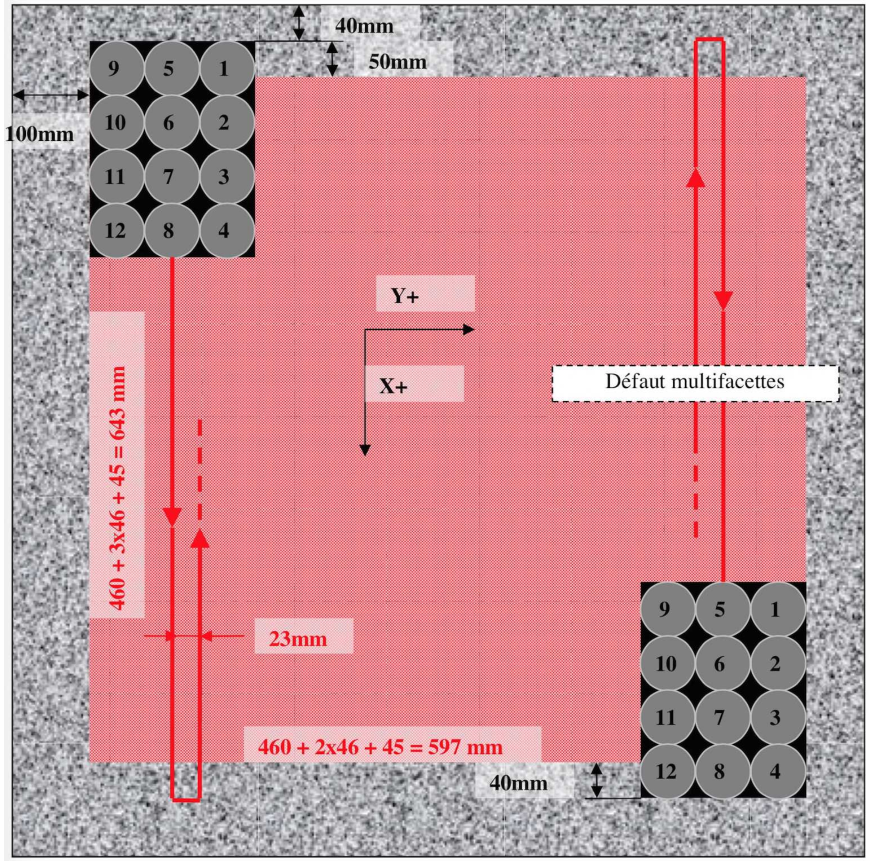
$\longrightarrow$

• Standard imaging techniques require:
- suitable conditions for wave propagation (ideally, homogeneous medium),
- controlled and known sources.

Ultrasound echography in concrete



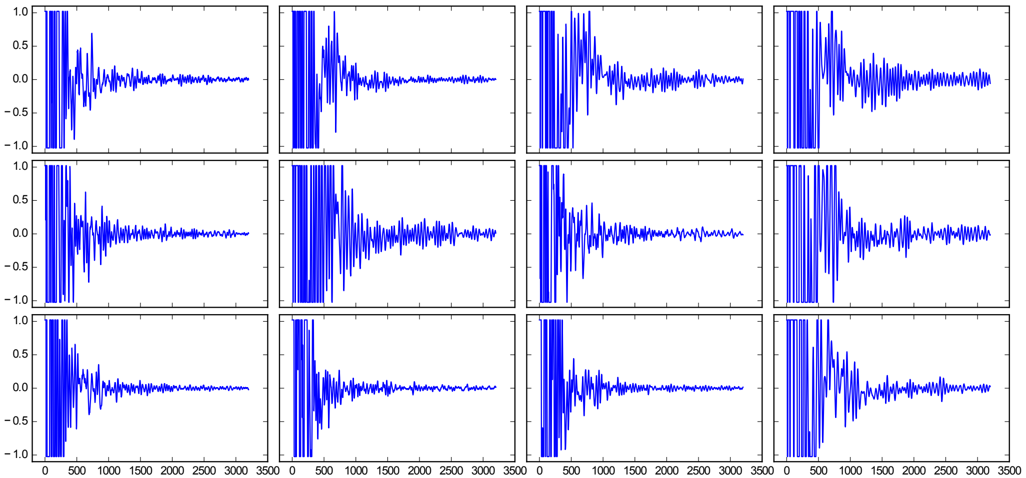
Experimental set-up



Acquisition geometry (top view)

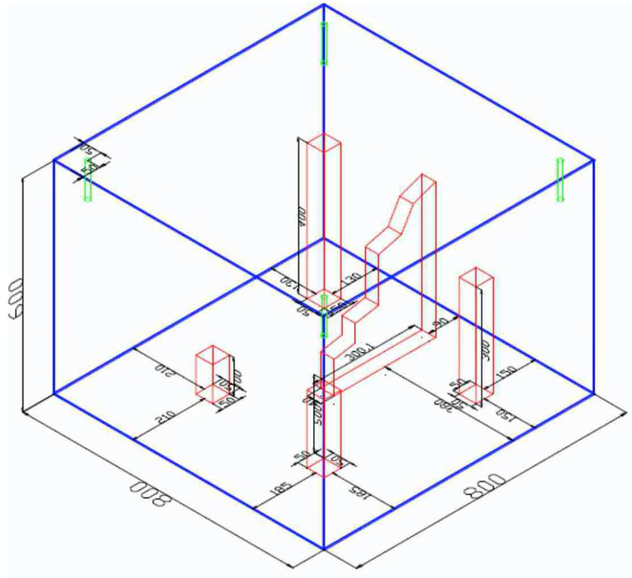
Concrete: highly scattering medium for ultrasonic waves.

Ultrasound echography in concrete



Data

$\xrightarrow{?}$

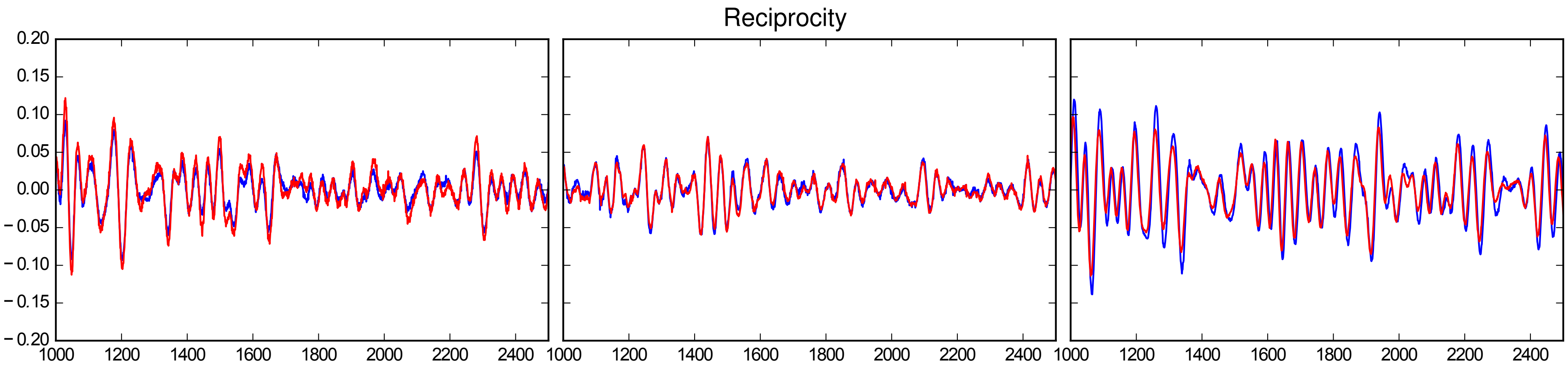


Real configuration

The recorded signals are very “noisy” due to scattering.

$\hookrightarrow$ Standard imaging techniques fail.

Ultrasound echography in concrete



Reciprocity: $u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) = u(t, \vec{\boldsymbol{x}}_s; \vec{\boldsymbol{x}}_r)$ for (almost) all pairs $(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_s)$.

$\hookrightarrow$ The data set is good !

Wave propagation in random media

- Wave equation:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} \frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}) = F(t, \vec{\boldsymbol{x}})$$

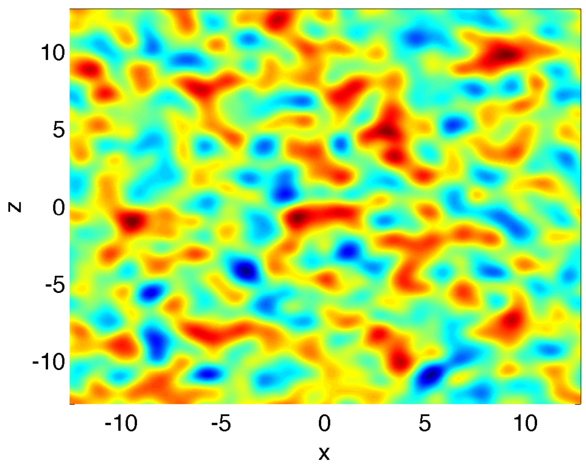
- Time-harmonic source in the plane $z = 0$: $F(t, \vec{\boldsymbol{x}}) = \delta(z) f(\boldsymbol{x}) e^{-i\omega t}$ (with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$).

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_o^2}\left(1 + \mu(\vec{\boldsymbol{x}})\right)$$

c_o is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



Wave propagation in the random paraxial regime

- Consider the time-harmonic wave equation (with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$, $\Delta = \Delta_\perp + \partial_z^2$)

$$(\Delta_\perp + \partial_z^2)\hat{u} + \frac{\omega^2}{c_o^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = -\delta(z) f(\boldsymbol{x}).$$

- Consider the paraxial regime "$\lambda \ll l_c, r_o \ll L$":

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big), \qquad f(\boldsymbol{x}) \to f\big(\frac{\boldsymbol{x}}{\varepsilon^2}\big).$$

The function $\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z)$ is solution of

$$(\Delta_\perp + \partial_z^2)\hat{u}^\varepsilon + \frac{\omega^2}{c_o^2 \varepsilon^8}\big(1 + \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big)\big)\hat{u}^\varepsilon = -\delta(z) f\big(\frac{\boldsymbol{x}}{\varepsilon^2}\big).$$

- The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = \frac{i\varepsilon^4 c_o}{2\omega} \exp\big(i\frac{\omega z}{\varepsilon^4 c_o}\big)\hat{\phi}^\varepsilon\big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_o}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_o^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 2i\frac{\omega}{c_o}\delta(z) f(\boldsymbol{x}).$$

Wave propagation in the random paraxial regime

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = \frac{i\varepsilon^4 c_o}{2\omega} \exp\Big(i\frac{\omega z}{\varepsilon^4 c_o}\Big)\hat{\phi}^\varepsilon\Big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\Big)$$

satisfies

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- $\hat{\phi}^\varepsilon$ converges in distribution in $C^0([0,L], L^2(\mathbb{R}^2))$ (or $C^0([0,L], H^k(\mathbb{R}^2))$) to $\hat{\phi}$ that is the unique solution of the Itô-Schrödinger equation [1]

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$, $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$, and $\hat{\phi}(z=0, \boldsymbol{x}) = f(\boldsymbol{x})$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

Moment calculations in the random paraxial regime

Consider

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z = 0) = f(\boldsymbol{x})$.

• By Itô's formula,

$$\frac{d}{dz}\mathbb{E}[\hat{\phi}] = \frac{ic_o}{2\omega}\Delta_\perp \mathbb{E}[\hat{\phi}] - \frac{\omega^2 \gamma(\mathbf{0})}{8c_o^2}\mathbb{E}[\hat{\phi}]$$

and therefore

$$\mathbb{E}\big[\hat{\phi}(\boldsymbol{x}, z)\big] = \hat{\phi}_{\rm hom}(\boldsymbol{x}, z) \exp\Big(- \frac{\gamma(\mathbf{0})\omega^2 z}{8c_o^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$ and $\hat{\phi}_{\rm hom}$ is the solution in the homogeneous medium.

• Strong damping of the coherent wave.

$\Longrightarrow$ Identification of the *scattering mean free path* $Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$.

$\Longrightarrow$ Coherent imaging methods (such as Kirchhoff migration, Reverse-Time migration) fail.

Moment calculations in the random paraxial regime

- The mean Wigner transform defined by

$$W(\boldsymbol{r},\boldsymbol{\xi},z) = \int_{\mathbb{R}^2} \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{q}\big)\mathbb{E}\Big[\hat{\phi}\big(\boldsymbol{r}+\frac{\boldsymbol{q}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{r}-\frac{\boldsymbol{q}}{2},z\big)\Big]\,d\boldsymbol{q},$$

is the angularly-resolved mean wave energy density.

By Itô's formula, it solves a *radiative transport-like equation*

$$\frac{\partial W}{\partial z} + \frac{c_o}{\omega}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{r}} W = \frac{\omega^2}{4(2\pi)^2 c_o^2}\int_{\mathbb{R}^2}\hat{\gamma}(\boldsymbol{\kappa})\Big[W(\boldsymbol{\xi}-\boldsymbol{\kappa})-W(\boldsymbol{\xi})\Big]d\boldsymbol{\kappa},$$

starting from $W(\boldsymbol{r},\boldsymbol{\xi},z=0)=W_0(\boldsymbol{r},\boldsymbol{\xi})$, the Wigner transform of the initial field f.

- The fields at nearby points are correlated and their correlations contain information about the medium.

$\Longrightarrow$ One should use (migrate) cross correlations for imaging in random media.

Beyond the second-order moments

- Fourth-order moments are useful to:
- quantify the statistical stability of correlation-based imaging methods.
- implement intensity-correlation-based imaging methods when only intensities can be measured (optics).

Moment calculations in the random paraxial regime

- Consider

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z = 0) = f(\boldsymbol{x})$.

- Let us consider the fourth-order moment:

$$\begin{aligned} M_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{q}_1, \boldsymbol{q}_2, z) \quad &= \quad \mathbb{E}\Big[\hat{\phi}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\hat{\phi}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big) \\ &\quad \times\overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big)\overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 - \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\Big] \end{aligned}$$

By Itô's formula,

$$\frac{\partial M_4}{\partial z} = \frac{ic_o}{\omega}\big(\nabla_{\boldsymbol{r}_1} \cdot \nabla_{\boldsymbol{q}_1} + \nabla_{\boldsymbol{r}_2} \cdot \nabla_{\boldsymbol{q}_2}\big) M_4 + \frac{\omega^2}{4c_o^2} U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) M_4,$$

with the generalized potential

$$\begin{aligned} U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) \quad &= \quad \gamma(\boldsymbol{q}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 - \boldsymbol{q}_1) \\ &\quad -\gamma(\boldsymbol{q}_2 + \boldsymbol{r}_2) - \gamma(\boldsymbol{q}_2 - \boldsymbol{r}_2) - 2\gamma(\boldsymbol{0}). \end{aligned}$$

$\Longrightarrow$ One can get a general (but complicated) characterization of the fourth-order moment [1].

[1] J. Garnier and K. Sølna, *ARMA* **220** (2016) 37.

Stability of the Wigner transform of the field

$$W(\boldsymbol{r},\boldsymbol{\xi},z) := \int_{\mathbb{R}^2} \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{q}\big)\hat{\phi}\big(\boldsymbol{r}+\frac{\boldsymbol{q}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{r}-\frac{\boldsymbol{q}}{2},z\big)d\boldsymbol{q}.$$

Let us consider two positive parameters $r_{\rm s}$ and $\xi_{\rm s}$ and define the smoothed Wigner transform:

$$W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) = \frac{1}{(2\pi)^2 r_{\rm s}^2\xi_{\rm s}^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} W(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{\xi}-\boldsymbol{\xi}',z)\exp\Big(-\frac{|\boldsymbol{r}'|^2}{2r_{\rm s}^2}-\frac{|\boldsymbol{\xi}'|^2}{2\xi_{\rm s}^2}\Big)d\boldsymbol{r}'d\boldsymbol{\xi}'.$$

- The coefficient of variation $C_{\rm s}$ of the smoothed Wigner transform is defined by:

$$C_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) := \frac{\sqrt{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)^2]-\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]^2}}{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]}.$$

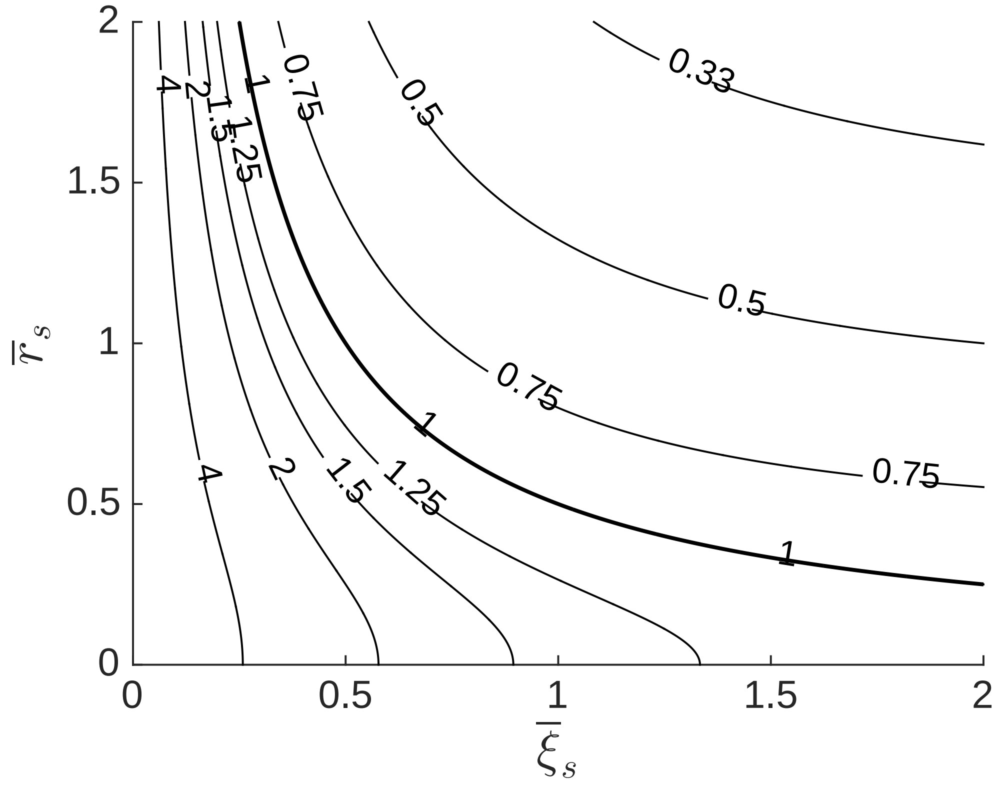
satisfies

$$C_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) \simeq \left(\frac{\frac{1}{\xi_{\rm s}^2\rho_z^2}+1}{\frac{4r_{\rm s}^2}{\rho_z^2}+1}\right)^{1/2}, \qquad \rho_z^2 = \frac{\ell_{\rm c}^2}{4Z_{\rm sca}z}\frac{r_o^2+\frac{8c_o^2z^3}{3\omega^2\ell_{\rm c}^2Z_{\rm sca}}}{r_o^2+\frac{2c_o^2z^3}{3\omega^2\ell_{\rm c}^2Z_{\rm sca}}},$$

when

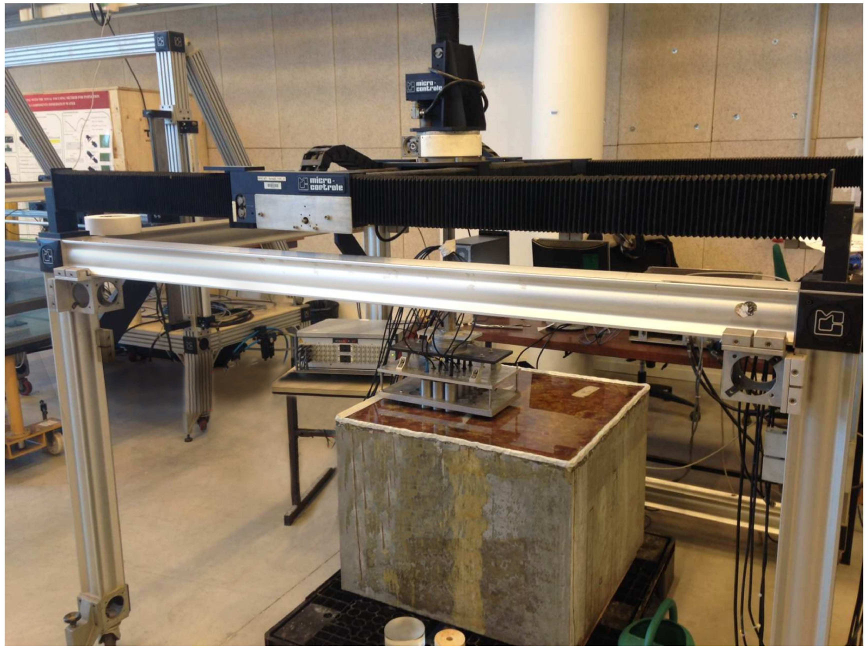
$$\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{0})\Big[1-\frac{|\boldsymbol{x}|^2}{\ell_{\rm c}^2}+o\Big(\frac{|\boldsymbol{x}|^2}{\ell_{\rm c}^2}\Big)\Big], \qquad z \gg Z_{\rm sca} = \frac{8c_o^2}{\gamma(\boldsymbol{0})\omega^2}.$$

Stability of the Wigner transform of the field

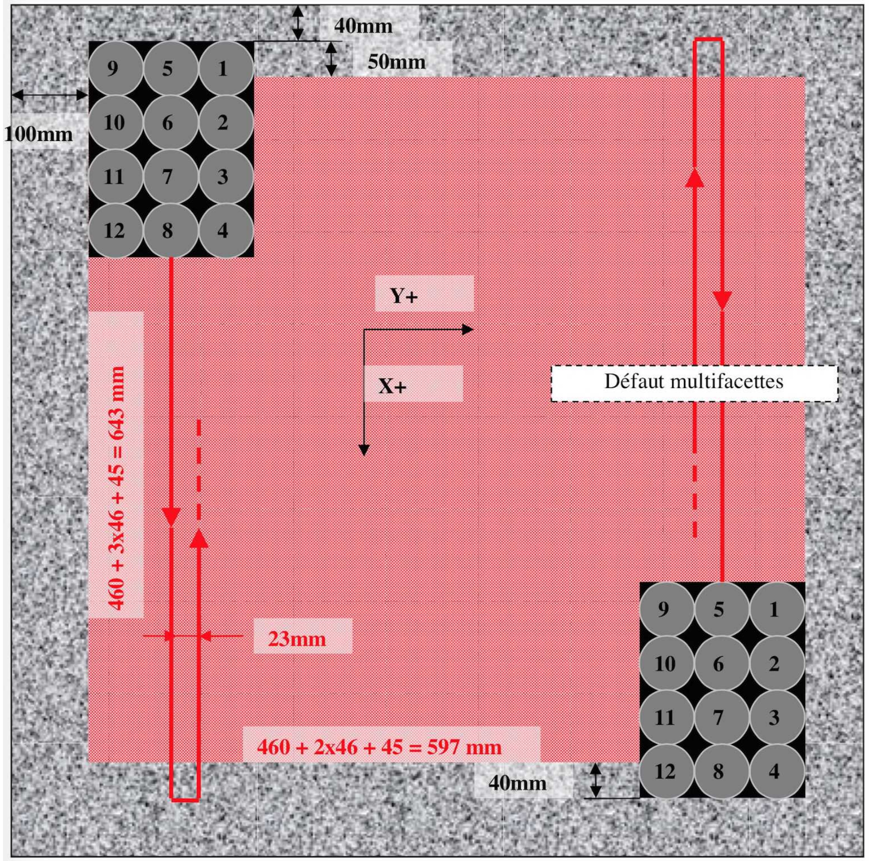


Contour levels of the coefficient of variation of the smoothed Wigner transform. Here $\overline{r}_\mathrm{s} = r_\mathrm{s}/\rho_z$ and $\overline{\xi}_\mathrm{s} = \xi_\mathrm{s}\rho_z$.

Application: Ultrasound echography in concrete



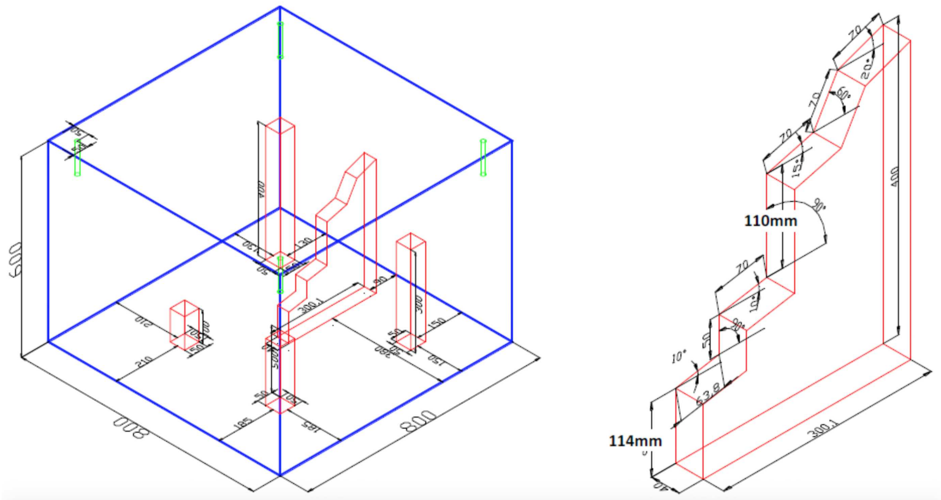
Experimental set-up

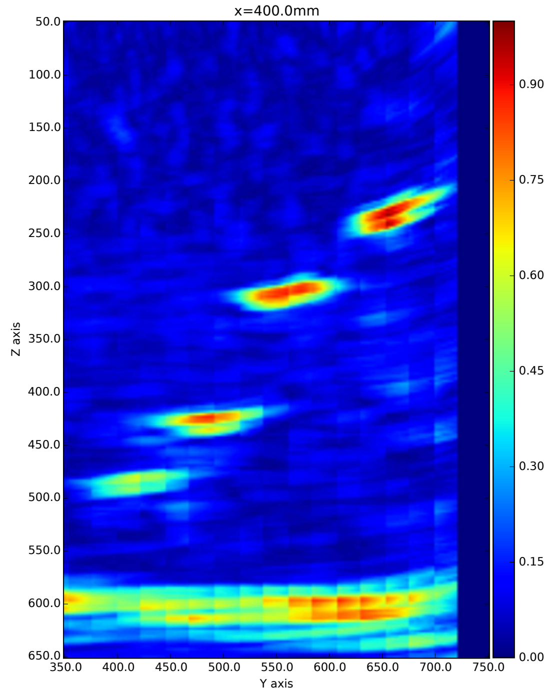


Acquisition geometry (top view)

Concrete: highly scattering medium for ultrasonic waves.

Application: Ultrasound echography in concrete





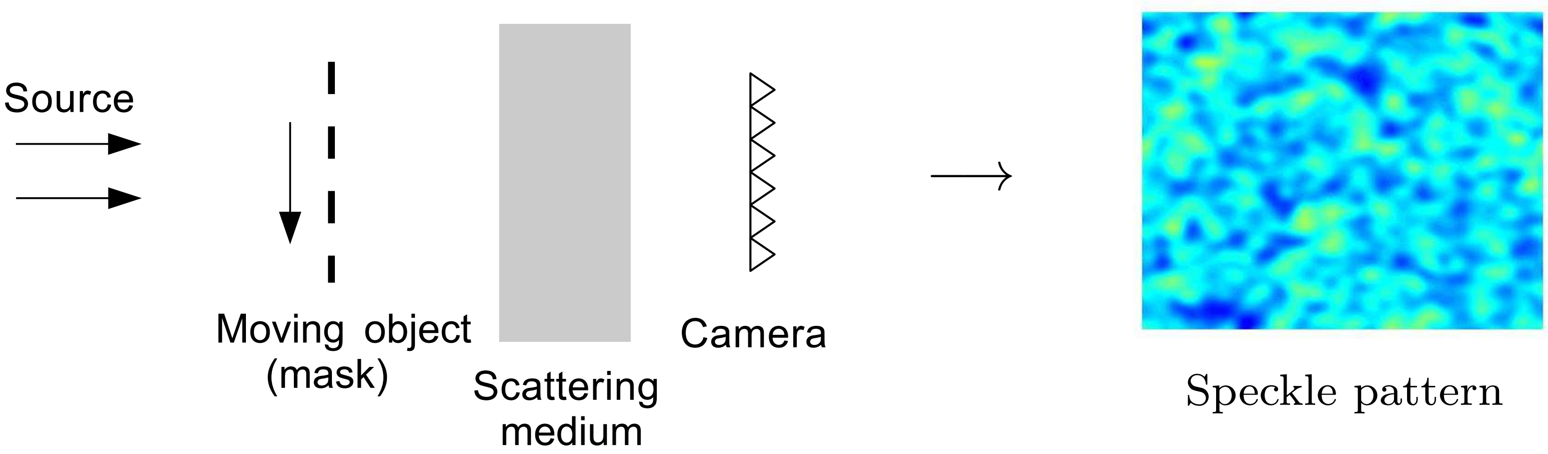
Real configuration   Image (2D slice)

Image obtained by travel-time migration of *well-chosen* cross correlations of data.

Beyond acoustics

- In optics: only intensities are measured (square moduli of complex amplitudes).

Speckle intensity correlation imaging through a scattering medium

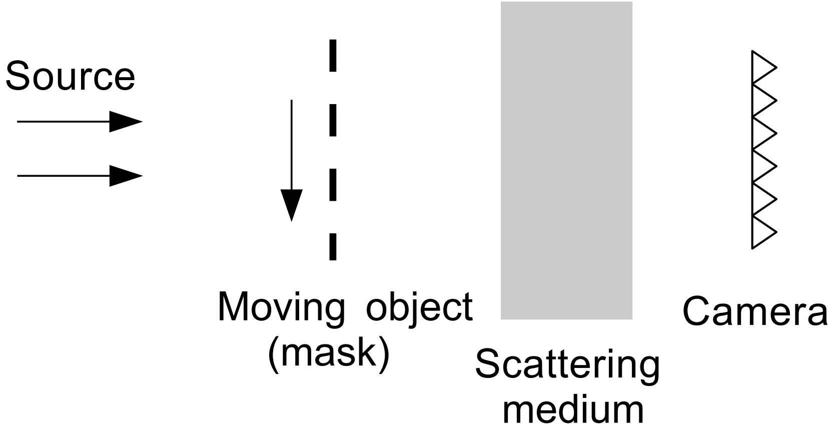


Experimental set-up [1]

- The light source is a time-harmonic plane wave.
- The object to be imaged is a mask that can be shifted transversally.
- For each position of the object the spatial intensity of the transmitted field (speckle pattern) can be recorded by the camera.

[1] J. A. Newmann and K. J. Webb, PRL **113**, 263903 (2014).

Speckle intensity correlation imaging through a scattering medium



- The field just after the object is of the form

$$U_{\boldsymbol{r}}(\boldsymbol{x}) = U(\boldsymbol{x} - \boldsymbol{r}),$$

for some function U (typically, the indicator function of the mask).

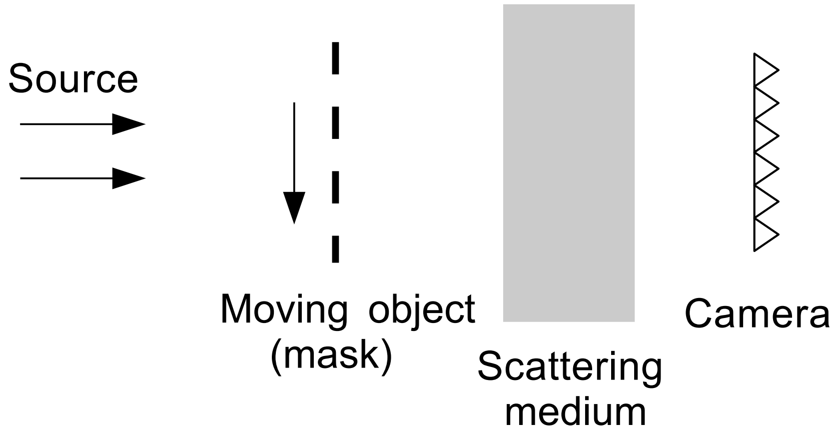
- The field in the plane of the camera is denoted by $E_{\boldsymbol{r}}(\boldsymbol{x})$.

The measured intensity correlation is

$$\begin{aligned} C_{\boldsymbol{r},\boldsymbol{r}'} &= \frac{1}{|A_0|}\int_{A_0} |E_{\boldsymbol{r}}(\boldsymbol{x})|^2 |E_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &\quad - \Big(\frac{1}{|A_0|}\int_{A_0} |E_{\boldsymbol{r}}(\boldsymbol{x})|^2 d\boldsymbol{x}\Big)\Big(\frac{1}{|A_0|}\int_{A_0} |E_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x}\Big), \end{aligned}$$

where A_0 is the spatial support of the camera.

Speckle intensity correlation imaging through a scattering medium

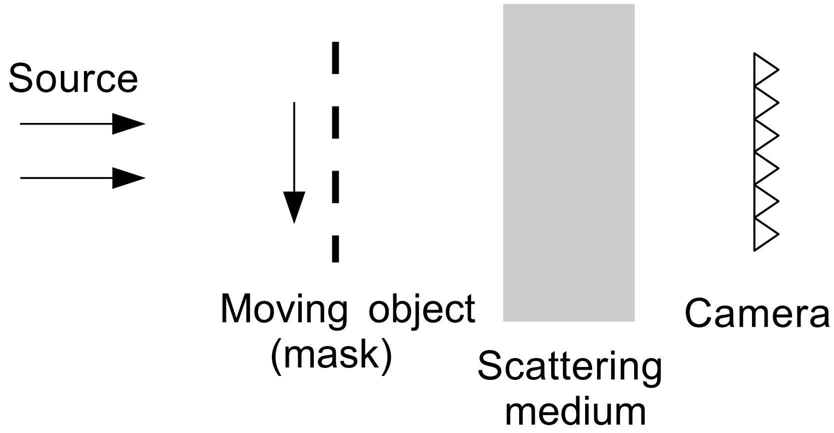


- Result:

$$\begin{aligned}\mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \;=\; & \int_{A_0} d\boldsymbol{X} \int d\boldsymbol{Y} \Big| \frac{1}{(2\pi)^2} \int \Big(\int U\big(\boldsymbol{x} + \frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big)\overline{U}\big(\boldsymbol{x} - \frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big) \exp\big(-i\boldsymbol{\zeta}\cdot\boldsymbol{x}\big) d\boldsymbol{x}\Big) \\ & \times \exp\Big(i\boldsymbol{\zeta}\cdot\big(\boldsymbol{X} - \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\big)\Big) \exp\Big(\frac{\omega^2}{4c_o^2}\int_0^L \gamma\big(\frac{c_o\boldsymbol{\zeta}}{\omega}z - \boldsymbol{Y}\big) - \gamma(\boldsymbol{0}) dz\Big) d\boldsymbol{\zeta}\Big|^2 \\ & - \Big|\frac{1}{(2\pi)^2}\int_{A_0} d\boldsymbol{X} \Big(\int U\big(\boldsymbol{x} + \frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big)\overline{U}\big(\boldsymbol{x} - \frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big) \exp\big(-i\boldsymbol{\zeta}\cdot\boldsymbol{x}\big) d\boldsymbol{x}\Big) \\ & \times \exp\Big(i\boldsymbol{\zeta}\cdot\big(\boldsymbol{X} - \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\big)\Big) \exp\Big(-\frac{\omega^2}{4c_o^2}\gamma(\boldsymbol{0})L\Big) d\boldsymbol{\zeta}\Big|^2,\end{aligned}$$

with $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0},0)\mu(\boldsymbol{x},z)]dz$.

Speckle intensity correlation imaging through a scattering medium



- Result: When $L \gg Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$ and

$$\frac{c_o^2 L^3}{\omega^2 Z_{\rm sca} \ell_c^2} \gg |A_0| (\sim \text{diam(camera)}^2) \gg \frac{Z_{\rm sca}\ell_c^2}{L}$$

we have

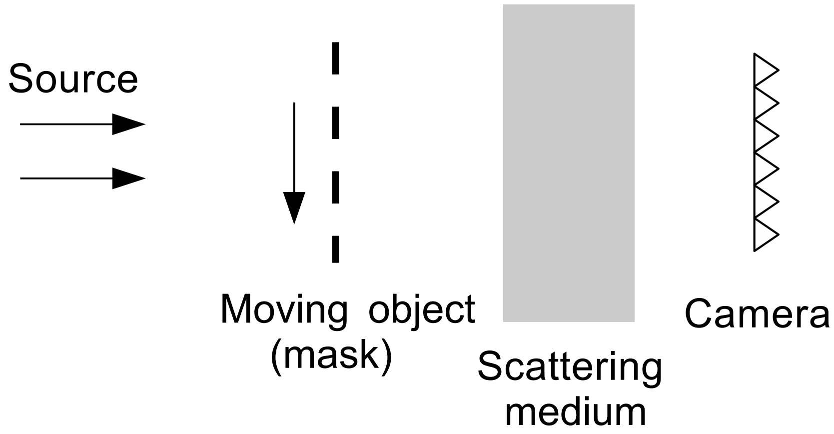
$$C_{\boldsymbol{r},\boldsymbol{r}'} \simeq \mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \approx \Big| \int |\hat{U}(\boldsymbol{\kappa})|^2 \exp\big(i\boldsymbol{\kappa}\cdot(\boldsymbol{r}'-\boldsymbol{r})\big) d\boldsymbol{\kappa}\Big|^2,$$

up to a multiplicative constant, where

$$\hat{U}(\boldsymbol{\kappa}) = \int U(\boldsymbol{x}) \exp\big(-i\boldsymbol{\kappa}\cdot\boldsymbol{x}\big) d\boldsymbol{x}.$$

$\hookrightarrow$ It is possible to reconstruct the incident field U.

Speckle intensity correlation imaging through a scattering medium



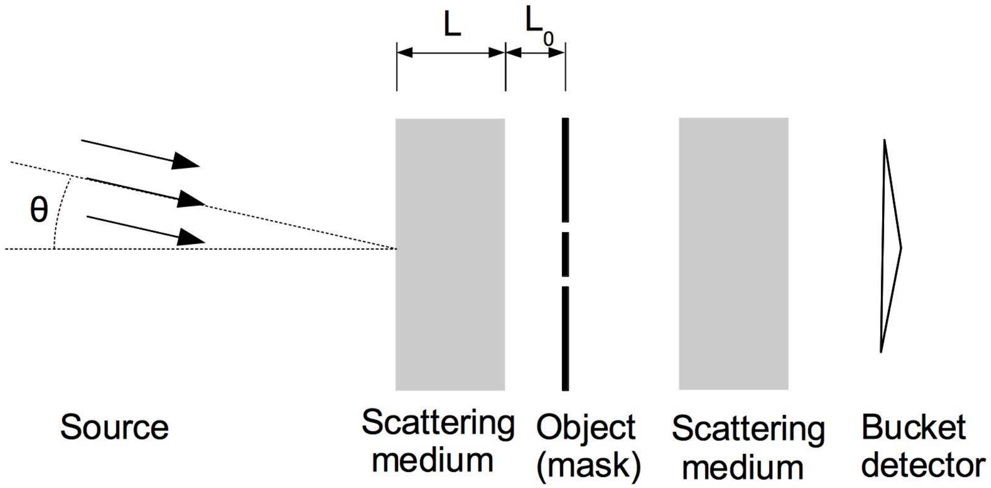
- We have

$$C_{\boldsymbol{r},\boldsymbol{r}'} \simeq \mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \approx \Big| \int |\hat{U}(\boldsymbol{\kappa})|^2 \exp\big(i\boldsymbol{\kappa}\cdot(\boldsymbol{r}'-\boldsymbol{r})\big) d\boldsymbol{\kappa} \Big|^2$$

$\hookrightarrow$ It is possible to reconstruct the incident field U by a two-step phase retrieval algorithm (Gerchberg-Saxon-type).

1) Given $C_{\boldsymbol{r},\boldsymbol{r}'}$, we know the modulus of the (I)FT of $|\hat{U}(\boldsymbol{\kappa})|^2$, and we know the phase of $|\hat{U}(\boldsymbol{\kappa})|^2$ (zero) $\rightarrow$ we can extract $|\hat{U}(\boldsymbol{\kappa})|^2$.
2) Given $|\hat{U}(\boldsymbol{\kappa})|^2$, we know the modulus of the FT of $U(\boldsymbol{x})$, and we know the phase of $U(\boldsymbol{x})$ (zero) $\rightarrow$ we can extract $U(\boldsymbol{x})$.

Speckle intensity correlation imaging through a scattering medium (II)



Experimental set-up

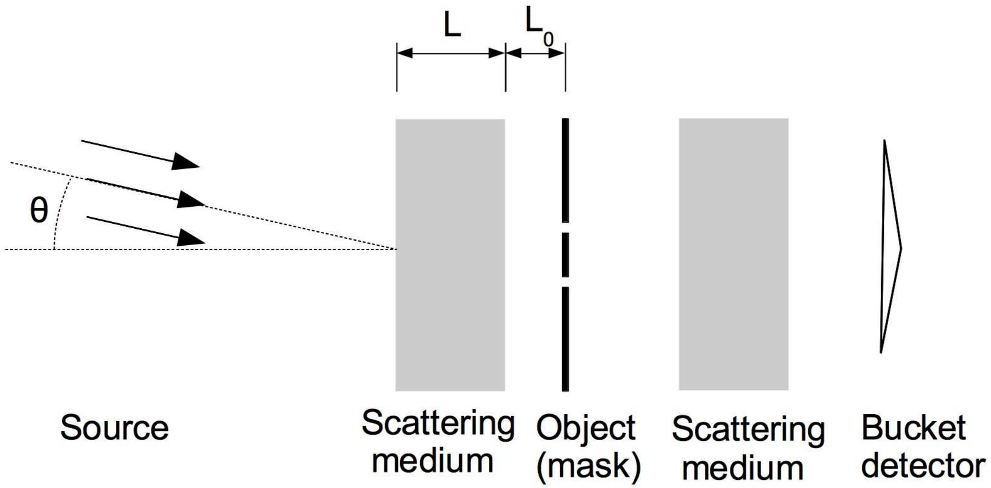
• A laser beam with incident angle $\boldsymbol{\theta}$ is shined on the scattering medium.

• The object to be imaged is a mask.

• The total intensity of the light that goes through the mask is collected by a bucket detector.

$\rightarrow$ For each incident angle $\boldsymbol{\theta}$ the total transmitted intensity $\mathcal{E}_{\boldsymbol{\theta}}$ is measured.

Speckle intensity correlation imaging through a scattering medium (II)



Consider:

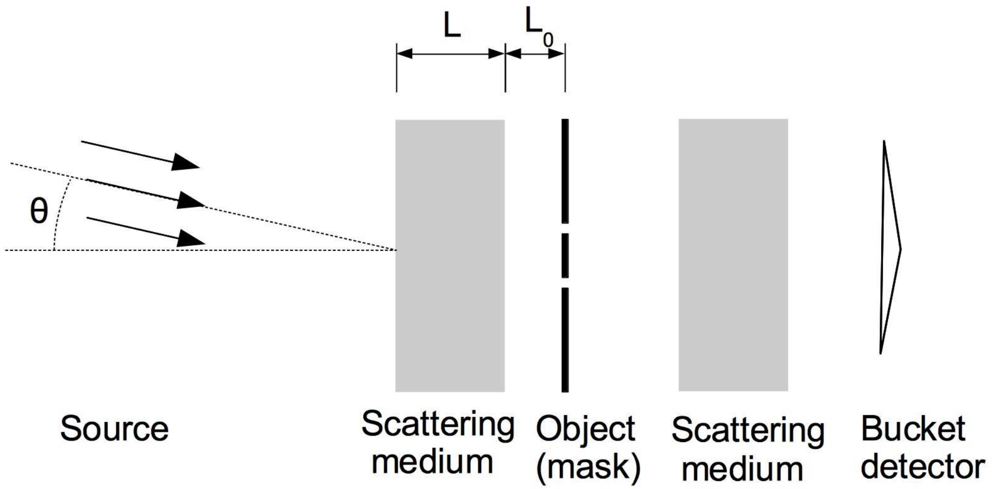
$$\mathcal{C}(\Delta\boldsymbol{\theta}) \simeq \frac{1}{\Theta}\int_{\Theta} \mathcal{E}_{\boldsymbol{\theta}}\mathcal{E}_{\boldsymbol{\theta}+\Delta\boldsymbol{\theta}} d\boldsymbol{\theta} - \Big(\frac{1}{\Theta}\int_{\Theta} \mathcal{E}_{\boldsymbol{\theta}} d\boldsymbol{\theta}\Big)^2$$

- Result:

$$\mathbb{E}[\mathcal{C}(\Delta\boldsymbol{\theta})] = \frac{1}{(2\pi)^2}\iint \exp\Big(\frac{\omega^2}{2c_o^2}\int_0^L \gamma\big(\boldsymbol{x}+\Delta\boldsymbol{\theta}(z+L_o)\big)dz\Big) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} |\hat{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} d\boldsymbol{x}$$
$$\times \exp\Big(-\frac{\omega^2\gamma(\mathbf{0})L}{2c_o^2}\Big) - |\hat{U}(\mathbf{0})|^2 \exp\Big(-\frac{\omega^2\gamma_o(\mathbf{0})L}{2c_o^2}\Big),$$

with $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$.

Speckle intensity correlation imaging through a scattering medium (II)



- Result: When $L \gg Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$, then

$$\mathcal{C}(\Delta\boldsymbol{\theta}) \simeq \mathbb{E}[\mathcal{C}(\Delta\boldsymbol{\theta})] \approx \left(\mathcal{K}_{\rho_L} \star (U \star U)\right)(\Delta\boldsymbol{\theta}\ell_o) \exp\Big(-\frac{L^2}{12\rho_L^2}|\Delta\boldsymbol{\theta}|^2\Big),$$

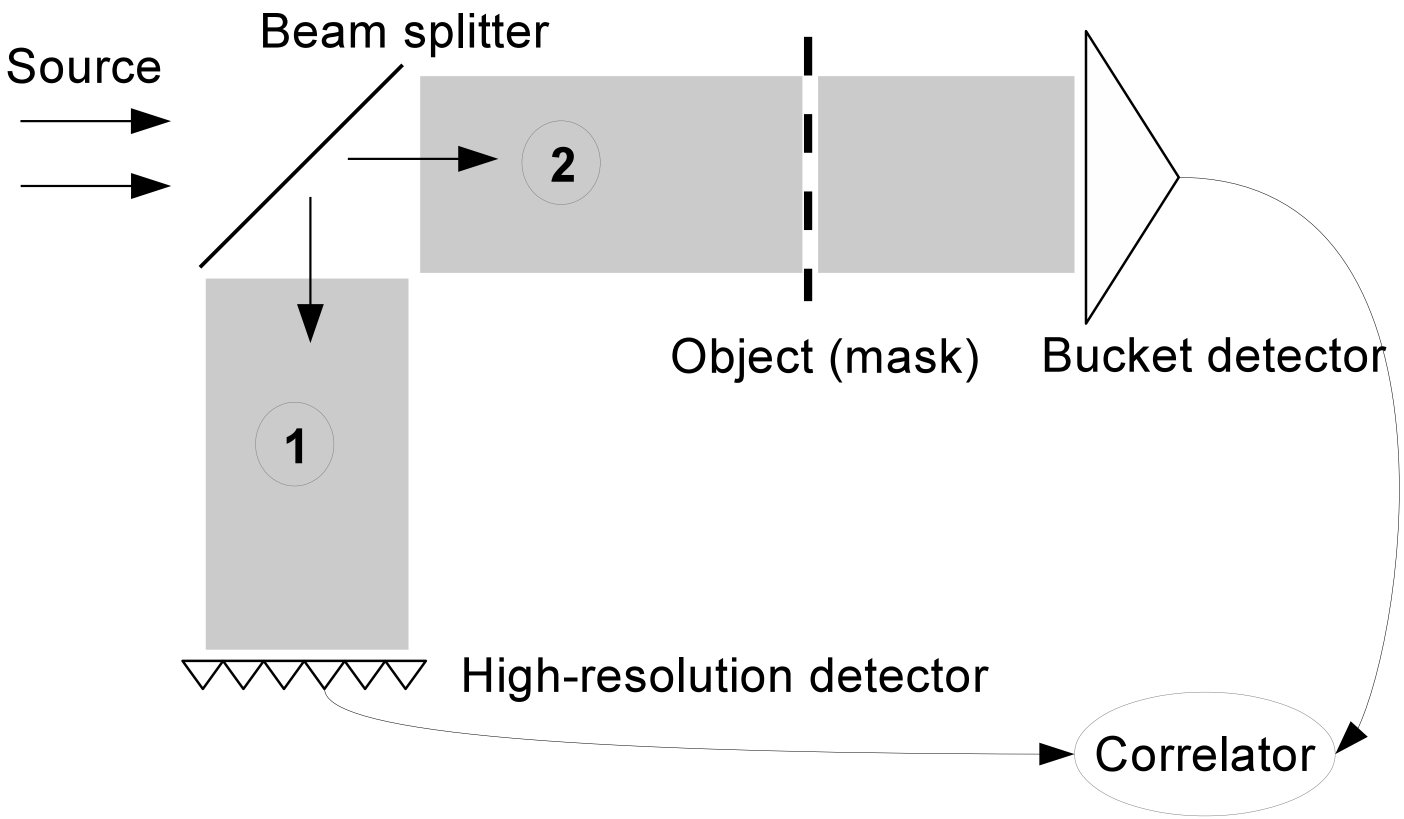
up to a multiplicative constant, where $f \star g(\boldsymbol{x}) = \int f(\boldsymbol{x}')g(\boldsymbol{x}+\boldsymbol{x}')d\boldsymbol{x}'$,

$$\mathcal{K}_{\rho_L}(\boldsymbol{x}) = \frac{1}{\pi\rho_L^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{\rho_L^2}\Big), \qquad \rho_L := \frac{2c_o\ell_c}{\omega\sqrt{\gamma(\mathbf{0})L}},$$

(ρ_L is the correlation radius of the speckle pattern at $z = L$), and $\ell_o = L_o + \frac{L}{2}$.

$\rightarrow$ If ρ_L is small and L_o is large enough, then one can extract $|\hat{U}(\boldsymbol{\kappa})|^2$ and then $U(\boldsymbol{x})$ by a phase retrieval algorithm.

Ghost imaging



- Noise source (laser light passed through a rotating glass diffuser).
- without object in path 1; a high-resolution detector measures the spatially-resolved intensity $I_1(t, \boldsymbol{x})$.
- with object (mask) in path 2; a single-pixel detector measures the spatially-integrated intensity $I_2(t)$.

Experiment: the correlation of $I_1(\cdot, \boldsymbol{x})$ and $I_2(\cdot)$ is an image of the object [1,2].

[1] A. Valencia et al., *PRL* **94**, 063601 (2005); [2] J. H. Shapiro et al., *Quantum Inf. Process* **1**, 949 (2012).

Ghost imaging

- Wave equation in paths 1 and 2:

$$\frac{1}{c_j^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u_j}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u_j = e^{-i\omega_o t} n(t,\boldsymbol{x})\delta(z) + c.c., \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \qquad j = 1,2$$

- Noise source (with Gaussian statistics):

$$\left\langle n(t,\boldsymbol{x})\overline{n(t,\boldsymbol{x}')}\right\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_o^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}')$$

- Wave fields: $u_j(t,\vec{\boldsymbol{x}}) = v_j(t,\vec{\boldsymbol{x}})e^{-i\omega_o t} + c.c., \qquad j = 1,2$
- Intensity measurements:

$$\begin{aligned} I_1(t,\boldsymbol{x}) &= |v_1(t,(\boldsymbol{x},L))|^2 \text{ in the plane of the high-resolution detector} \\ I_2(t) &= \int_{\mathbb{R}^2} |v_2(t,(\boldsymbol{x}',L+L_0))|^2 d\boldsymbol{x}' \text{ in the plane of the bucket detector} \end{aligned}$$

- Correlation:

$$C_T(\boldsymbol{x}) = \frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})I_2(t)dt - \Big(\frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})dt\Big)\Big(\frac{1}{T}\int_0^T I_2(t)dt\Big)$$

Ghost imaging in homogeneous media

- Resolution analysis in homogeneous media.
- Model for the object: Mask $\mathcal{T}(\boldsymbol{x})$ in the plane $z = L$.
- Result:

$$C_T(\boldsymbol{x}) \stackrel{T\to\infty}{\longrightarrow} C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}-\boldsymbol{z})|\mathcal{T}(\boldsymbol{z})|^2 d\boldsymbol{z}$$

with

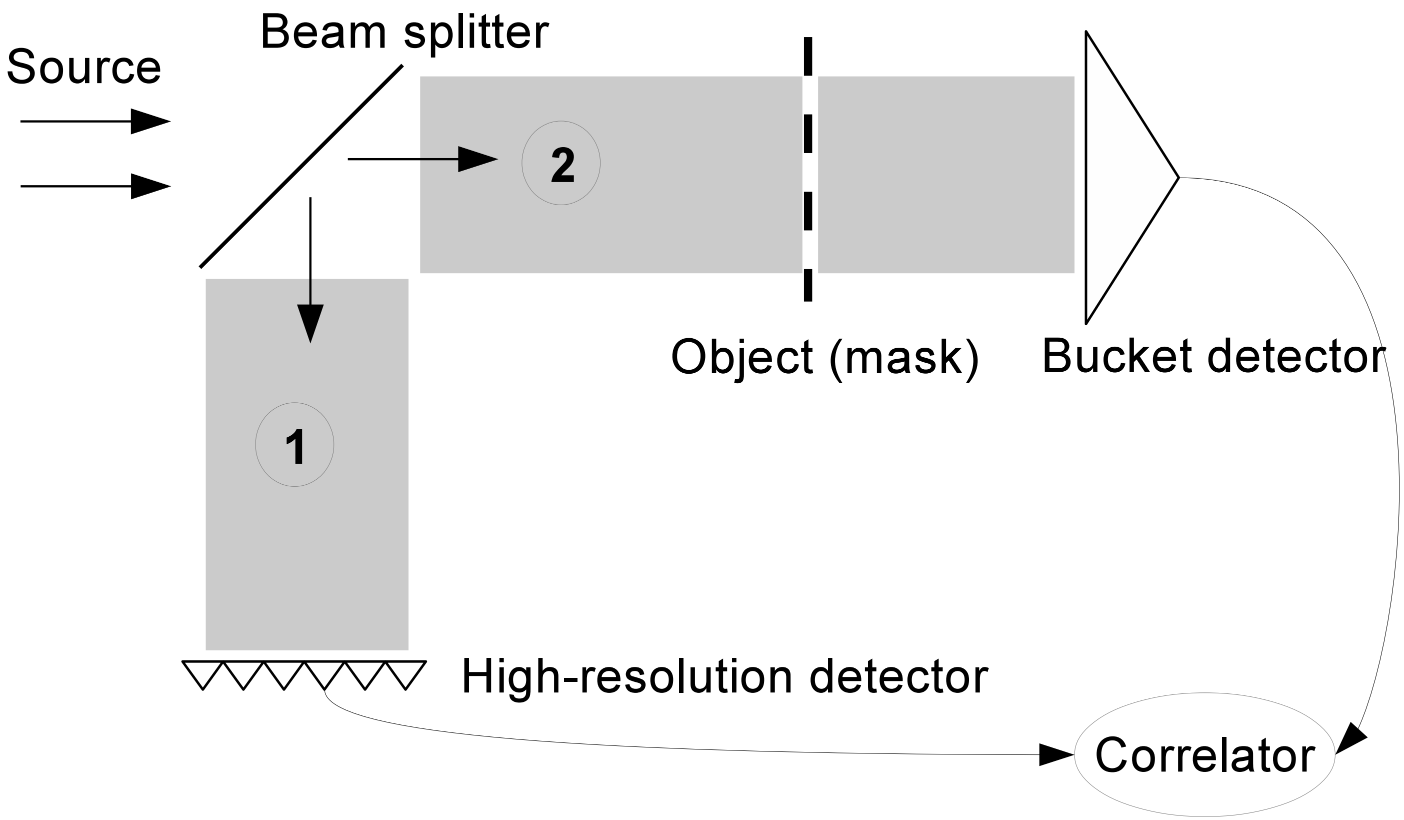
$$h(\boldsymbol{x}) = \frac{r_o^4}{2^8\pi^2 L^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi0}^2}\Big), \qquad \rho_{\rm gi0}^2 = \frac{c_o^2 L^2}{2\omega_o^2 r_o^2}$$

Resolution: $\rho_{\rm gi0} \sim \lambda_o L / r_o$ (Rayleigh resolution formula).

Sketch of ideal proof. Use the Gaussian summation rule (the fourth-order moments of Gaussian random fields can be expressed in terms of sums of products of second-order moments).
If $v(\boldsymbol{x})$ is a complex symmetric circular Gaussian random field, then

$$\mathrm{Cov}\big(|v(\boldsymbol{x})|^2, |v(\boldsymbol{x}')|^2\big) = \Big|\mathrm{Cov}\big(v(\boldsymbol{x}), \overline{v(\boldsymbol{x}')}\big)\Big|^2$$

Ghost imaging in heterogeneous media



The medium in paths 1 and 2 is heterogeneous (for instance, turbulent atmosphere).
They are two independent realizations with the same distribution.

Ghost imaging in heterogeneous media

- Resolution analysis in randomly heterogeneous media.
- If the propagation distance is larger than the scattering mean free path, then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y}) |\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

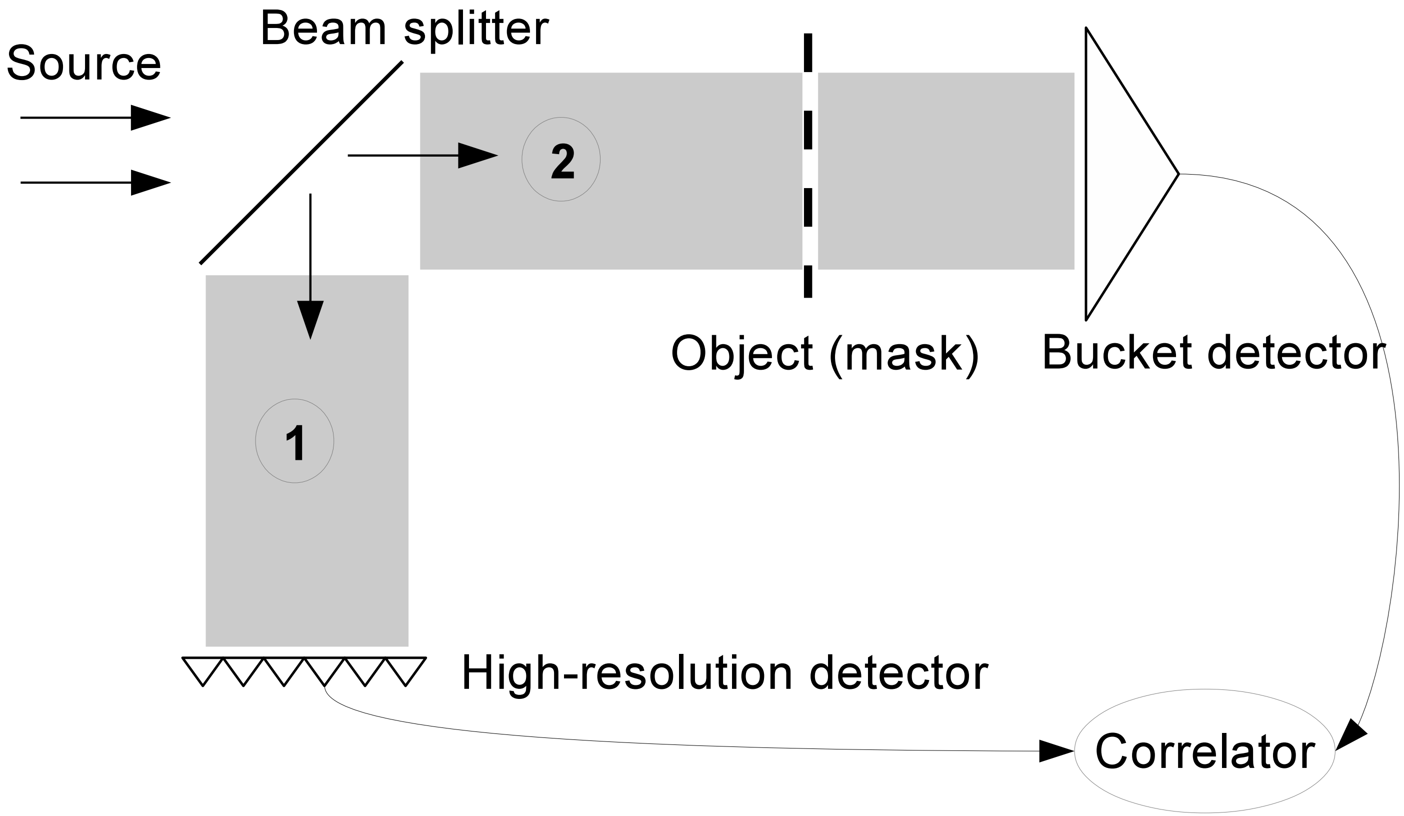
with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4 \rho_{\rm gi0}^2}{2^8 \pi^2 L^4 \rho_{\rm gi2}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi2}^2}\Big), \qquad \rho_{\rm gi2}^2 = \rho_{\rm gi0}^2 + \frac{4c_o^2 L^3}{3\omega_o^2 Z_{\rm sca} \ell_{\rm c}^2}, \qquad \rho_{\rm gi0}^2 = \frac{c_o^2 L^2}{2\omega_o^2 r_o^2}$$

↪ Scattering only slightly reduces the resolution !

This imaging method is robust with respect to medium noise. It gives an image even when $L/Z_{\rm sca} \gg 1$.

Ghost imaging in heterogeneous identical media



The medium in paths 1 and 2 is heterogeneous.

They are the *same realization*.

Ghost imaging in heterogeneous identical media

- Resolution analysis in randomly heterogeneous and identical media.
- If the propagation distance is larger than the scattering mean free path, then

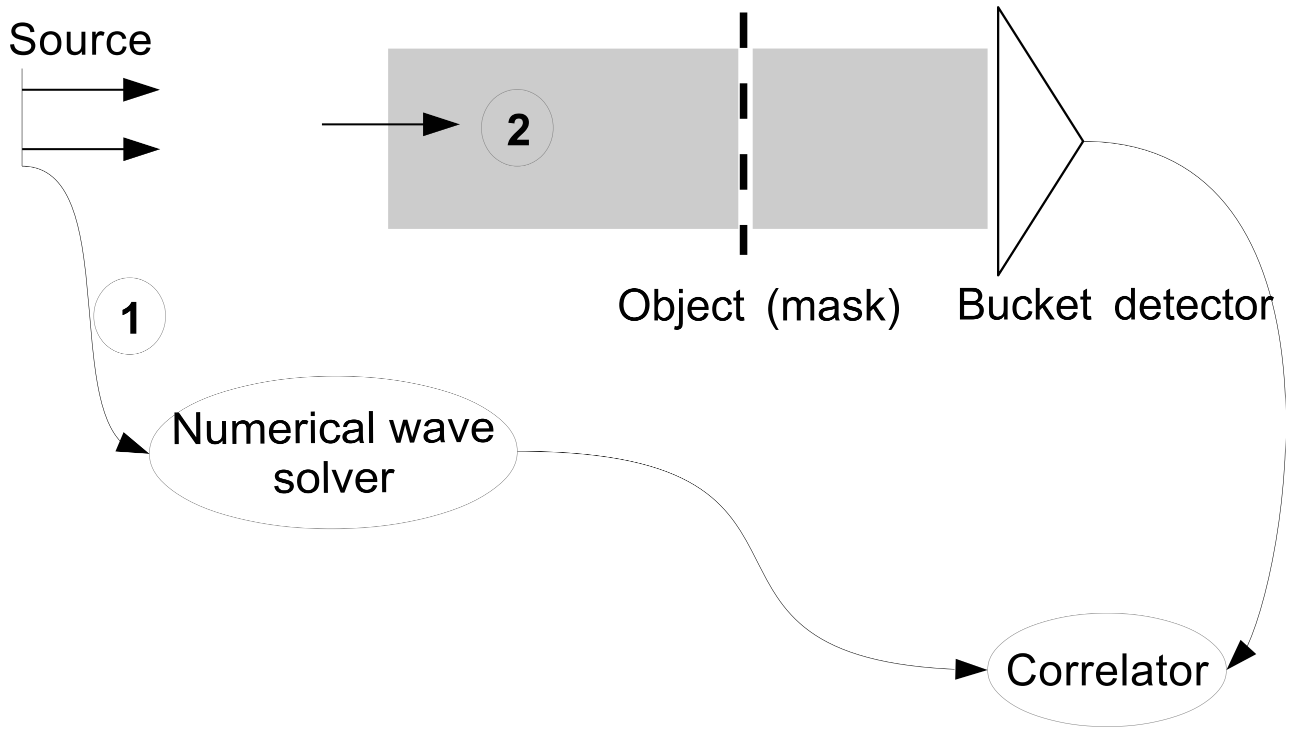
$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y}) |\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4}{2^8 \pi^2 L^4} \exp\Big(- \frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi3}^2} \Big), \qquad \frac{1}{\rho_{\rm gi3}^2} = \frac{1}{\rho_{\rm gi0}^2} + \frac{16L}{Z_{\rm sca}\ell_{\rm c}^2}$$

$\hookrightarrow$ the radius of the convolution kernel is reduced by scattering and can even be smaller than the Rayleigh resolution formula: enhanced resolution compared to the homogeneous case (similar phenomenon observed in time-reversal experiments) !

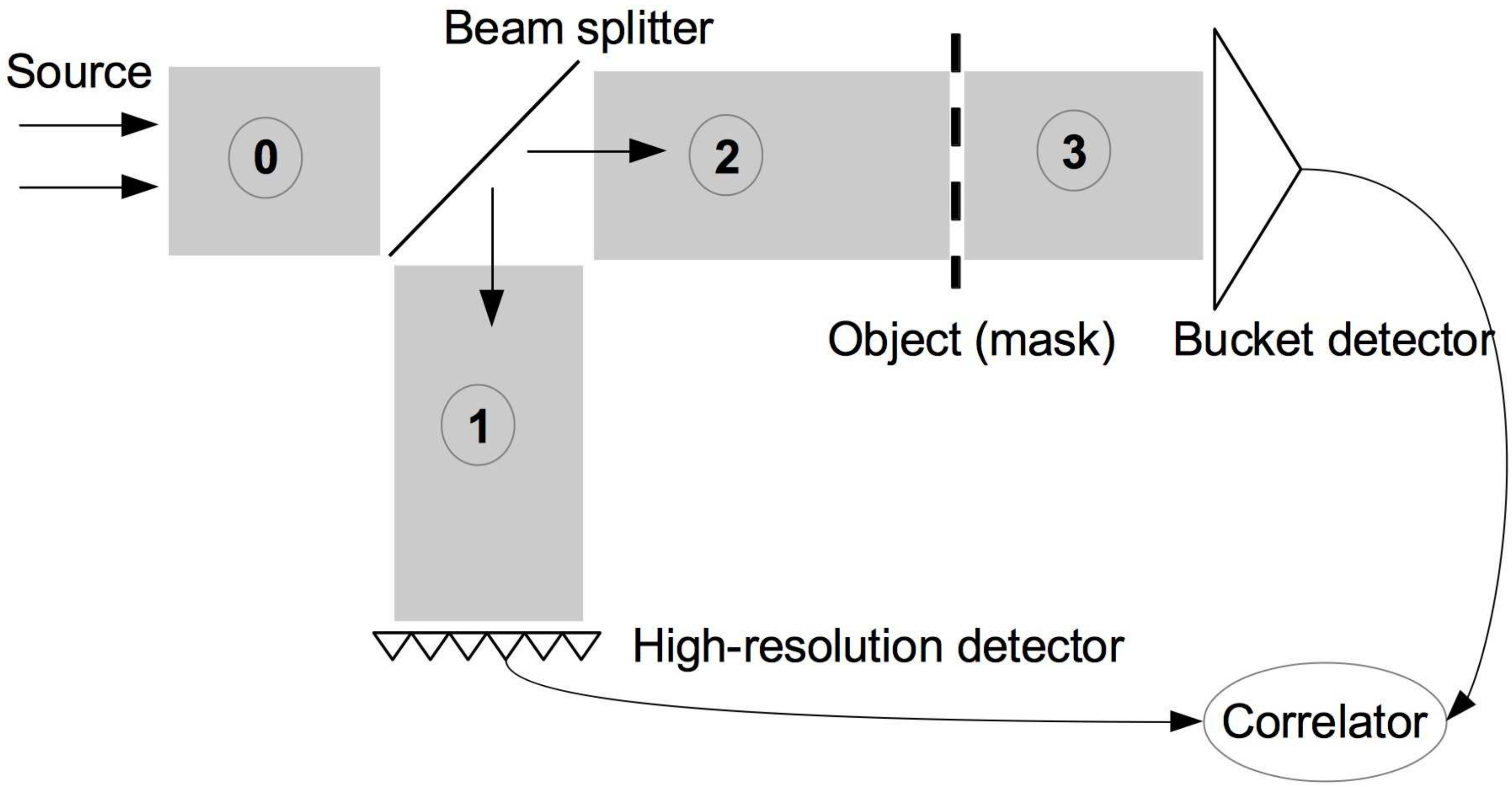
Ghost imaging with a virtual high-resolution detector



- The medium in path 2 is randomly heterogeneous.
- There is no other measurement than $I_2(t)$.
- The realization of the source is known (use of a Spatial Light Modulator) and the medium is taken to be homogeneous in the "virtual path 1" $\to$ one can *compute* the field (and therefore its intensity $I_1(t, \boldsymbol{x})$) in the "virtual" output plane of path 1.

$\hookrightarrow$ a *one-pixel camera* can give a high-resolution image of the object!

On the role of the random medium



Random medium in region 0 is *good*.

Random medium in regions 1 and 2 is *bad* (unless they are the same realization).

Random medium in region 3 plays *no role*.

Conclusion

- Correlation-based imaging allows for imaging in randomly scattering media.
- One needs to process *well-chosen* cross correlations of the data.

- Fourth-order moment of the wave field is useful.
- Application: Speckle intensity correlation-based imaging. Many modalities !