Bulk-Boundary Correspondence for Apperiodic Structures

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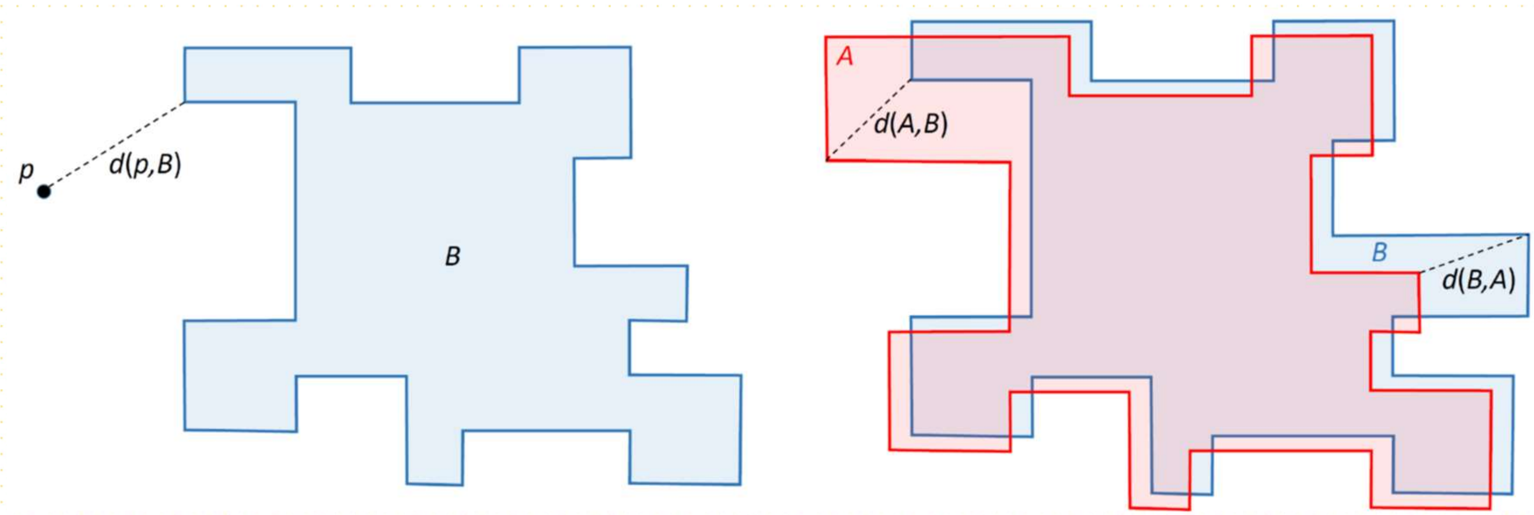
The space of compact subsets $K(\mathbb{R}^{\alpha})$



(Hausdorff metric) $d_H(A, B) = \max\{d(A,B), d(B,A)\}$

$$(K(\mathbb{R}^{\alpha}), d_H) = \text{metric space}$$

The space of closed sets $\mathcal{C}(\mathbb{R}^d)$

Proposition. For $\Lambda \subset \mathbb{R}^d$ closed, define

$$\Lambda(r) = (\Lambda \cap B(0,r)) \cup \partial B(0,r)$$

Then:

$$D(\Lambda, \Lambda') = \inf\left\{ \frac{1}{1+r},\ d_H(\Lambda(r), \Lambda(r')) < \frac{1}{r} \right\}$$

defines a metric on $\mathcal{C}(\mathbb{R}^d)$.

Facts: $\mathcal{C}(\mathbb{R}^d)$ is bounded, compact, complete.

Patterns and the Space of Patterns

Definition: $\omega \subset \mathbb{R}^d$ is a Delone set if $\exists\, r_{min}, r_{max}$

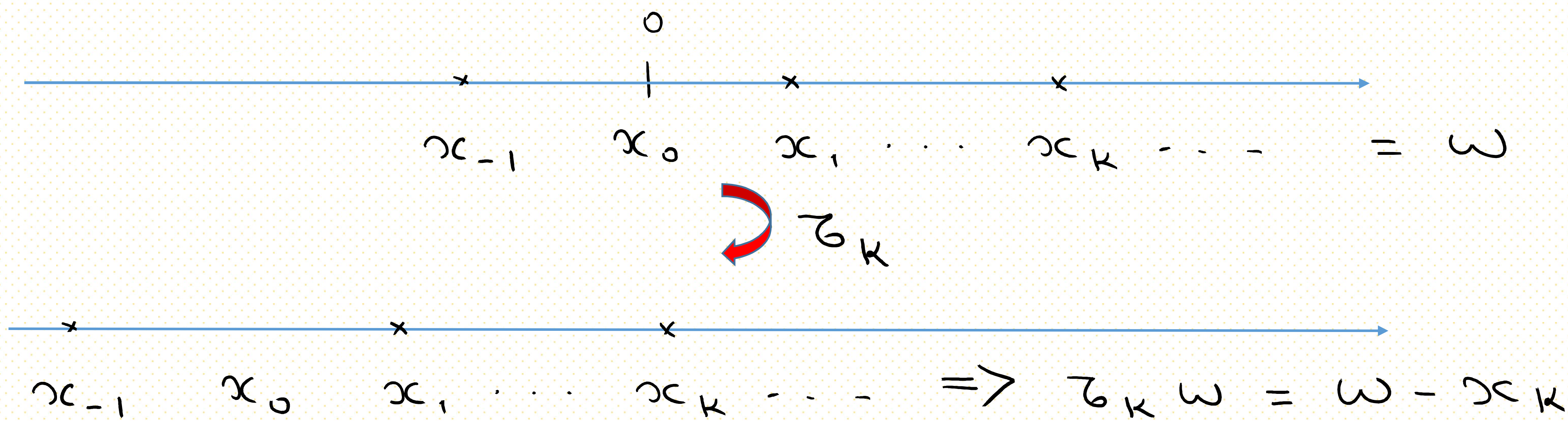
1) $B(x, r_{min}) \cap \omega = \{x\} \quad \forall\ x \in \omega$

2) $B(x, r_{max}) \cap \omega \neq \emptyset \quad \forall\ x \in \mathbb{R}^d$

The Delone sets are closed $\Rightarrow \omega \in \mathcal{C}(\mathbb{R}^d)$.

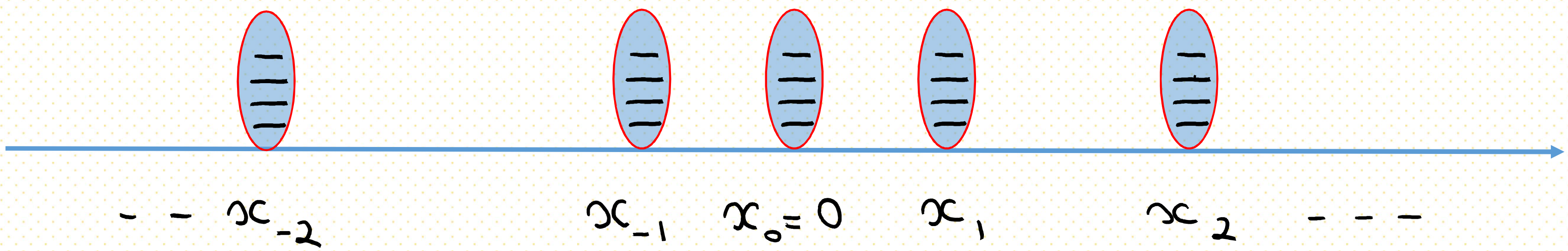
Point pattern = Delone set (for us!)

The action of $\mathbb{Z}^d$ on point patterns



Such action always exists when the points can be labeled by $\mathbb{Z}^d$ in a meaningful way.

Dynamics over Point Patterns



- identical discrete resonators
- internal degrees of freedom $\mathbb{C}^N$

Coupling => Collective modes

The Hilbert space:

$$\mathcal{H} = \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) = \mathrm{Span}\{\xi \otimes |n\rangle,\ \xi \in \mathbb{C}^N,\ n \in \mathbb{Z}^d\}$$

Obs: Hilbert space is exactly the same for all patterns

The Hamiltonian: (H_ω continuous of ω in strong top)

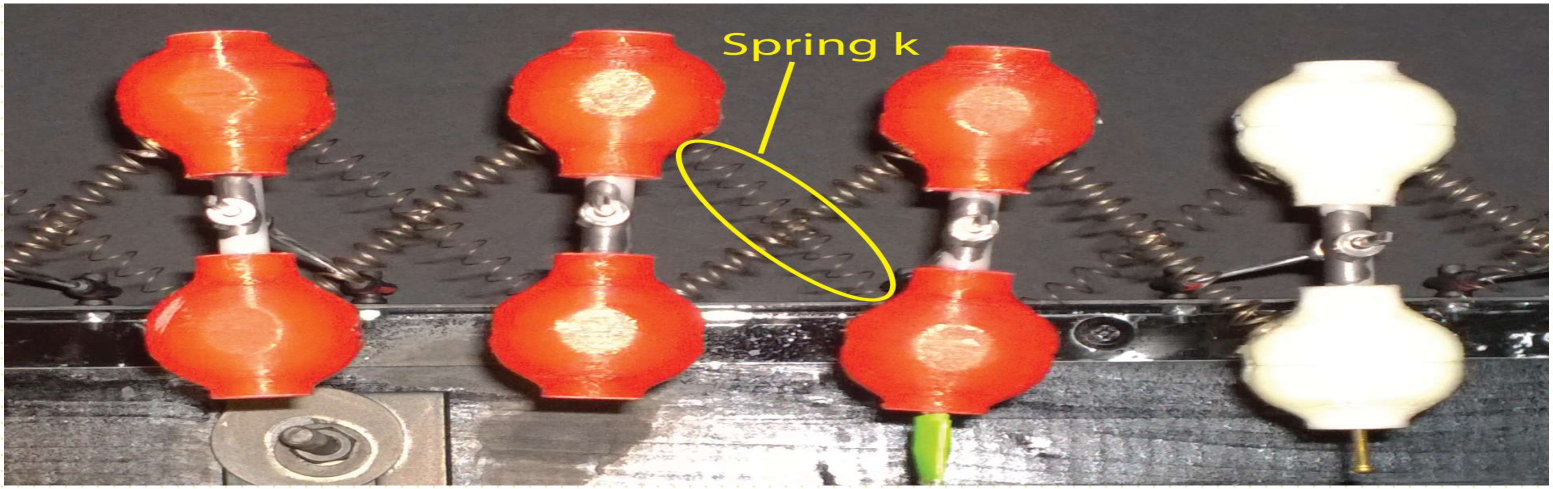
$$H_\omega = \sum_{n,m} \hat{w}_{n,m}(\omega) \otimes |n\rangle\langle m|$$

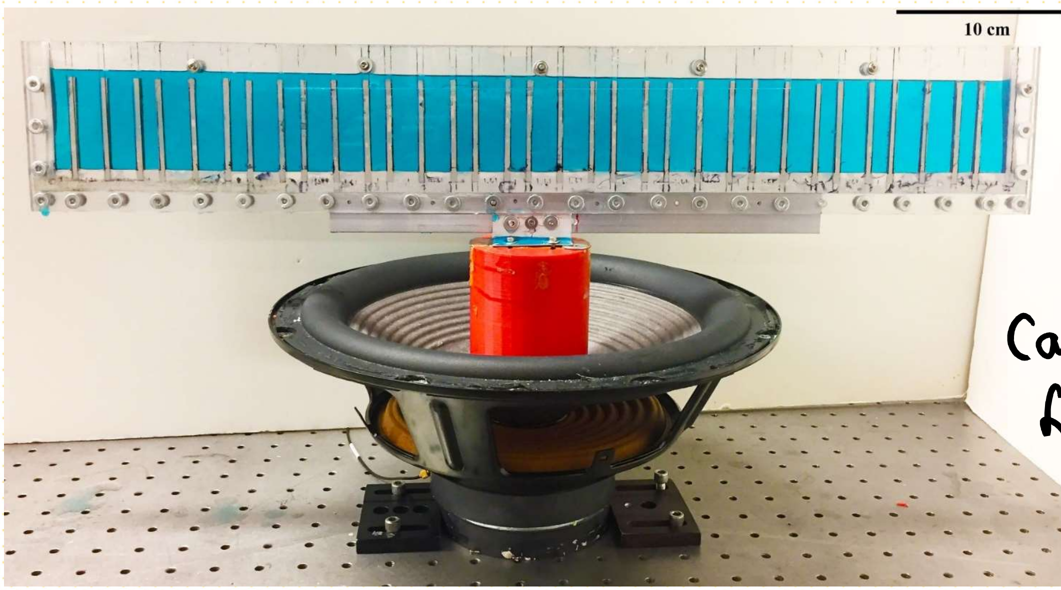
Example:

(to be used throughout)

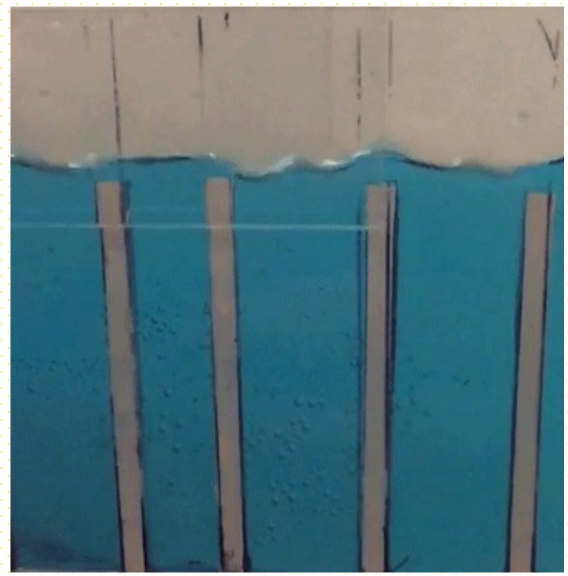
$$H_\omega = \sum_{n,m} e^{-\beta|x_n - x_m|} |n\rangle\langle m|$$

Coupling by evanescent tails

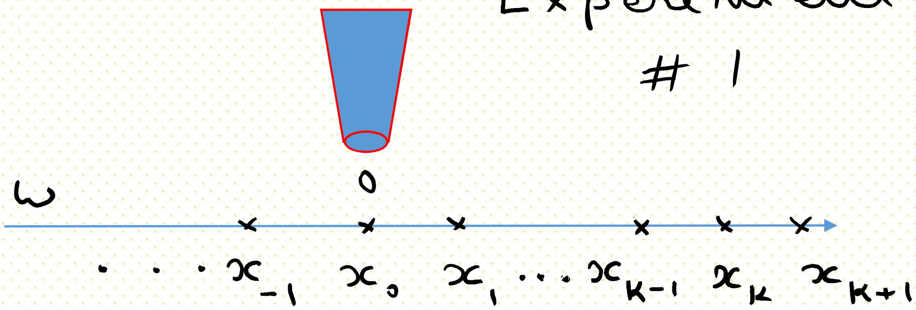




Camelia Prodan Lab, NJIT



Experiment #1

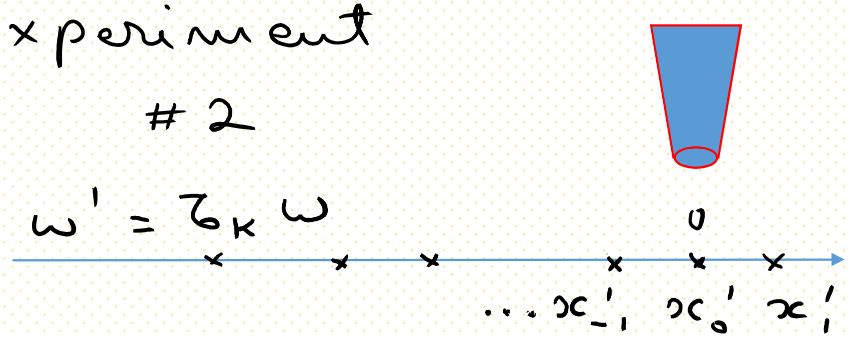


$$\Psi_0 = \sum_i c_i |i\rangle \;\longrightarrow\; \Psi(t) = e^{-itH_\omega}\Psi_0$$

$$\Downarrow$$

H_ω (derived from dynamics)

Experiment #2



$$\Psi_0 = \sum_i c_i |i\rangle \;\longrightarrow\; \Psi'(t) = e^{-itH_{\omega'}}\Psi_0$$

$$\Downarrow$$

$$H_{\omega'} = H_{\tau_K \omega}$$

Consistency between the two observations:

$$H_{\tau_K \omega} = T_K H_\omega T_K^*, \quad T_K |i\rangle = |i+K\rangle$$

Conclusions :

1) The dynamics of collective states is determined by

$$\{ H_{\tau_k \omega} \}_{k \in \mathbb{Z}^d}, \text{ a whole family !}$$

2) The covariant property $H_{\tau_k \omega} = T_k H_\omega T_k^*$

$$\Rightarrow \hat{w}_{n,m}(\tau_k \omega) = \hat{w}_{n-k,\, m-k}(\omega) \text{ (continuous of } \omega)$$

3) $$H_\omega = \sum_{n,m} w_{n,m}(\omega) \otimes |n\rangle\langle m| = \sum_q \sum_n \hat{w}_{0,q}(\tau_n \omega) \otimes |n\rangle\langle n| T_q$$

The Hull and the Minimal Algebra of Observables

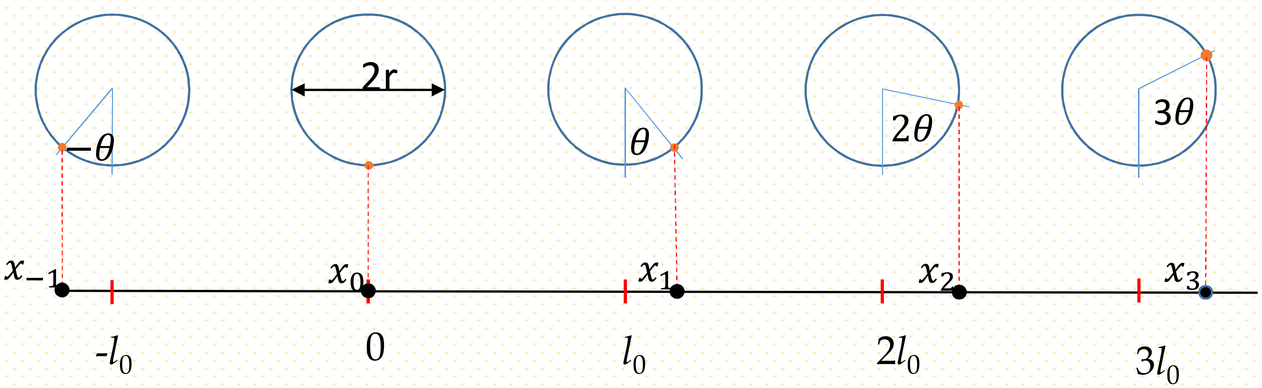
Starting from the original ω, $\tau_k \omega$ provide an orbit in the topological space $(\mathscr{C}(\mathbb{R}^{\alpha}), D)$

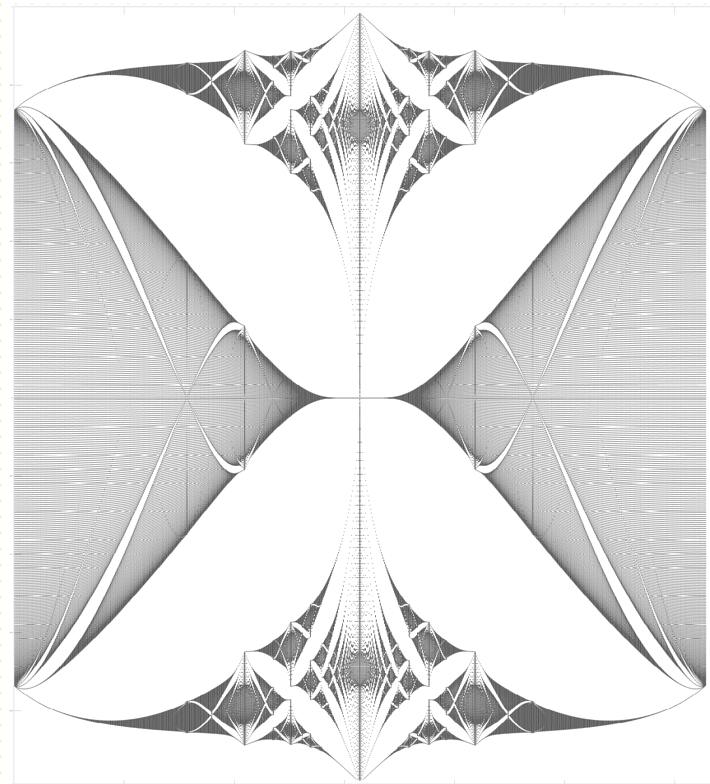
$$\text{Hull:} \quad \Omega = \overline{\{ \tau_k \omega , \ k \in \mathbb{Z}^d \}} \subset \mathscr{C}(\mathbb{R}^{\alpha})$$

The hull is invariant w.r.t. translations

$$\Rightarrow (\Omega, \mathbb{Z}, \tau)$$ classical topological dynamical system

(θ = irrational)

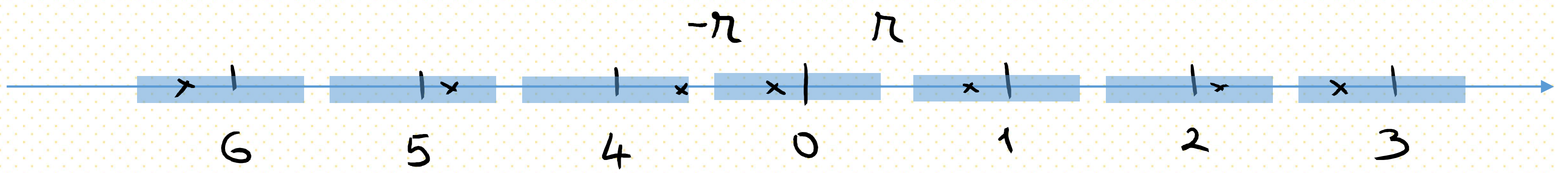




$$\mathcal{C}(\mathbb{R}) \ni \tau_k \omega \longleftrightarrow k\theta \in \mathbb{T}^1$$

$\{\tau_k \omega\}_{k\in\mathbb{Z}}$ generates a dense orbit in $\mathbb{T}^1$ ➡ $\Omega = \mathbb{T}^1$

(a disordered crystal)

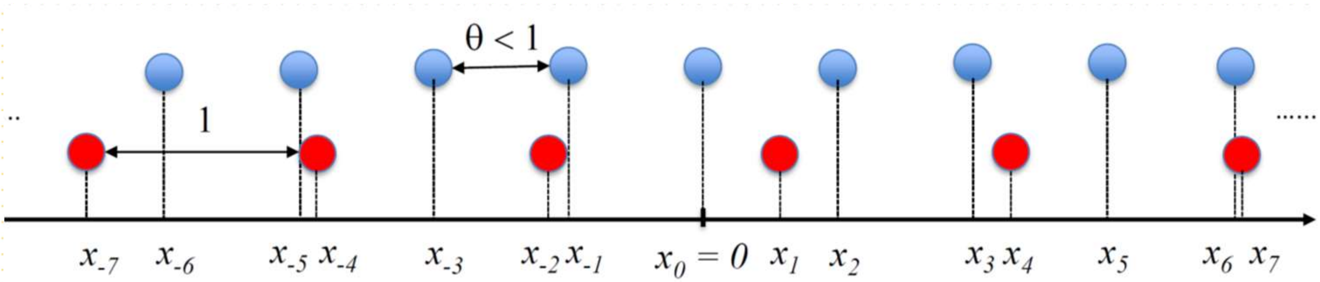


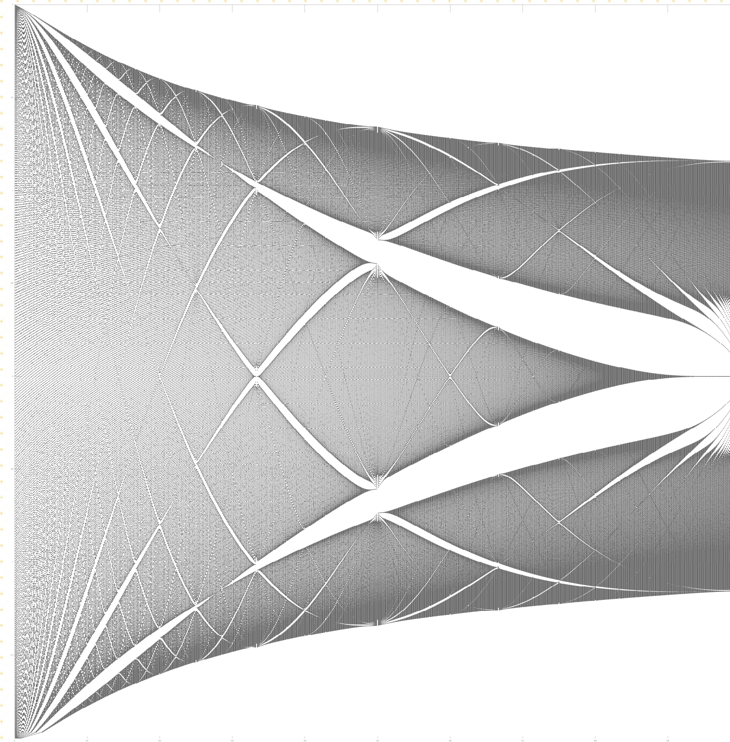
$\{\alpha_n\}_n$ randomly, independtly, uniformly

$$x_n = n + r\alpha_n \,, \quad r < 0.5$$

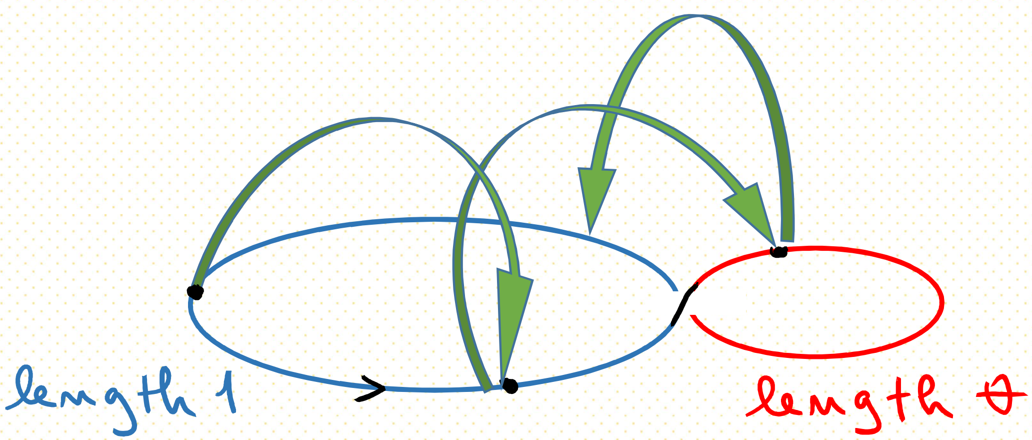
Then: $\tau_k \omega$ generates a dense orbit in $[-1,1]^{\mathbb{Z}}$

$$\Omega = [-1,1]^{\mathbb{Z}} \quad \text{(topologically trivial)}$$





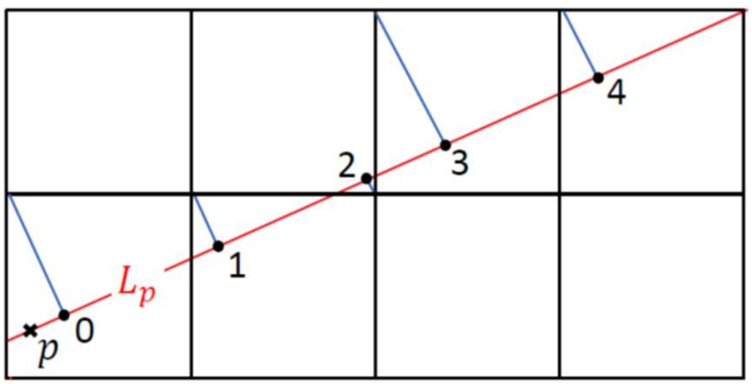
The hull :

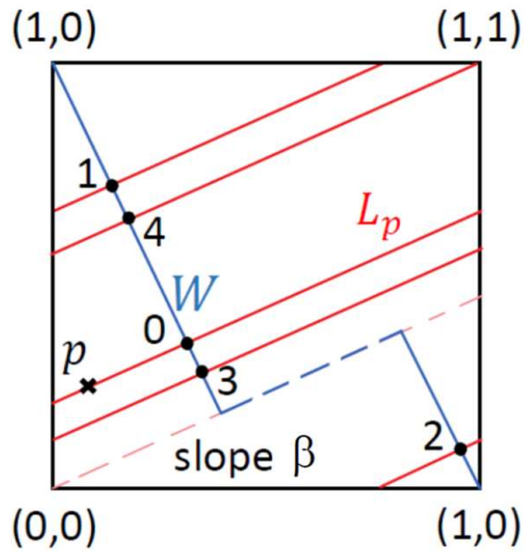


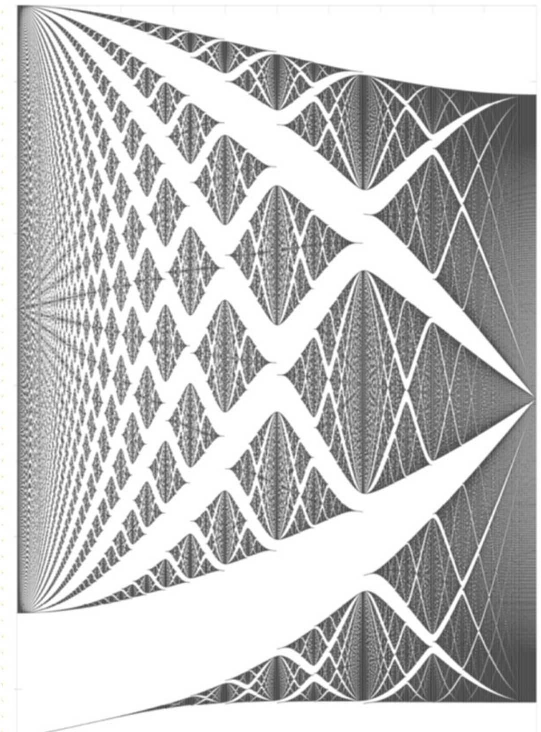
discrete translations

= hoppings by θ

Quasi-crystals by cut-and-project

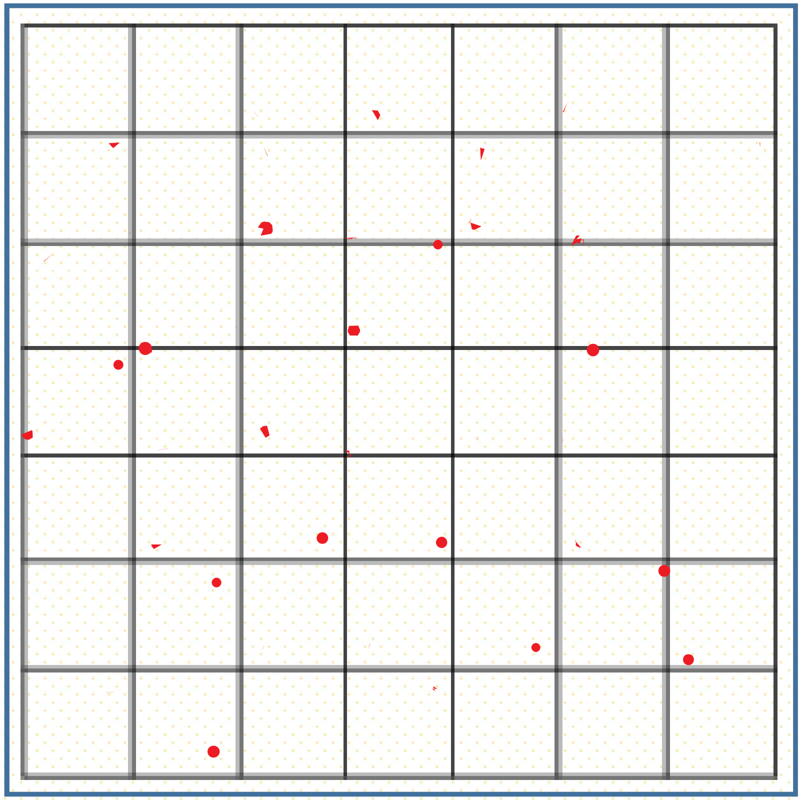


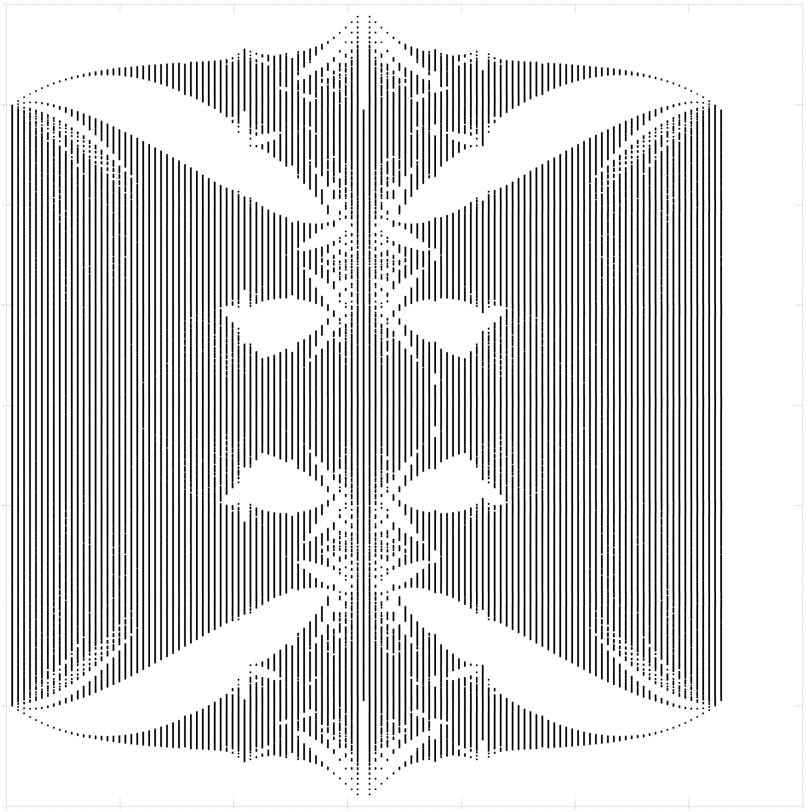




The hull is the Cantorized circle at $\{n\theta, n \in \mathbb{Z}\}$

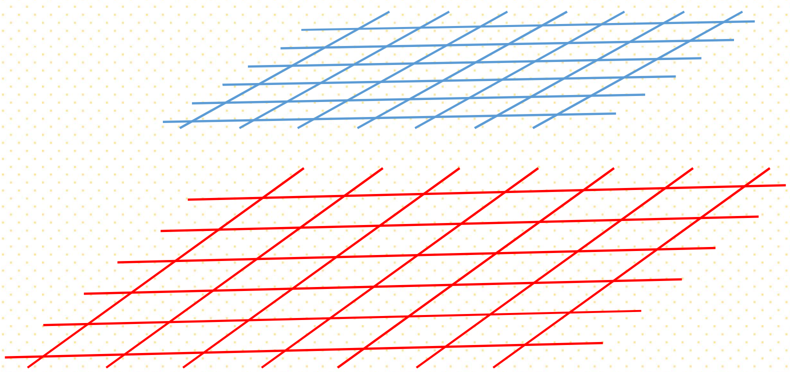
$$x_{nm} = (n, m) + r\left(\sin(n\theta_1), \sin(m\theta_2)\right)$$



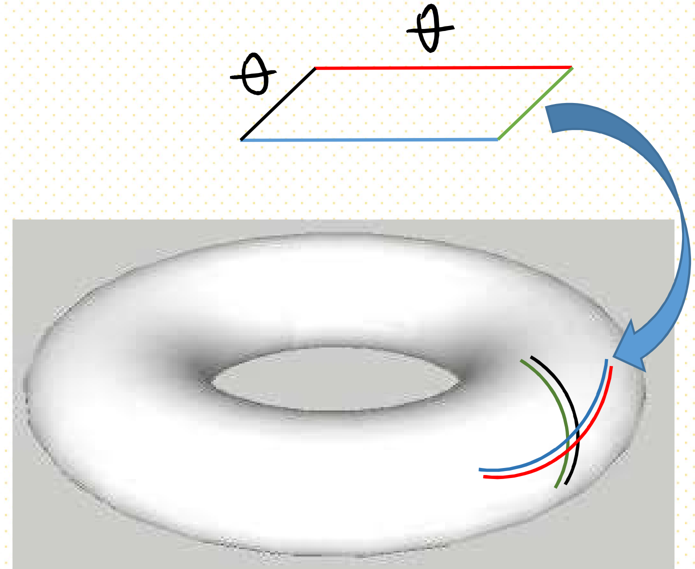


$\{\tau_k \omega\}_{k \in \mathbb{Z}^2}$ generates a dense orbit in $\mathbb{T}^2$ $\Rightarrow$ $\Omega = \mathbb{T}^2$

Lattice constant θ



Lattice constant 1



Glued as collored coded

Ω

The algebra: Starting point $(\Omega, \tau, \mathbb{Z}^d)$

$$\mathcal{A}_d = C^*\big(C(\Omega), u_1, \dots, u_d\big)$$

universal algebra generated by $f : \Omega \to \mathbb{C}$ plus u_j's

$$f\, u_q = u_q\,(f \circ \tau_q)\,, \quad u_q = u_1^{q_1} \cdots u_d^{q_d}\,, \quad q \in \mathbb{Z}^d$$

Generic element: $$a = \sum_q a_q\, u_q\,, \quad a_q \in C(\Omega)$$

The canonical representation:

$$\mathcal{A}_d \xrightarrow{\pi_\omega} \text{Operators over } \ell^2(\mathbb{Z}^d)$$

$$\pi_\omega(u_i) = T_i \quad (\text{translation in } i\text{-th direction})$$

$$\pi_\omega(f) = \sum_{\vec{m}} f(\tau_{\vec{m}}\omega)\,|m\rangle\langle m|$$

All covariant families $\{H_\omega\}$ can be generated like

$$\mathbb{C}(N)\otimes\mathcal{A}_d \ni \sum_{\vec{q}} \hat{w}_{\vec{q}}\, u_{\vec{q}} \xrightarrow{\pi_\omega} H_\omega = \sum_{\vec{q},\vec{m}} \hat{w}_{\vec{q}}(\tau_{\vec{m}}\omega)\otimes|\vec{m}\rangle\langle\vec{m}|\,T_{\vec{q}}$$

Remarks about the meaning of $\mathcal{A}_d$

1) $\mathcal{A}_d$ provides the environment where we can deform the system. This can be done in several ways:

a) changing the coefficients $\hat{w}_{\vec{q}}(\omega)$

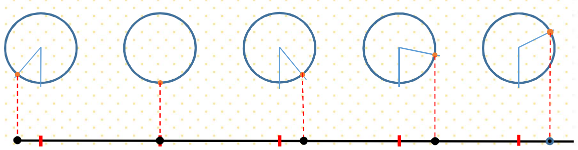
b) changing the pattern but leaving Ω the same

2) The element $h = \sum_q \hat{w}_q u_q$ encodes the whole $\{H_\omega\}_{\omega\in\Omega}$

but note that $\tau(h) = \tau(H_\omega)$ (indep of ω)

Bulk - Boundary Principle at work

Example 1 :



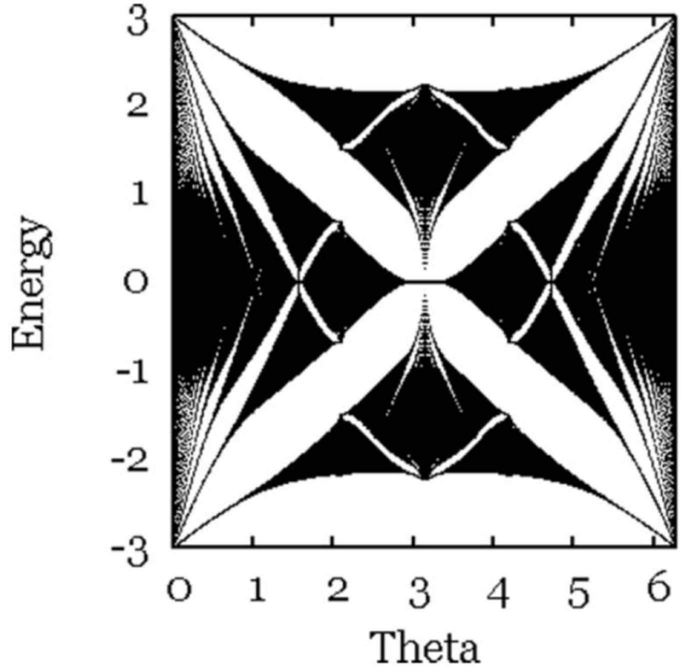
$$h = \sum_q w_q u^q$$

$$w_0(\omega) = |x_1 - x_0|$$

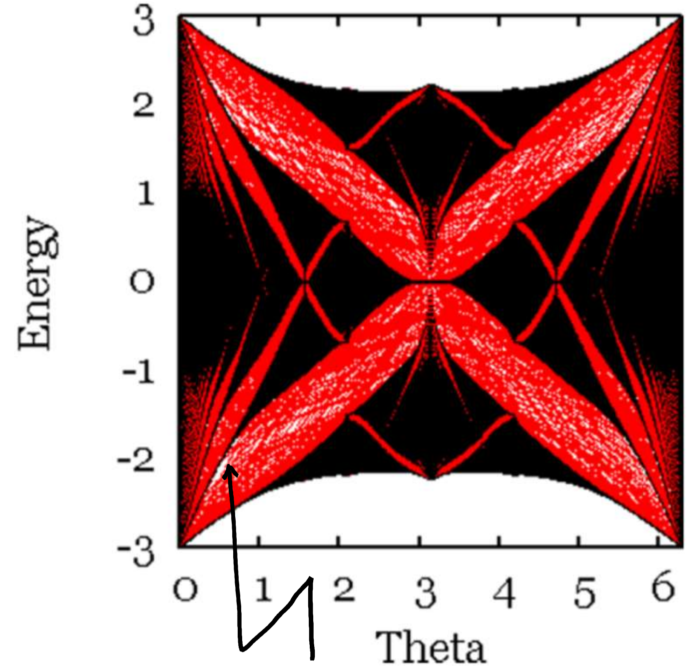
$$w_{\pm 1} = 1$$

$$w_q = 0 \text{ in rest}$$

Bulk

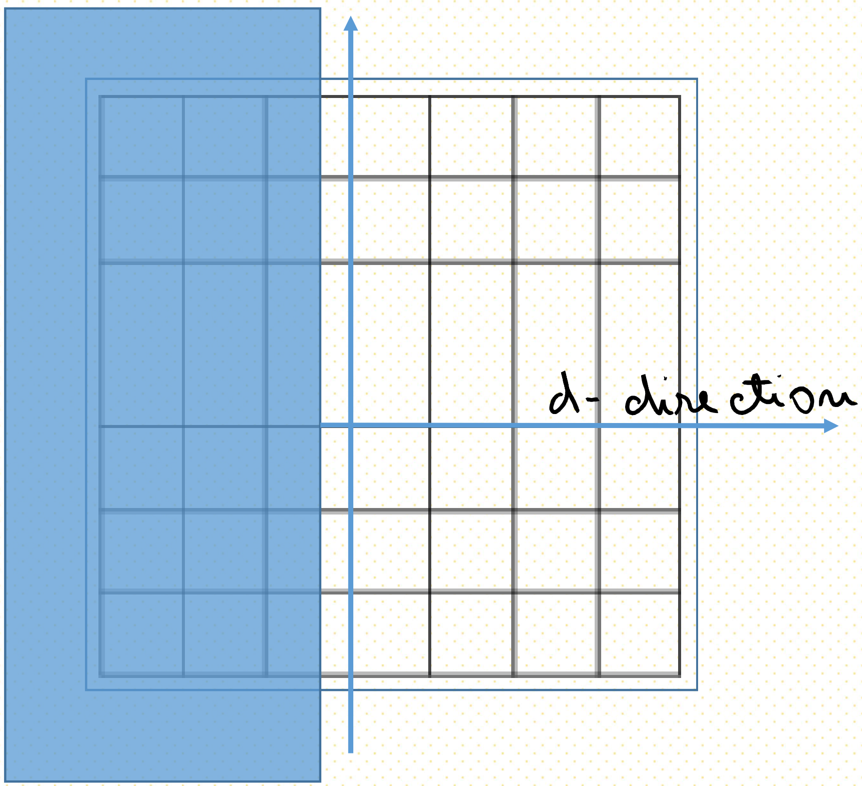


Ribbon



Edge
Spectrum

Systems with boundary



Hilbert space

$$\hat{\mathcal{H}} = \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$$

T_d - becomes partial isometry

$$T_d^* T_d = I \,, \quad T_d T_d^* = I - P_0$$

$$P_0 = \sum_{n_d = 0} |n\rangle\langle n|$$

The Algebra of Observables for Half-Space

$$\hat{A}_d = C^*\left(C(\Omega), \hat{u}_1, \ldots, \hat{u}_d\right)$$

where the only change is:

$$\hat{u}_d \hat{u}_d^* = 1 \quad , \quad \hat{u}_d^* \hat{u}_d = 1 - e$$

A generic element takes the form:

$$\hat{a} = \sum_{\vec{q} \in \mathbb{Z}^{d-1}} \sum_{n, m \in \mathbb{N}} \hat{a}_{\vec{q},n,m} \, \hat{u}_{\vec{q}} \, \hat{u}_d^n \left(\hat{u}_d^*\right)^m$$

Canonical Representation

$$C(\Omega) \ni \hat{f} \xrightarrow{\hat{\Pi}_\omega} \sum_{\vec{x}, n} \hat{f}(\tau_{\vec{x}, n} \omega) \, |\vec{x}, n\rangle\langle \vec{x}, n|$$

$$\hat{u}_j \to T_j \,, \quad j = 1, \ldots, d-1 \,, \quad \hat{u}_d \to \hat{T}_d \,, \quad e \longrightarrow P_0$$

Example: H_ω with Dirichlet boundary condition

$$\hat{h}_D = \sum_{\vec{q} \in \mathbb{Z}^{d-1}} \left(\sum_{n<0} w_{\vec{q}, n} \, u_{\vec{q}} \, (u_d^*)^{|n|} + \sum_{n>0} w_{\vec{q}, n} \, u_{\vec{q}} \, u_d^n \right)$$

The Boundary Algebra

The special elements

$$\tilde{a} = \sum_{\vec{q} \in \mathbb{Z}^{d-1}} \sum_{n, m \in \mathbb{N}} \tilde{a}_{\vec{q}, n, m} \, \hat{u}_q \, u_d^n \, e \, (u_d^*)^m$$

are mapped by $\widehat{\pi}_\omega$ in boundary terms. They form:

(ideal) $\qquad \tilde{A}_d = \hat{A}_d \, e \, \hat{A}_d \subset \hat{A}_d$

Example: $\hat{h}' = \hat{h}_D + \tilde{h} \longrightarrow H_\omega$ with a different BC.

Important remarks about half-space algebra.

1) $\hat{h}$ encodes the whole family $\{\hat{H}_\omega\}_{\omega\in\Omega}$

2) This time $\sigma(\hat{H}_\omega)$ depends on ω More precisely

$$\sigma(\hat{H}_{\tau_{\vec{q}_\parallel}\omega}) = \sigma(\hat{H}_\omega)$$

$$\sigma(\hat{H}_{\tau_{\vec{q}_\perp}\omega}) = \sigma(\hat{H}'_\omega)$$

where $\hat{H}'_\omega$ is with boundary moved up by $\vec{q}_\perp$

Conclusion: $\sigma(\hat{h}) = \bigcup_{\omega\in\Omega} \sigma(\hat{H}_\omega) = \bigcup_{\text{Boundary}} \sigma(\hat{H}_\omega)$

Elements of K-Theory

Both, the bulk spectrum and the emergence of edge spec can be rationalize using K-Theory:

$K_0(\mathcal{A})$ classifies projections

$p \in \mathbb{C}(\infty) \otimes \mathcal{A}$, $p^2 = p^* = p$

$p \sim p'$ if homotopic

$$[p]_0 + [p']_0 = \begin{bmatrix} p & 0 \\ 0 & p' \end{bmatrix}_0$$

$K_1(\mathcal{A})$ classifies unitaries

$u \in \mathbb{C}(\infty) \otimes \mathcal{A}$, $u^* u = u u^* = 1$.

$u \sim u'$ if homotopic

$$[u]_1 [u']_1 = [u \cdot u']_1$$

The Engine of Bulk-Boundary correspondance

There exists the exact sequence

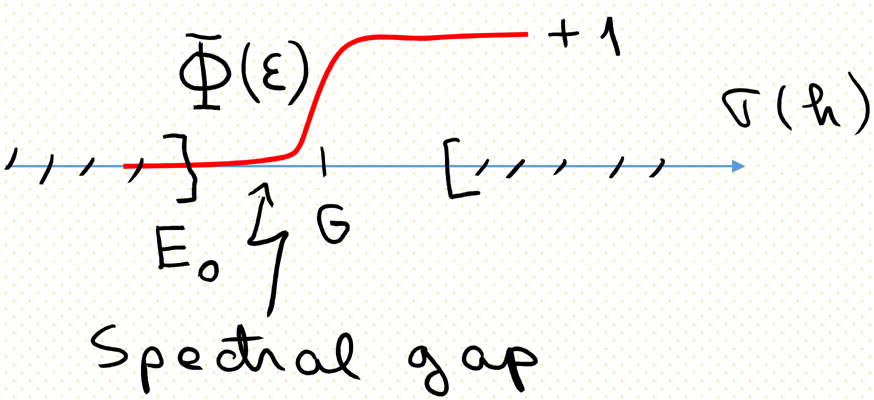
$$0 \longrightarrow \tilde{\mathcal{A}}_d \xrightarrow{i} \hat{\mathcal{A}}_d \xrightarrow{ev} \mathcal{A}_d \longrightarrow 0 \qquad \left(ev(\hat{u}_j) = u_j \right)$$

which automatically leads to an exact sequence:

$$\boxed{\begin{array}{ccccc} K_0(\tilde{\mathcal{A}}_d) & \longrightarrow & K_0(\hat{\mathcal{A}}_d) & \longrightarrow & K_0(\mathcal{A}_d) \\ \mathrm{Ind} \uparrow & & & & \downarrow \mathrm{Exp} \\ K_1(\mathcal{A}_d) & \longleftarrow & K_1(\hat{\mathcal{A}}_d) & \longleftarrow & K_1(\tilde{\mathcal{A}}_d) \end{array}}$$

How the Exponential Map Works

$$h \in \mathcal{A}_d$$



$$p = \chi_{(-\infty,G)}(h) \in K_0(\mathcal{A}_d)$$

Boundary $(\hat{h} \in \hat{\mathcal{A}}_d)$

$$\mathrm{Exp}[p]_0 = [\,e^{-2\pi i \Phi(\hat{h})}\,]_1$$

If $\mathrm{Exp}[p]_0 \neq [1]_1$

$$\Downarrow$$

$$\boxed{\sigma(\hat{h}) \cap [G-\delta, G+\delta] \neq \emptyset}$$

Example 1: $x_n = n + r \sin(n\theta)$

$$\Rightarrow \Omega = \mathbb{T}^1 \;, \quad \tau_k x = (x + k\theta) \bmod 2\pi$$

Then: $C(\Omega) = C^*(v)$, $v : \mathbb{T}^1 \to \mathbb{C}$, $v(x) = e^{ix}$

The algebra of observables:

$$\mathcal{A} = C^*(C(\Omega), u) = C^*(v, u)$$

Commutation relation:

$$v u = u (v \circ \tau_1) = u \left(e^{i(x+\theta)} \right) = e^{i\theta} u v$$

Conclusion: $\mathcal{A}$ = rotational algebra

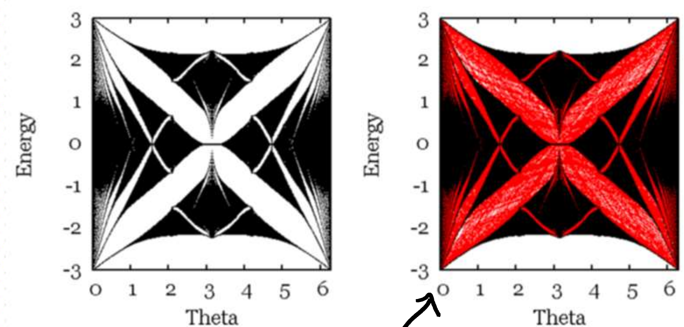
$K_0(\mathcal{A})$ generated by $[1]_0$ and $[e]_0$ $(\mathrm{Ch}[e]_0 = 1)$

$K_1(\tilde{\mathcal{A}}) = K_1(C(\Omega))$ generated by $[v]_1$.

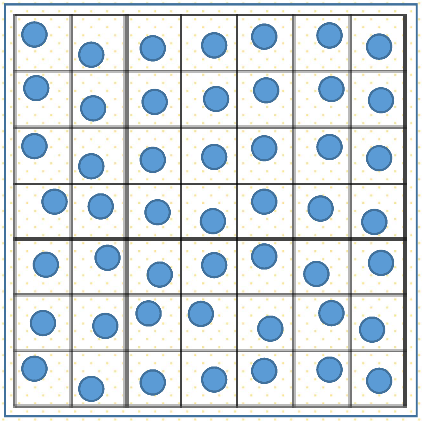
$\mathrm{Exp}[e]_0 = [v]_1$ and $\mathrm{Ch}[e]_0 = \mathrm{Winding}[v]_1$

Then, for any gap

$[P_{gap}]_0 = n[1]_0 + m[e_0]$



$\mathrm{Exp}[P_{gap}]_0 = [v^m]_1 \neq [1]_1$ Edge States!



$$P_{nm} = (n, m) + (n\theta_1, m\theta_2) \bmod (1,1)$$

$$\Omega = \mathbb{T}^2$$

$$\tau_{\vec{q}} : \Omega \to \Omega$$

$$\begin{cases} \tau_{1,0}(x_1, x_2) = (x_1 + \theta_1, x_2) \\ \tau_{0,1}(x_1, x_2) = (x_1, x_2 + \theta_2) \end{cases}$$

$$C(\Omega) = C(\mathbb{T}^2) = C^*(u_1, u_2)$$

$$u_i : \mathbb{T}^2 \longrightarrow \mathbb{C}$$

$$\begin{cases} u_1(x_1, x_2) = e^{ix_1} \\ u_2(x_1, x_2) = e^{ix_2} \end{cases}$$

$$\mathcal{A} = C^*(C(\Omega), u_3, u_4) = C^*(u_1, u_2, u_3, u_4)$$

Commutation Relations:

$$u_1 u_3 = u_3 (u_1 \circ \tau_{1,0}) = u_3 \, e^{i(x_1 + \theta_1)} = e^{i\theta_1} u_3 u_1$$

$$u_2 u_4 = u_4 (u_2 \circ \tau_{0,1}) = u_4 \, e^{i(x_2 + \theta_2)} = e^{i\theta_2} u_4 u_2$$

Conclusion: $\mathcal{A}$ is just the noncommutative torus

$$\mathcal{A} = C^*(u_1, \dots, u_4), \quad u_i u_j = e^{i\theta_{ij}} u_j u_i, \quad \begin{cases} \theta_{1,3} = \theta_1 \\ \theta_{2,4} = \theta_2 \end{cases}$$

K_0 - Group and its Generators:

$$\mathcal{J} \subseteq \{1, \dots, d\} = \text{set of indices}$$

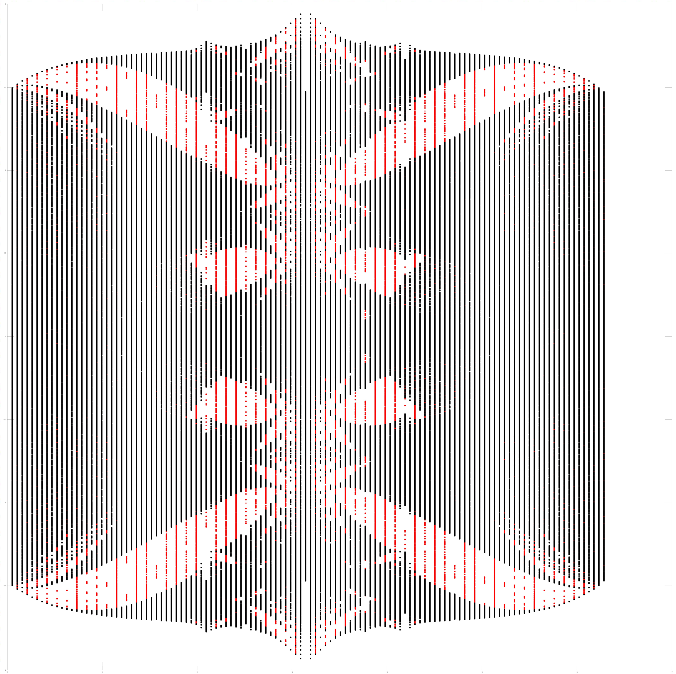
$\Rightarrow$ $K_0(\mathcal{A})$ is generated by

$$[e_{\mathcal{J}}]_0, \quad 2^3\text{- generators}$$

For top generator $(\mathcal{J} = \{1, \dots, 4\})$

$$Ch_2[e_{top}] = 1 = \text{Winding}\{\text{Exp}[e_{top}]_1\}$$

As expected, there are many gaps in $\sigma(H_\omega)$



$$[P_G]_0 = \sum_{J \subseteq \{1,\dots,4\}} \underbrace{\alpha_J}_{\text{integers}} [e_J]_0$$

and there is a P_g with $\alpha_{Top} \neq 0$. Then:

$$Ch_{top}[P_g]_0 = \alpha_{top} = \mathrm{Wind}_3 \, \mathrm{Exp}[P_g]_0 \neq 0$$

$\Rightarrow$ Edge modes appear!