K-theory via the emergent topology of insulators

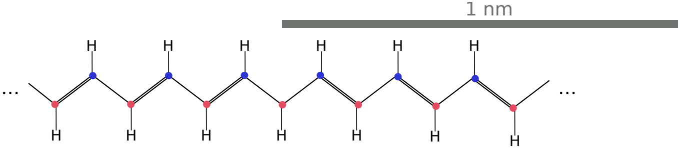
Mathematical and Physical Aspects of Topologically Protected States, Columbia University.

Terry A. Loring

May, 2017

Infinite SSH chain with defect

Su Schrieffer Heeger model of polyacetalene:



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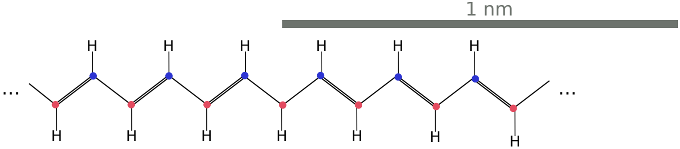
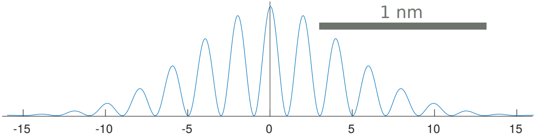


Defect between A and B phase, leads to a soliton, with $|\Psi|$ as in:



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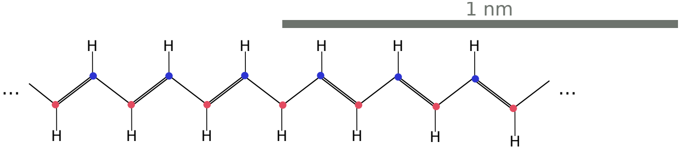

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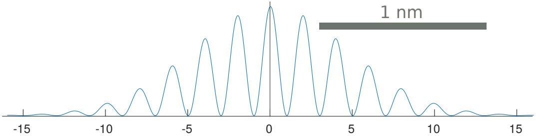

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Or

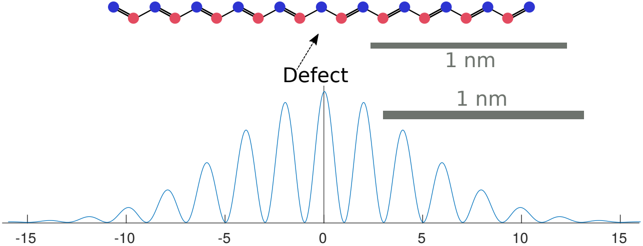
$$\langle \psi | H | \psi \rangle = 0, \ \langle \psi | X | \psi \rangle = 0,$$

and

$$\Delta^2_{\psi}(X) = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2 \approx 0.$$

Finite SSH chains

Finite polyacetalene chain:



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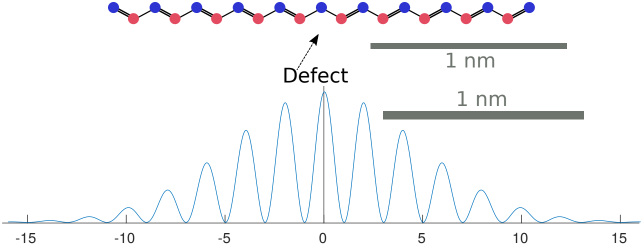

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Using symmetry we can enforce $H\psi_0 = 0$.

Finite SSH chains

Alternately, find a null vector ψ_0 for the Hamiltonian (with $\alpha = 2.85$ and $\beta = 2.15$)

$$H = \begin{bmatrix}
0 & \alpha \\
\alpha & 0 & \beta \\
& \beta & 0 & \alpha \\
& & \alpha & 0 & \beta \\
& & & \beta & 0 & \alpha \\
& & & & \alpha & 0 & \beta \\
& & & & & \beta & 0 & \alpha \\
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& & & & & & & & & & & & & & & & & \alpha & 0 & \beta \\
& & & & & & & & & & & & & & & & & & \beta & 0 & \alpha \\
& & & & & & & & & & & & & & & & & & & \alpha & 0
\end{bmatrix}$$

Finite SSH chains

and compute $X\psi$ etc. for the position observable

$$X = \begin{bmatrix} -2.0 &&&&&&&&&&&&&&&&&&&& \\ & -1.8 &&&&&&&&&&&&&&&&&&& \\ && -1.6 &&&&&&&&&&&&&&&&&& \\ &&& -1.4 &&&&&&&&&&&&&&&&& \\ &&&& -1.2 &&&&&&&&&&&&&&&& \\ &&&&& -1.0 &&&&&&&&&&&&&&& \\ &&&&&& -0.8 &&&&&&&&&&&&&& \\ &&&&&&& -0.6 &&&&&&&&&&&&& \\ &&&&&&&& -0.4 &&&&&&&&&&&& \\ &&&&&&&&& -0.2 &&&&&&&&&&& \\ &&&&&&&&&& 0.0 &&&&&&&&&& \\ &&&&&&&&&&& 0.2 &&&&&&&&& \\ &&&&&&&&&&&& 0.4 &&&&&&&& \\ &&&&&&&&&&&&& 0.6 &&&&&&& \\ &&&&&&&&&&&&&& 0.8 &&&&&& \\ &&&&&&&&&&&&&&& 1.0 &&&&& \\ &&&&&&&&&&&&&&&& 1.2 &&&& \\ &&&&&&&&&&&&&&&&& 1.4 &&& \\ &&&&&&&&&&&&&&&&&& 1.6 && \\ &&&&&&&&&&&&&&&&&&& 1.8 & \\ &&&&&&&&&&&&&&&&&&&& 2.0 \end{bmatrix}$$

(Exactly) Compatible Observables

Recall $XY = XY$ implies “enough” common eigenvalues. Must restrict (λ_1, λ_2) to the joint spectrum $\sigma(X, Y)$, determined by any of the following.

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$$\langle X \rangle_{\mathbf{v}} = \lambda_1,\ \Delta^2_{\mathbf{v}}(X) = 0,$$

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$$\lambda_1 + i\lambda_2 \in \sigma(X + iY)$$

$$s_{\min}(X_{\lambda_1} + iY_{\lambda_2}) = 0$$

$$\lambda_{\min} \begin{bmatrix} 0 & X_{\lambda_1} - iY_{\lambda_2} \\ X_{\lambda_1} + iY_{\lambda_2} & 0 \end{bmatrix} = 0$$

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Here s_{min} is smallest singular value, λ_{min} is magnitude of the eigenvalue closest to zero. Also, $X_\lambda = X - \lambda I$ and $Y_\lambda = Y - \lambda I$.

Almost Compatible Observables

Given $X_j X_k \approx X_k X_j$, prefer to use multivariate pseudospectrum:

Definition

Given Hermitian matrices $X_1, \ldots, X_d$ define

$$\Lambda_\epsilon(X_1, \ldots, X_d) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^d \,\middle|\, \left\| \left(\sum (X_j - \lambda_j) \otimes \Gamma_j \right)^{-1} \right\| \geq \epsilon^{-1} \right\}$$

with the convention $0^{-1} = \infty$ and $\left\| (\text{singular})^{-1} \right\| = \infty$ and where $\Gamma_1, \ldots, \Gamma_d$ are Dirac matrices.

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When $\epsilon = 0$ this is called the Clifford spectrum.

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Assume $\delta = \|XY - XY\|$ is small, not zero.

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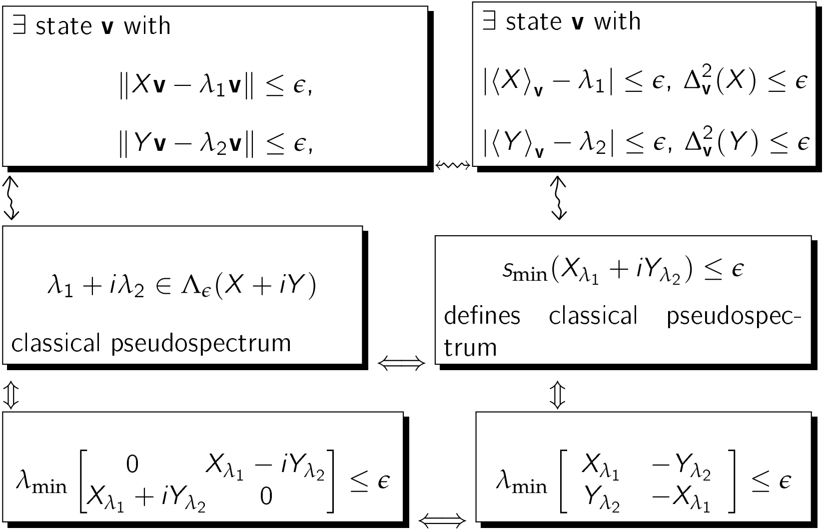

∃ state $\mathbf{v}$ with

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$$\|Y\mathbf{v} - \lambda_2\mathbf{v}\| \leq \epsilon,$$

$\leftrightsquigarrow$

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$\updownarrow$ $\updownarrow$

$$\lambda_1 + i\lambda_2 \in \Lambda_\epsilon(X + iY)$$

classical pseudospectrum

$\Longleftrightarrow$

$$s_{\min}(X_{\lambda_1} + iY_{\lambda_2}) \leq \epsilon$$

defines classical pseudospectrum

$\Updownarrow$ $\Updownarrow$

$$\lambda_{\min}\begin{bmatrix} 0 & X_{\lambda_1} - iY_{\lambda_2} \\ X_{\lambda_1} + iY_{\lambda_2} & 0 \end{bmatrix} \leq \epsilon$$

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Think of pseudospectrum as a function

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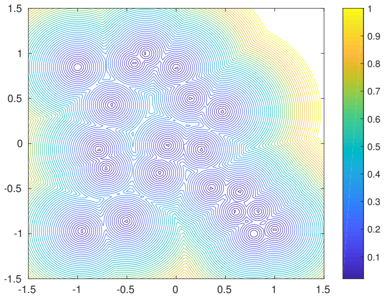
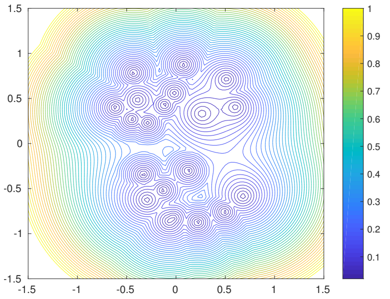
Random examples, $\|X\| = \|Y\| = 1$, matrix size 20:



$\|XY - YX\| = 0.00001$

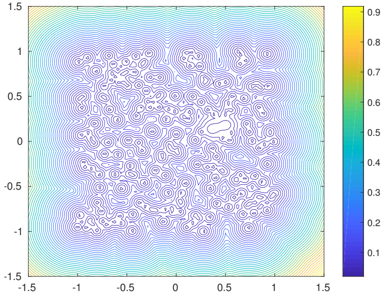


$\|XY - YX\| = 0.93$

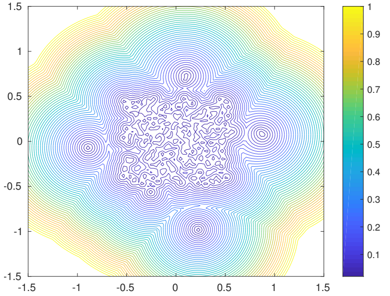
Almost Compatible Observables

Bigger random examples, $\|X\| = \|Y\| = 1$, matrix size 200.

Examples with large commutator are hard to generate. Must avoid standard matrix distributions.

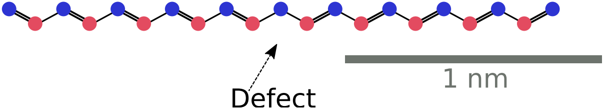


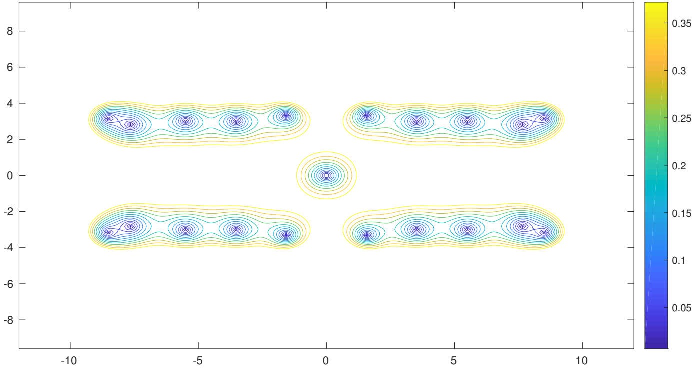
$\|XY - YX\| = 0.15$



$\|XY - YX\| = 0.88$

Pseudospectrum of X and H in SSH models





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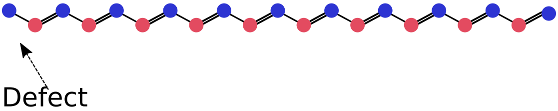

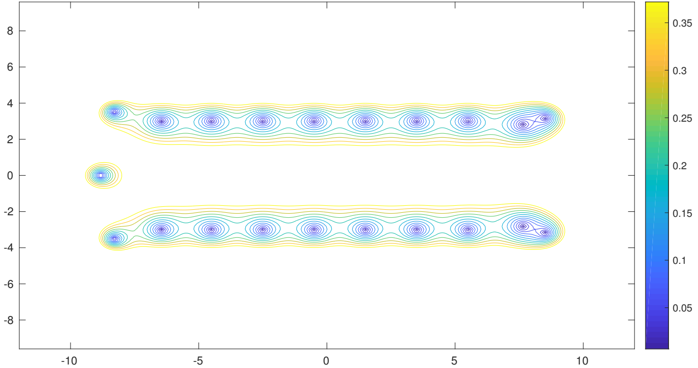

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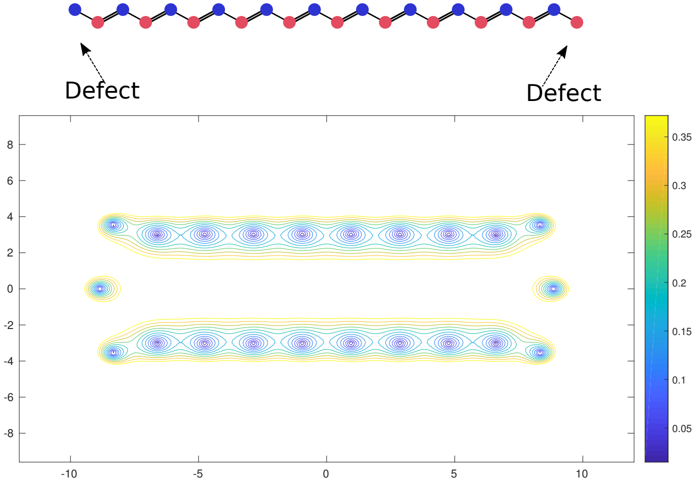

Even dimensional space. Chiral symmetry: $\Gamma\,|{\color{blue}\bullet}\rangle = |{\color{blue}\bullet}\rangle$, $\Gamma\,|{\color{red}\bullet}\rangle = -\,|{\color{red}\bullet}\rangle$, and $\Gamma X = X\Gamma$, $\Gamma H = -H\Gamma$.

K-Theory, Counting Zero Modes

In chiral situation,

$$s_{\min}\left(X_{\lambda_1}+iH_{\lambda_2}\right)=s_{\min}\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma\right)$$

and $\left(X_{\lambda_1}+iY_{\lambda_2}\right)\Gamma$ is Hermitian.

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Eigenvalues in conjugate pairs.
Real eigenvalues can be single.

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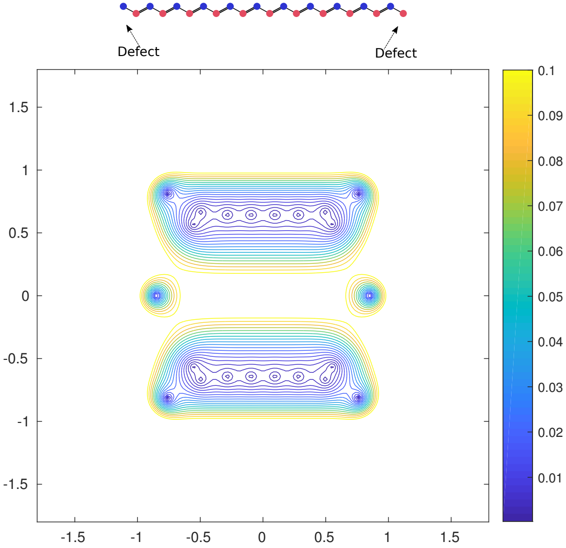
Using $\Gamma_t\Gamma\left|\bullet\right\rangle=e^{\pi it}\left|\bullet\right\rangle$ we can animate this.

22 site SSH chain with end defects



$$\Lambda_{\epsilon}\left(X_{\lambda_1}+iH_{\lambda_2}\right)$$

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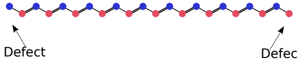

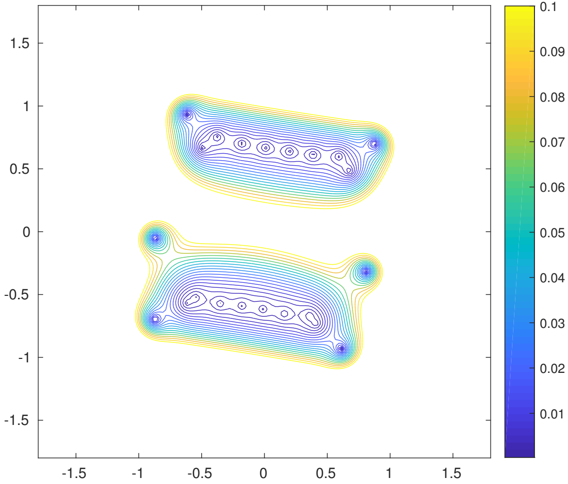

$$\Lambda_\epsilon\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma^{0.1}\right)$$

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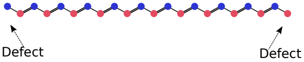

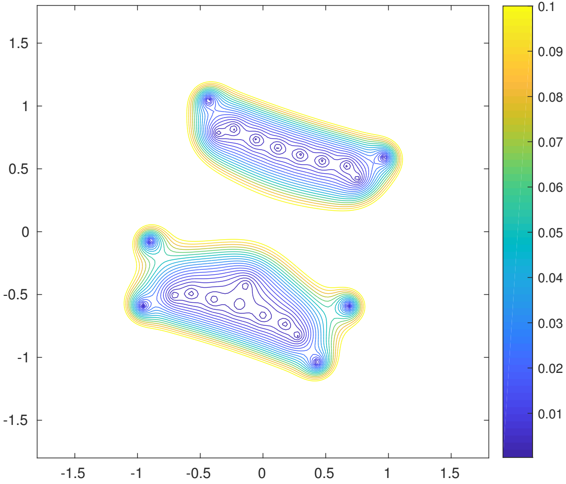

$$\Lambda_\epsilon\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma^{0.2}\right)$$

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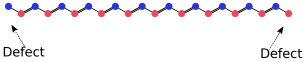

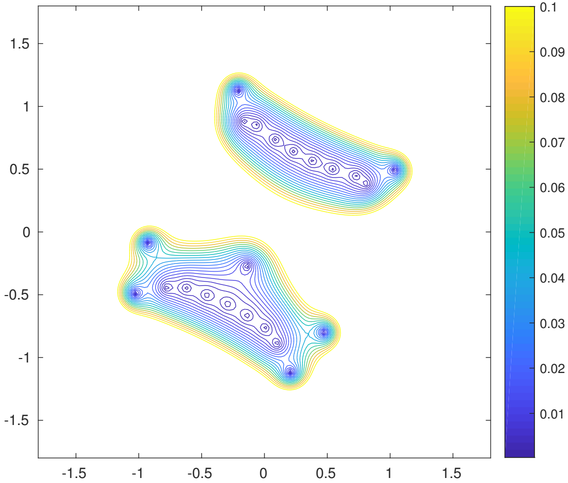

$$\Lambda_\epsilon\left(\left(X_{\lambda_1} + iH_{\lambda_2}\right)\Gamma^{0.3}\right)$$

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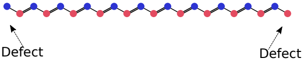

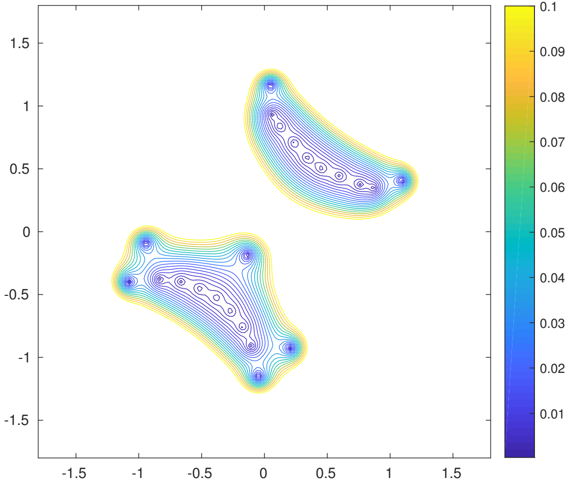

$$\Lambda_\epsilon\left(\left(X_{\lambda_1} + iH_{\lambda_2}\right)\Gamma^{0.4}\right)$$

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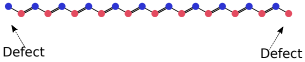

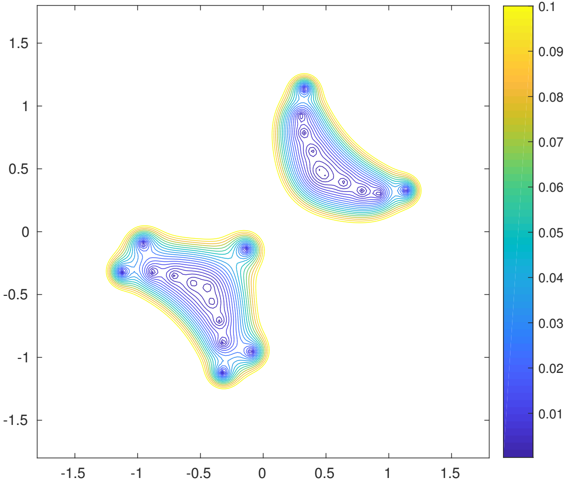

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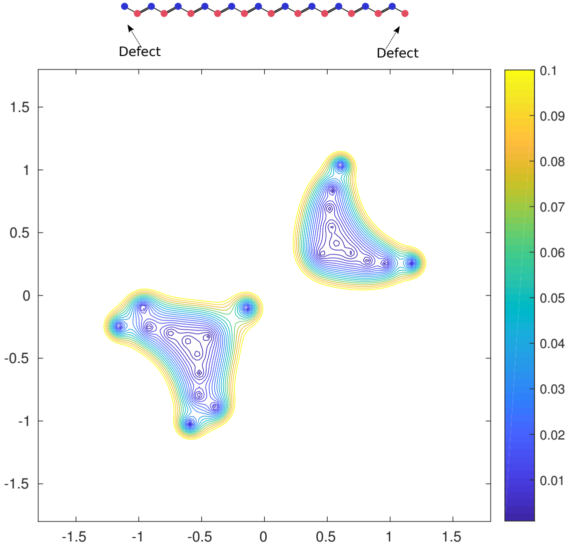

$$\Lambda_\epsilon\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma^{0.6}\right)$$

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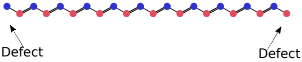
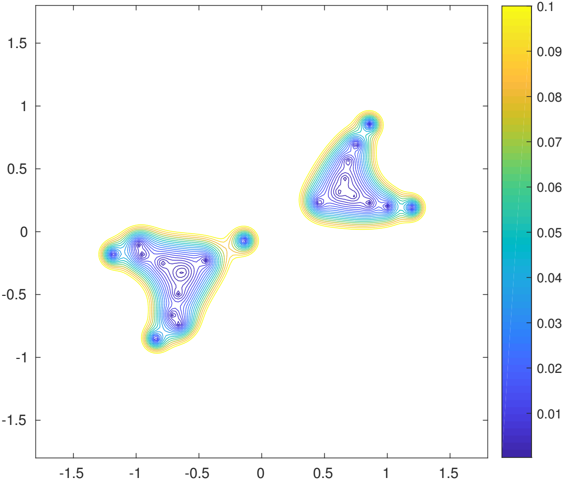
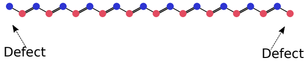
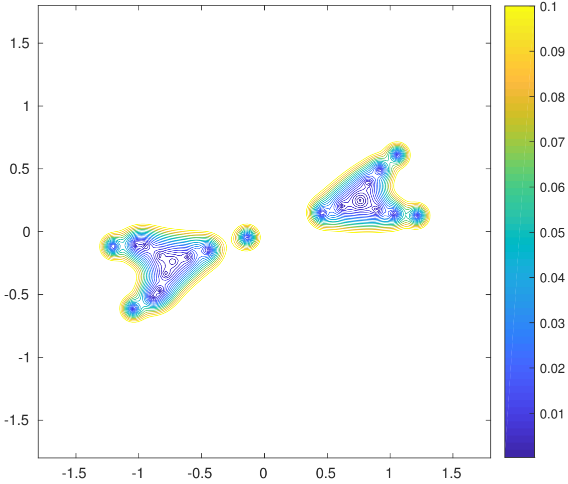




$$\Lambda_\epsilon\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma^{0.7}\right)$$

22 site SSH chain with end defects





$\Lambda_\epsilon\left(\left(X_{\lambda_1} + iH_{\lambda_2}\right)\Gamma^{0.8}\right)$

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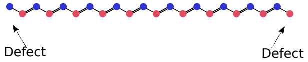

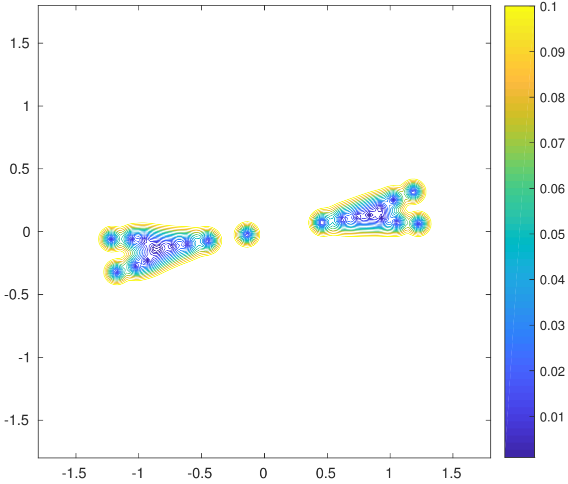

$$\Lambda_{\epsilon}\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma^{0.9}\right)$$

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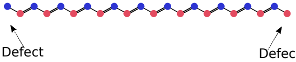
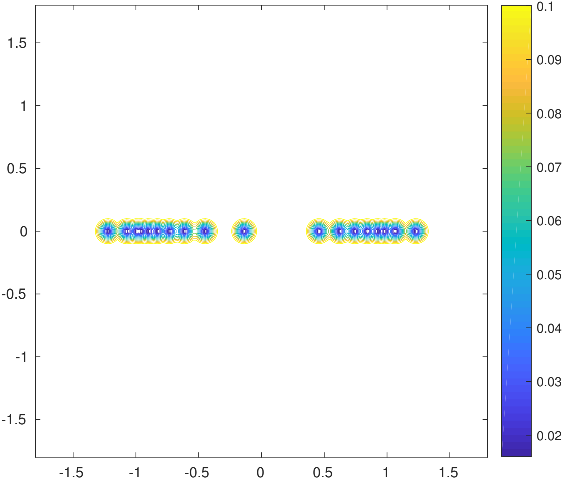
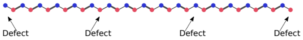
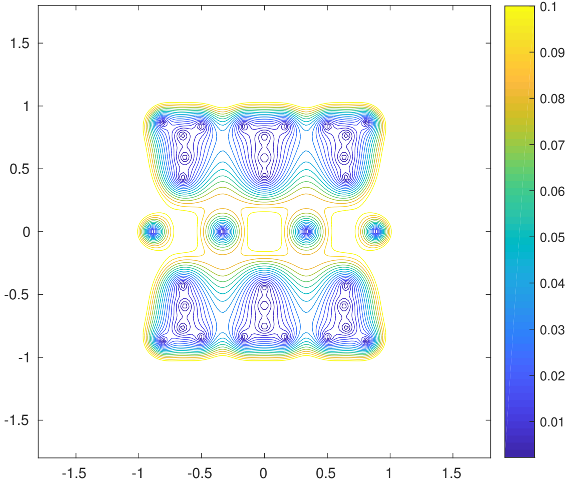




$$\Lambda_\epsilon\left(\left(X_{\lambda_1}+iH_{\lambda_2}\right)\Gamma\right)$$

34 site SSH chain, ABA phases





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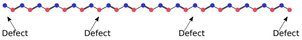

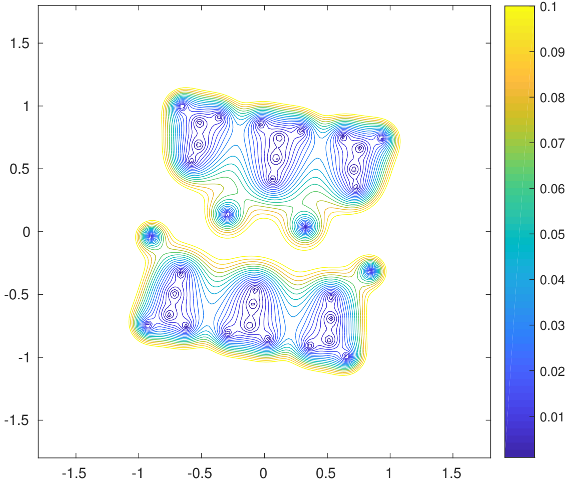

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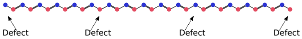

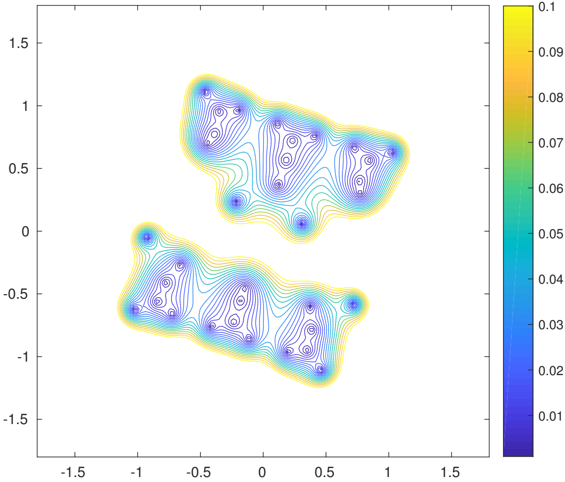

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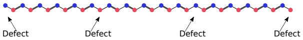

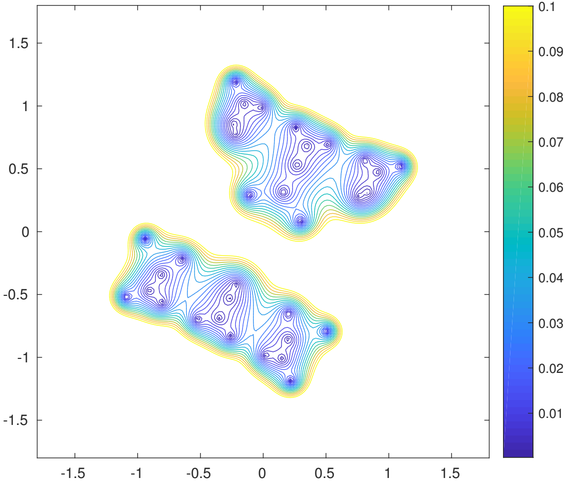

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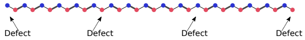

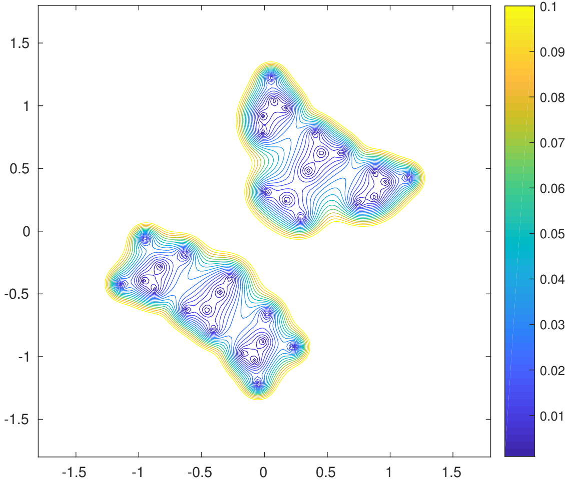

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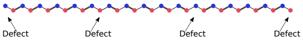

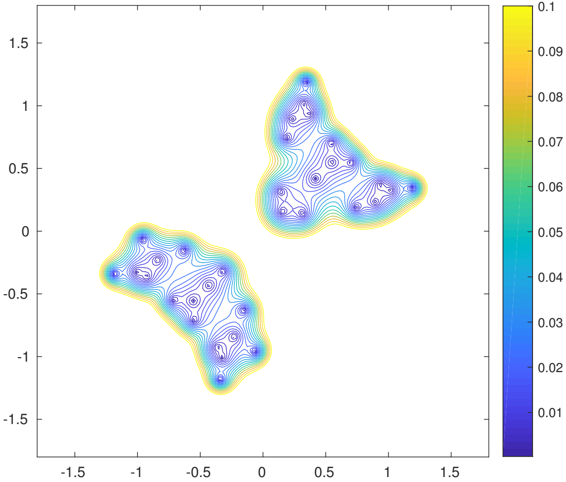

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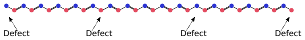

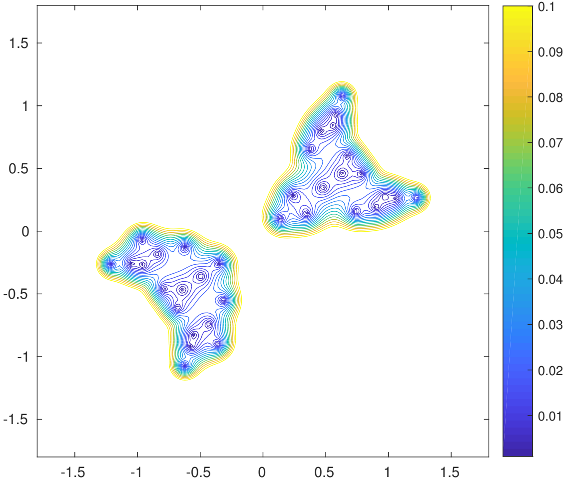

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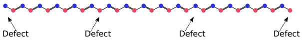

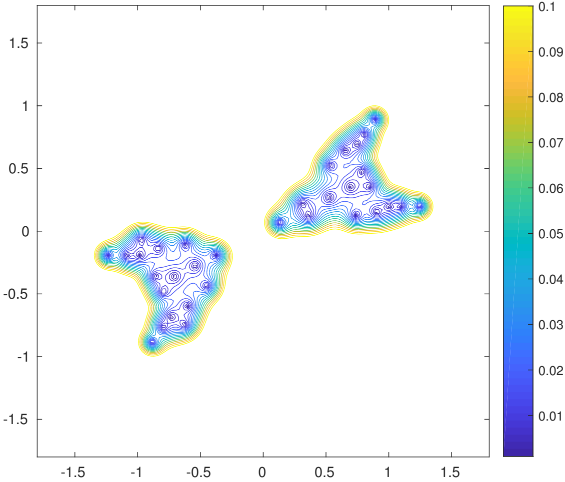

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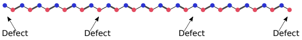

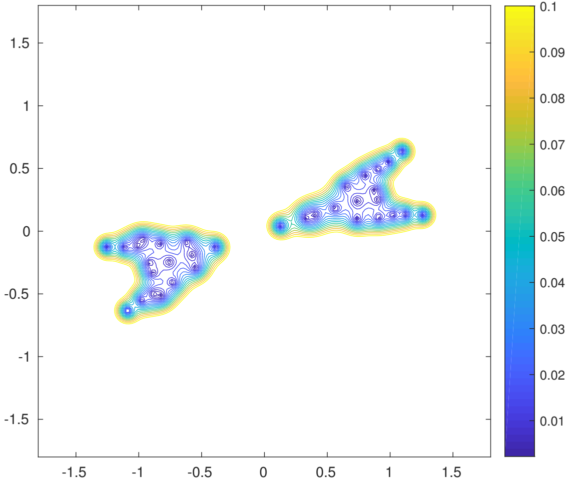

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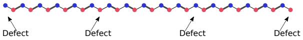

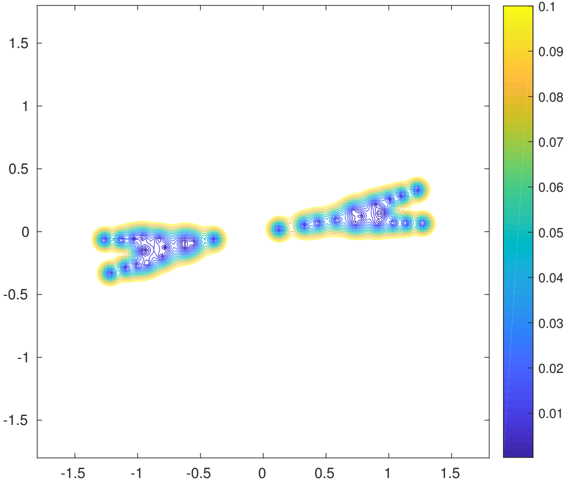

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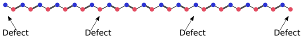

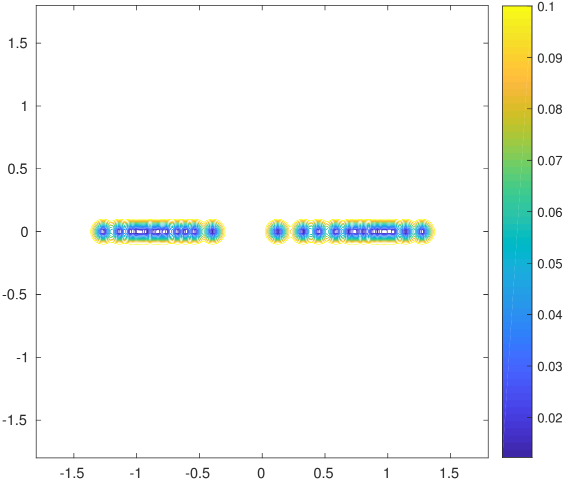

K-Theory, Counting Zero Modes

The eigenvalues of the *Hermitian* matrix $\left(X_{\lambda_1} + iH_{\lambda_2}\right)\Gamma$ tell us about the real eigenvalues of $X_{\lambda_1} + iH_{\lambda_2}$. Helps find $\mathbf{v}$ with $H\mathbf{v} \approx 0$ and $X\mathbf{v} \approx \lambda_1\mathbf{v}$.

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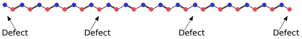
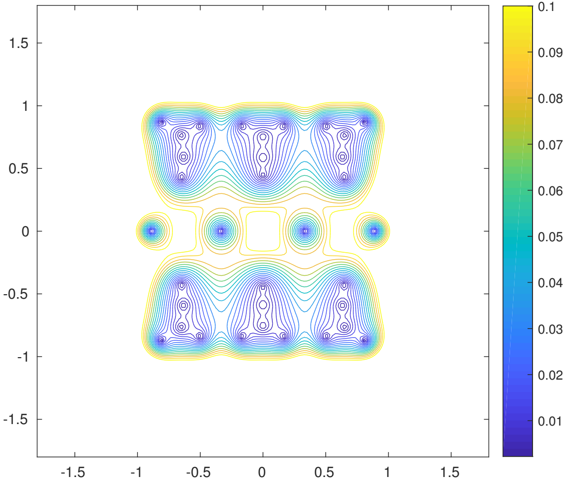
34 site SSH chain, ABA phases, K-theory to the left





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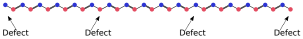

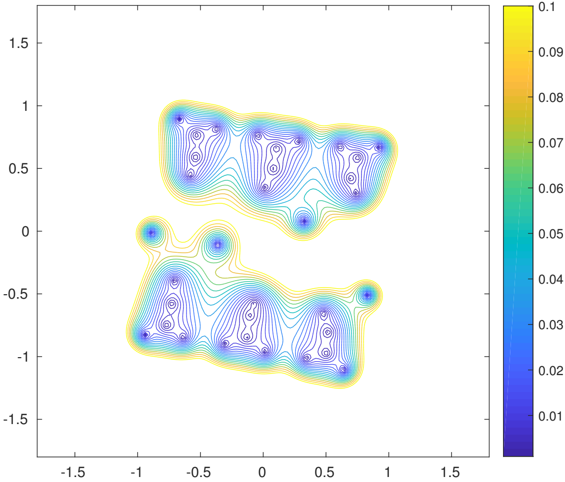

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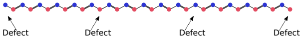

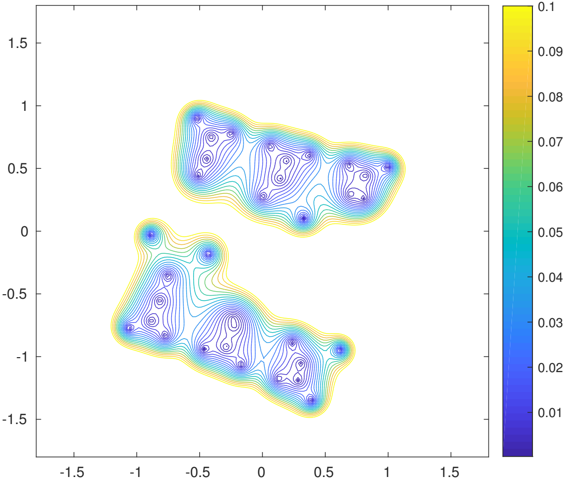

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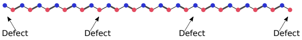

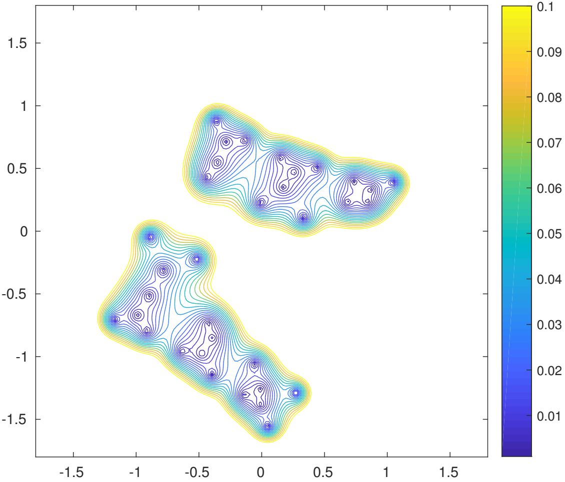

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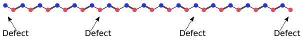

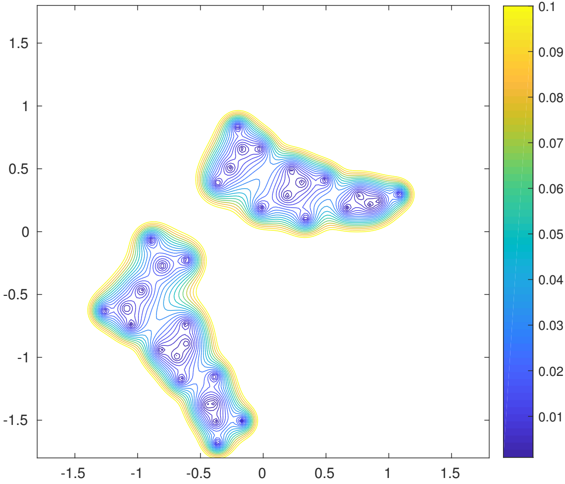

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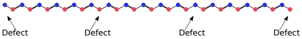

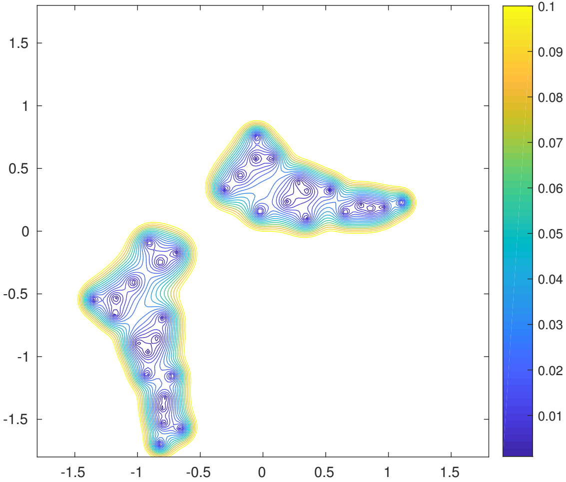

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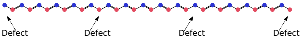

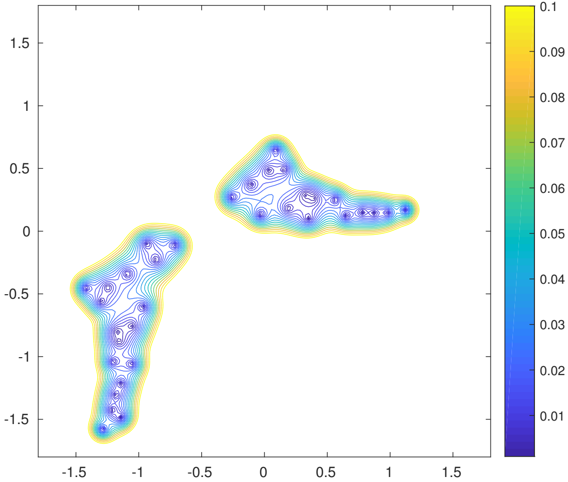

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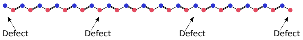

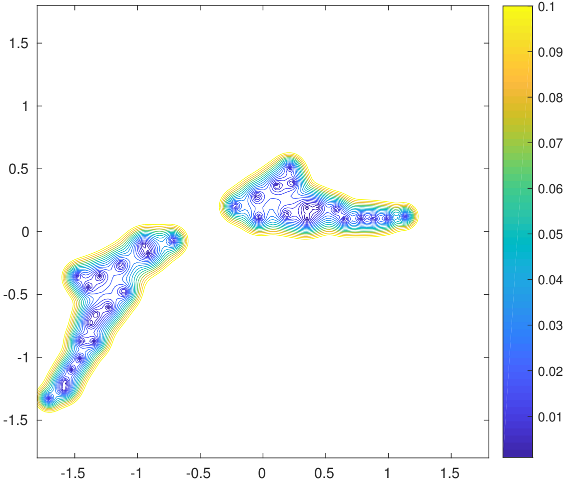

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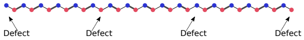

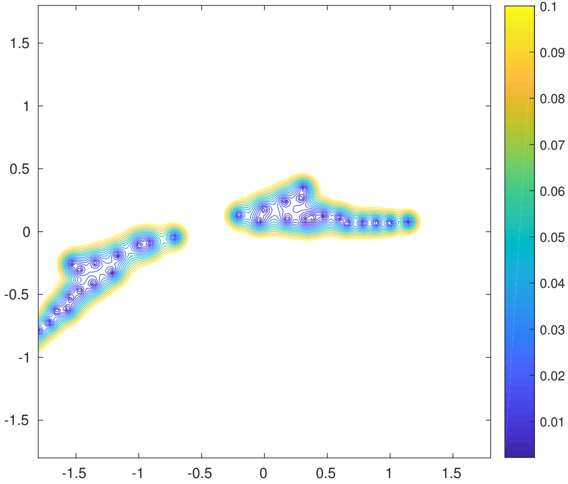

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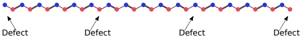

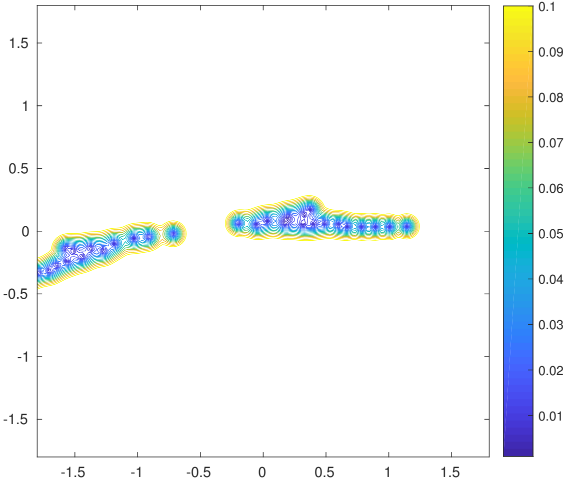

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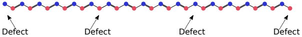

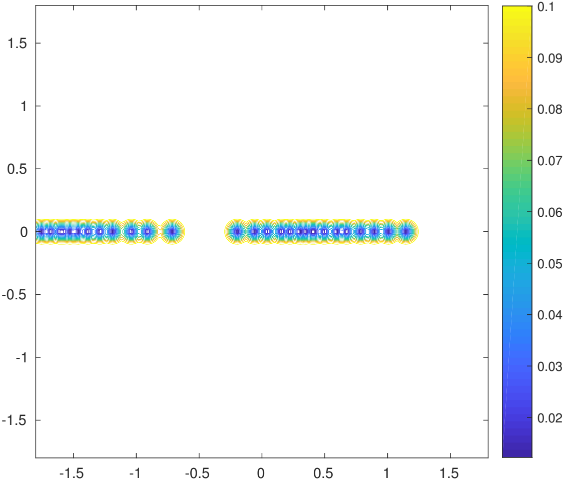

Infinite length chiral systems

- Hilbert space is now $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}$.
- Chiral symmetry is determined by $\Gamma = 1 \otimes \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
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Let $\hat{\Pi}$ denote the projection of $\mathcal{H}$ onto

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- If $\sigma(H) = \pm 1$ then $\Pi A \Pi$ represents a general unitary in the Calkin algebra, while

  $$\hat{\Pi} H \hat{\Pi} = \begin{bmatrix} 0 & \Pi A \Pi \\ \Pi A^* \Pi & 0 \end{bmatrix}$$

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To relate to the finite systems, we need two basic things.

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$$(\mathbb{K}(\mathcal{H}_+))^\sim = \mathbb{K}(\mathcal{H}_+) + \mathbb{C}I.$$

- Work with the boundary map in K-theory associated to

$$0 \to \mathbb{K}(\mathcal{H}_+) \to \mathbb{B}(\mathcal{H}_+) \to \mathbb{B}(\mathcal{H}_+) / \mathbb{K}(\mathcal{H}_+) \to 0,$$

  specifically

$$\partial : K_1\left(\mathbb{B}(\mathcal{H}_+) / \mathbb{K}(\mathcal{H}_+)\right) \to K_0(\mathbb{K}(\mathcal{H}_+)).$$

Infinite length chiral systems

Given radius ρ, define a finite system on sites at X position less that ρ, put on Dirichet boundary conditions, with new observables H_ρ and X_ρ.

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Theorem

(2017 with Schulz-Baldes) Assuming H is invertible, with gap $g = \left\| H^{-1} \right\|^{-1}$, if

$$\|[X, H]\| \leq \frac{g^3}{18\|H\|\kappa}$$

and

$$\frac{2g}{\kappa} \leq \rho,$$

then

$$\frac{1}{2}\mathrm{sig}\left(\left(\kappa X_\rho + iH_\rho\right)\Gamma\right) = \mathrm{ind}\left(\Pi A \Pi\right).$$

Pictures of K-theory for graded C^*-algebras

In addition to a C^*-algebra A, have $a \mapsto a^\sigma$ implementing and action of $\mathbb{Z}/2$, determining even ($a^\sigma = a$) and odd ($a^\sigma = -a$).

For example, $A = \mathbf{M}_{2n}$ and for a matrix a define $a = \Gamma a \Gamma$ with $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

Pictures of K-theory for graded C^*-algebras

Trout picture of K_0	Van Daele of K_1	
$u^*u = uu^* = 1$ $u^\sigma = u^*$	$u^*u = uu^* = 1$ $u^* = u$ $u^\sigma = -u$	
u^{-1} exists $u^\sigma = u^*$	u^{-1} exists $u^* = u$ $u^\sigma = -u$	

As always, compute homotopy classes, and stabilize by using a in A, $\mathbf{M}_2(A)$, $\mathbf{M}_3(A)$, etc.

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If we use Hermitian and anti-Hermitian parts, $a = x + iy$, then this becomes more familiar.

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Trout picture of K_0	Van Daele of K_1	
$x^2 + y^2 = 1$ $xy = yx$ $x^\sigma = x$ $y^\sigma = -y$	$x^2 = 1$ $x^\sigma = -x$	Can lead to bad numerics
$(x + iy)^{-1}$ exists	x^{-1} exists $x^\sigma = -x$	No formula for boundary map

As always, compute homotopy classes, and stabilize by using a in A, $\mathbf{M}_2(A)$, $\mathbf{M}_3(A)$, etc.

Spectral Flattening, etc.

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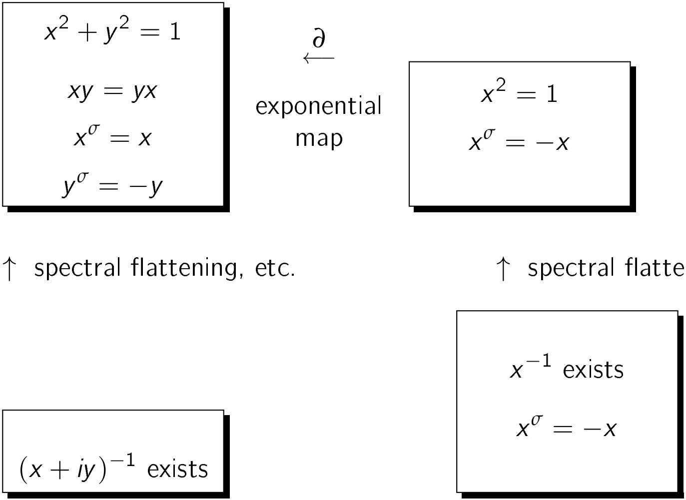
$$H \rightsquigarrow \frac{H}{|H|}.$$

In $\mathbb{B}(\mathcal{H}_+)$ and approximating finite matrices, we need first to rescale so $-1 \leq X \leq 1$. Then we attenuate H near boundaries, so

$$H \rightsquigarrow \left(1 - X^2\right)^{\frac{1}{4}} \frac{H}{|H|} \left(1 - X^2\right)^{\frac{1}{4}}.$$

Pictures of K-theory for graded C^*-algebras



$$x^2 + y^2 = 1$$

$$xy = yx$$

$$x^\sigma = x$$

$$y^\sigma = -y$$

$\xleftarrow{\partial}$

exponential
map

$$x^2 = 1$$

$$x^\sigma = -x$$

↑ spectral flattening, etc.

↑ spectral flattening

$(x + iy)^{-1}$ exists

x^{-1} exists

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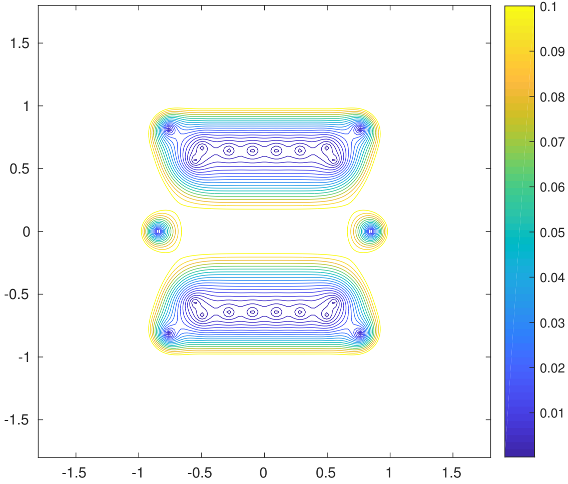
Spectral flattening and curving



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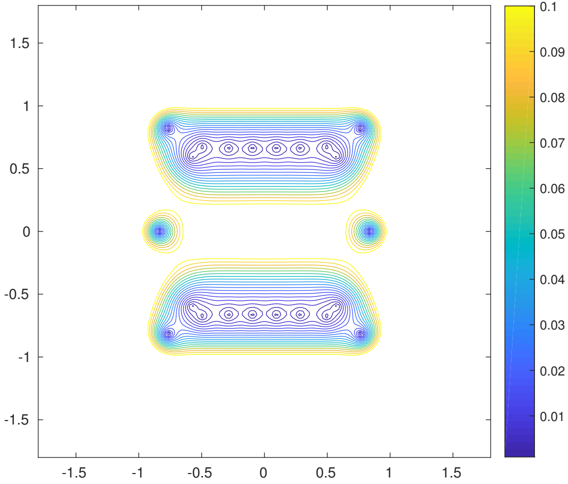

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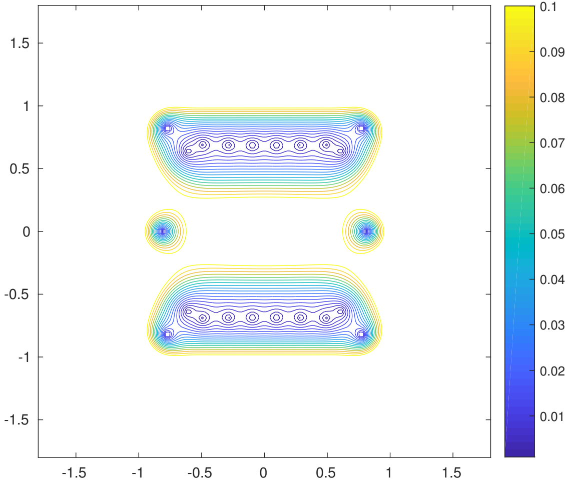

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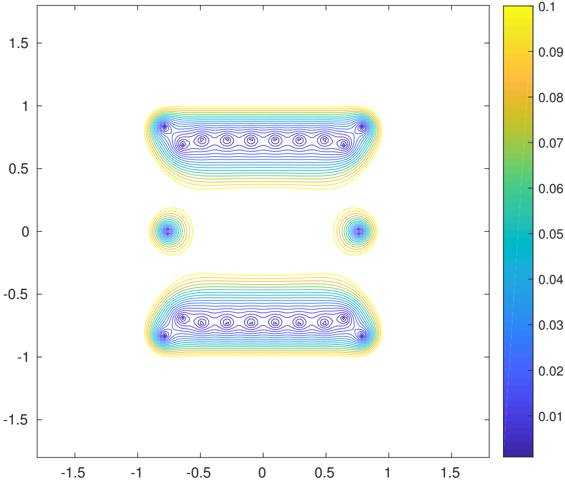

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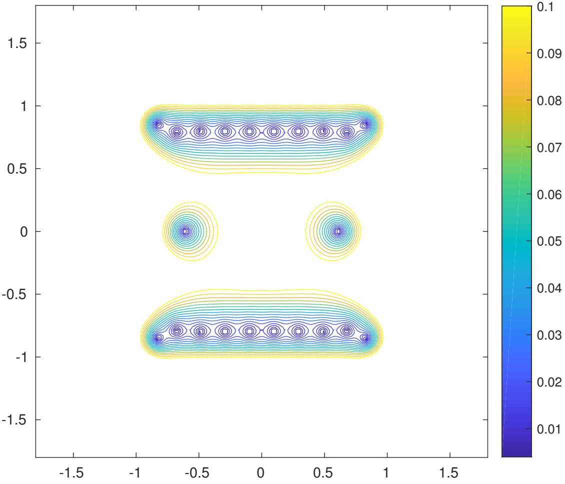

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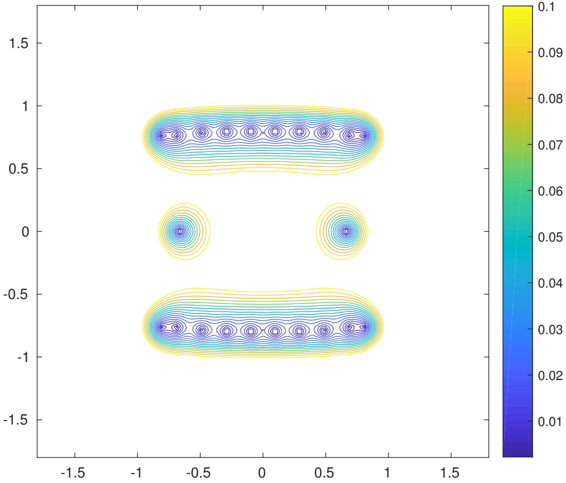

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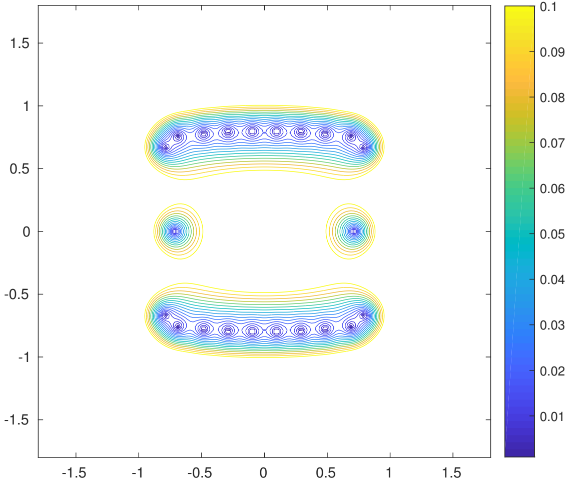

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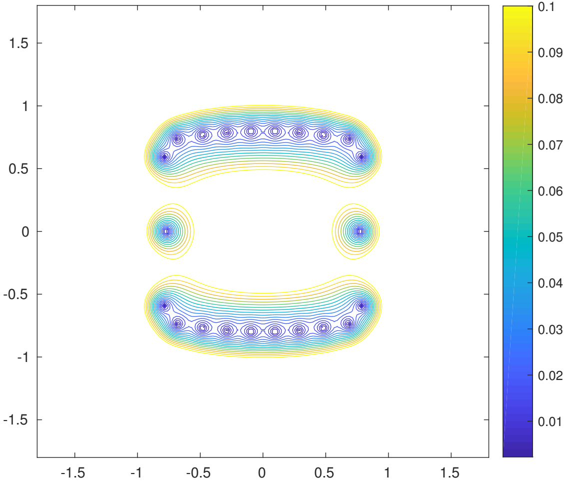

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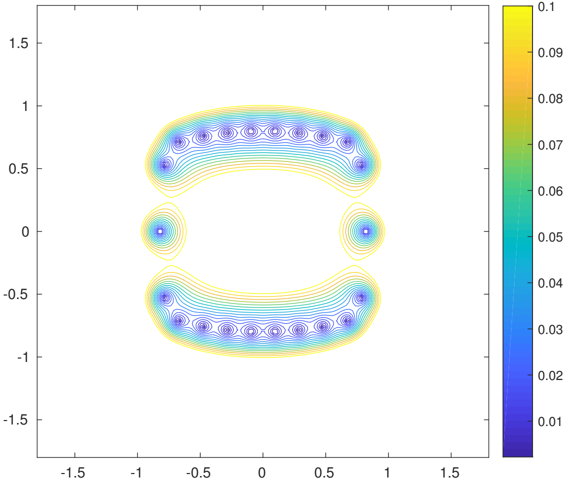

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