Chiral Anomaly, Topological Field Theory, and Topological States of Matter

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Plan of lecture

Credits

R. Morf (mentor in matters of the QHE) - Various collaborations with, among others: Alekseev, Bieri, Boyarsky, Cheianov, Graf, Kerler, Levkivskyi, Pedrini, Ruchayskiy, Schweigert, Studer, Sukhorukov, Thiran, Walcher, Werner, Zee.

Contents

Introduction

Abstract

Starting with a description of the goals of the analysis and a brief survey of the chiral anomaly, I will review some basic elements of the theory of the quantum Hall effect in 2D electron gases. I will discuss the role of anomalous chiral edge currents and of anomaly inflow in 2D insulators with explicitly or spontaneously broken parity and time reversal, i.e., in incompressible Hall fluids and Chern insulators, respectively. The topological Chern-Simons actions yielding the correct response equations for the 2D bulk of such materials will be exhibited.

I will then analyse chiral edge spin-currents and the bulk response equations in time-reversal invariant 2D topological insulators.

A short digression into the theory of 3D topological insulators, including "axionic insulators", will follow.

To conclude some open problems and an outlook towards other related areas of theoretical physics will be presented.

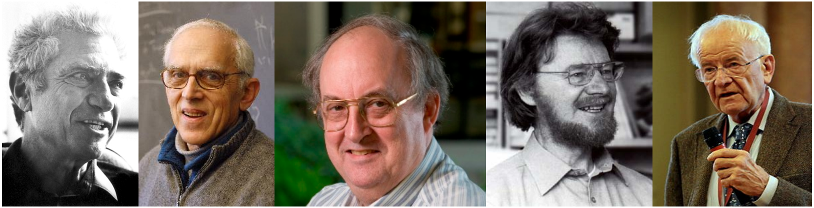
General goals of analysis

- Classify bulk- and surface states of (condensed) matter, using concepts and results from gauge theory, current algebra & GR: Effective actions (= generating functionals of connected current Green fcts.! transport coefficients!), gauge-invariance, anomalies & their cancellation, "holography", etc.
- Complement Landau Theory of Phases and Phase Transitions by a Gauge Theory of Phases of Matter.

Applications

- Fractional Quantum Hall Effect (1989 – 2012)
- Topological Insulators and -Superconductors (1993 – 2015)
- Higher-dimensional cousins of QHE $\Rightarrow$ Cosmology: Primordial magnetic fields in the Universe, matter-antimatter asymmetry, dark matter & dark energy, etc. (1999 – $\cdots$)

The chiral anomaly



Anomalous axial currents (for massless fermions):
In 2D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{2\pi} E, \quad \alpha := \frac{e^2}{\hbar}, \quad [j_5^0(\vec{y}, t), j^0(\vec{x}, t)] \overset{(ACC)}{=} i\alpha\delta'(\vec{x} - \vec{y})$$

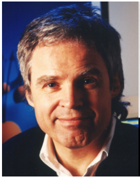
In 4D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{\pi} \vec{E} \cdot \vec{B},$$

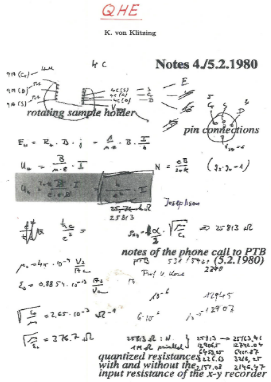
and

$$[j_5^0(\vec{y}, t), j^0(\vec{x}, t)] \overset{(ACC)}{=} i\frac{\alpha}{\pi} \vec{B}(\vec{y}, t) \cdot \nabla_{\vec{y}} \delta(\vec{x} - \vec{y})$$

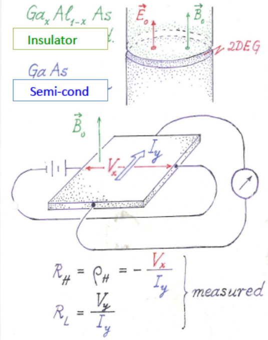
1. Anomalous Chiral Edge Currents in Incomp. Hall Fluids

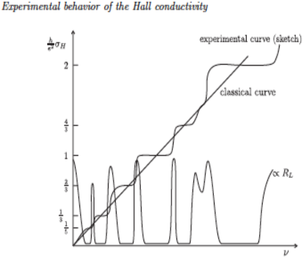



From von Klitzing's lab journal ($\Rightarrow$ 1985 Nobel Prize in Physics):



Setup & basic quantities





2D EG confined to $\Omega \subset xy$ - plane , in mag. field $\vec{B}_0 \perp \Omega$; ν such that $R_L = 0$. Response of 2D EG to small perturb. em field, $\vec{E} \| \Omega$, $\vec{B} \perp \Omega$, with $\vec{B}^{tot} = \vec{B}_0 + \vec{B}$, $\quad B := |\vec{B}|$, $\quad \underline{E} := (E_1, E_2)$.

Field tensor: $\quad F := \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix} = dA, \quad$ (A: vector pot.)

Electrodynamics of 2D incompressible e^{-}-gases

Def.:

$$j^{\mu}(x) = \langle J^{\mu}(x) \rangle_{A}, \quad \mu = 0, 1, 2.$$

(1) Hall's Law

$$\underline{j}(x) = \sigma_H \big(\underline{E}(x)\big)^{*}, \quad (R_L = 0!) \ \rightarrow \ \text{broken } P, T \tag{1}$$

(2) Charge conservation

$$\frac{\partial}{\partial t}\rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0 \tag{2}$$

(3) Faraday's induction law

$$\frac{\partial}{\partial t} B_3^{tot} + \underline{\nabla} \wedge \underline{E}(x) = 0 \tag{3}$$

Then

$$\frac{\partial \rho}{\partial t} \overset{(2)}{=} -\underline{\nabla} \cdot \underline{j} \overset{(1)}{=} -\sigma_H \underline{\nabla} \wedge \underline{E} \overset{(3)}{=} \sigma_H \frac{\partial B}{\partial t} \tag{4}$$

ED of 2D e^--gases, ctd.

Integrate (4) in t, with integration constants chosen as follows:

$$j^0(x) := \rho(x) + e \cdot n, \;\; B(x) = B_3^{tot}(x) - B_0 \qquad \Rightarrow$$

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x) \tag{5}$$

$$\text{Eqs. (1) and (5)} \quad \Rightarrow \boxed{j^\mu(x) = \sigma_H \; \varepsilon^{\mu\nu\lambda} \; F_{\nu\lambda}(x)} \tag{6}$$

Now

$$0 \overset{(2)}{=} \partial_\mu j^\mu \overset{(3),(6)}{=} \varepsilon^{\mu\nu\lambda}(\partial_\mu \sigma_H) F_{\nu\lambda} \;\; \neq 0, \tag{7}$$

wherever $\sigma_H \neq$ *const*., e.g., at $\partial\Omega$. – Actually, j^μ is *bulk* current density, (j^μ_{bulk}), $\neq$ conserved *total* electric current density:

$$j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{edge}, \quad \partial_\mu j^\mu_{tot} = 0, \;\text{but}\;\; \partial_\mu j^\mu_{bulk} \overset{(7)}{\neq} 0 \tag{8}$$

Anomalous chiral edge currents

We have that

$$\text{supp } j^{\mu}_{edge} = \text{supp}(\underline{\nabla}\sigma_H) \supseteq \partial\Omega, \qquad \underline{j}_{edge} \perp \underline{\nabla}\sigma_H.$$

"Holography": On $\text{supp}(\underline{\nabla}\sigma_H)$,

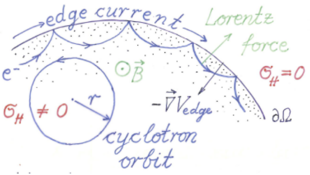
$$\boxed{\partial_\mu j^{\mu}_{edge} \overset{(8)}{=} -\partial_\mu j^{\mu}_{bulk}|_{\text{supp}(\underline{\nabla}\sigma_H)} \overset{(6)}{=} -\sigma_H E_{\|}|_{\text{supp}(\underline{\nabla}\sigma_H)}} \tag{9}$$

Chiral anomaly in 1+1 dimensions!

Edge current, $j^{\mu}_{edge} \equiv j^{\mu}_{5}$, is anomalous chiral current in $1+1$ D: At edge,

$$\frac{e}{c} B^{tot} v_{\|} = (\underline{\nabla} V_{edge})^{*}, \quad V_{edge}: \text{ confining edge pot.}$$

Skipping orbits, hurricanes, and fractional charges



Analogous phenomenon in classical physics: Hurricanes!

$$\vec{B} \to \vec{\omega}_{\text{earth}}, \text{ Lorentz force } \to \text{ Coriolis force, } V_{edge} \to \text{ air pressure.}$$

The chiral anomaly in $(1+1)$D says that

$$\partial_\mu j_5^\mu = -\frac{e^2}{h}\Big(\sum_{\text{i.e.m., }\alpha} Q_\alpha^2\Big)E_\parallel \quad \overset{\text{with (9)}}{\Rightarrow} \quad \boxed{\sigma_H = \frac{e^2}{h}\sum_\alpha Q_\alpha^2,} \qquad (10)$$

where eQ_α is the electric charge of the edge current corresponding to a clockwise-chiral edge mode α; (similar contributions from anti-clockwise modes, but with reversed sign!) $\to$ Bert Halperin's edge currents!

Edge- and bulk effective actions

Apparently, if $\sigma_H \notin \frac{e^2}{h}\mathbb{Z}$ then there exist fractionally charged quasi-particles propagating along supp$(\underline{\nabla}\sigma_H)$!
Chiral edge current d. J^μ_{edge} = generator of $U(1)$- current algebra (free massless fields!) Green functions of J^μ_{edge} obtained from 2D anomalous effective action $\Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|}) = \cdots$, where $A_{\|}$ is restriction of vector potential, A, to boundary $\partial\Omega \times \mathbb{R}$.
Anomaly of $\sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|})$ – consequence of fact that J^μ_{edge} is *not* cons. – is cancelled by the one of bulk effective action, $S_{\Omega\times\mathbb{R}}(A)$:

$$\begin{aligned} j^\mu_{bulk}(x) &= \langle J^\mu(x)\rangle_A \equiv \frac{\delta S_{\Omega\times\mathbb{R}}(A)}{\delta A_\mu(x)} \\ &\overset{(6)!}{=} \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad x \notin \partial\Omega \times \mathbb{R} \end{aligned}$$

$$\Rightarrow \quad \boxed{S_{\Omega\times\mathbb{R}}(A) = \frac{\sigma_H}{2}\int_{\Omega\times\mathbb{R}} A \wedge \mathrm{d}A + \sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|})} \qquad (11)$$

Chern-Simons action on manifold with boundary!

Edge- and bulk effective actions – ctd.

The 2D anomalous effective action is given by

$$\Gamma_{\partial\Omega\times\mathbb{R}}(a) := \int_{\partial\Omega\times\mathbb{R}} \left[a_+ a_- - 2a_\pm \frac{\partial_\mp^2}{\Box} a_\pm\right] \mathrm{d}^2 u,$$

where $a = a_+ \mathrm{d}u^+ + a_- \mathrm{d}u^- \equiv A_{||}$ ("light-cone coordinates"). *Exercise:* Check that the anomaly of the (bulk) Chern-Simons action, which is a boundary term, is cancelled by the one of $\Gamma_{\partial\Omega\times\mathbb{R}}(A_{||})$.

Whatever we have said about 2D electron gases in a homogeneous external magnetic field exhibiting the QHE can be extended to so-called Chern insulators, which have the property that Parity and Time Reversal are broken *even* in the absence of an external magnetic field, (e.g., because of magnetic impurities in the bulk). The low-energy physics of quasi-particles in the bulk of a Chern insulator resembles the one of two-component relativistic Dirac fermions coupled to the electromagnetic vector potential, with an effective action given by (11), with $A = A_{\text{tot}}$, and $\sigma_H = e^2/2h$ ($=$ Chern class of a certain vector bundle of Dirac fermion wave functions over Brillouin zone $\mathbb{T}^2$).

Classification of "abelian" Hall fluids & Chern insulators

Next, I sketch a general classification of $2D$ insulators with broken T and P in topologically protected states exhibiting quasi-particles with abelian braid statistics; ("non-abelian states" discussed elsewhere).

As above, J denotes the *total* electric current density (bulk + edge), which is conserved, ie., $\partial_\mu J^\mu = 0$. We consider a general ansatz for J:

$$J = \sum_{\alpha=1}^{N} Q_\alpha J_\alpha,$$

where the J_α are separately conserved current densities corresp. to different quasi-particle species, and the coeffs. $Q_\alpha \in \mathbb{R}$ are "charges". On a 3D space-time $\Lambda = \Omega \times \mathbb{R}$, a conserved current density J can be derived from a vector potential: If j denotes the 2-form dual to J then conservation of J implies that $\mathrm{d}j = 0$, and hence

$$j = \mathrm{d}B,$$

where the 1-form B is the vector potential of j and is determined up to a gradient of a scalar function β; i.e., B and $B + \mathrm{d}\beta$ yield the same j.

Chern-Simons effective action of conserved currents

Henceforth we use units where $\frac{e^2}{h} = 1$. For a 2D insulator, the field theory of the conserved currents $\left(J_\alpha\right)_{\alpha=1}^{N}$ in the limit of very large distances and low frequencies must be topological: If P and T are broken the "most relevant" term in the action is the Chern-Simons action

$$S_\Lambda(\underline{B}, A) := \sum_{\alpha=1}^{N} \int_\Lambda \{\frac{1}{2} B_\alpha \wedge \mathrm{d}B_\alpha + A \wedge Q_\alpha \mathrm{d}B_\alpha\} + \text{boundary terms}, \quad (^*)$$

where A is the electromagnetic vector potential, and the boundary terms must be added to cancel the anomalies under the gauge transformations, $B \to B + \mathrm{d}\beta$ and $A \to A + \mathrm{d}\chi$, of the Chern-Simons action (1^{st} term on right side).

Carrying out the Gaussian functional integral, we find that

$$\int \exp\left(iS_\Lambda(\underline{B}, A)\right) \prod_{\alpha=1}^{N} \mathcal{D}B_\alpha = \exp\left(i\frac{\sigma_H}{2} \int_\Lambda A \wedge \mathrm{d}A + \sigma_H \Gamma_\Lambda(A_\parallel)\right), \quad (^{**})$$

where $\sigma_H = \sum_{\alpha=1}^{N} Q_\alpha^2.$

Classification of 2D "abelian" topological insulators with broken P and T – bulk degrees of freedom

Physical states of the topological field theory with action given by $(^*)$ can be constructed by inserting Wilson lines into the Gaussian functional integral on the left side of $(^{**})$. The operator measuring the electric charge contained in a region $\mathcal{O}$ of the sample space Ω is given by

$$\mathcal{Q}_{\mathcal{O}} = \int_{\mathcal{O}} J^0 \mathrm{d}^2 x = \sum_{\alpha=1}^{N} Q_\alpha \int_{\partial\mathcal{O}} B_\alpha .$$

If a Wilson line is supposed to create a state describing n electrons or holes contained in $\mathcal{O}$ from the ground state of the insulator then its electric charge, as measured by $\mathcal{Q}_{\mathcal{O}}$, is $-n + 2k$, $k = 1, \ldots, n$. If n is odd the statistics of this excitation is Fermi-Dirac statistics, if n is even it is Bose-Einstein statistics. This relation between the electric charge of an excitation and its statistics implies that the charge quantum numbers of Wilson lines creating multi-electron-hole excitations must belong to an odd-integral lattice, Γ, of rank N, and that $Q := (Q_1, \ldots, Q_N) \in \Gamma^*$. Hence $\sigma_H = \sum_{\alpha=1,\ldots,N} Q_\alpha^2$ is a rational number!

Classification, ctd. – edge degrees of freedom

Chiral anomaly (10) $\Rightarrow$ several (N) species of gapless quasi-particles propagating along edge $\leftrightarrow$ described by N chiral scalar Bose fields $\{\varphi^\alpha\}_{\alpha=1}^N$ with propagation speeds $\{v_\alpha\}_{\alpha=1}^N$, such that

1. Chiral electric edge current operator & Hall conductivity

$$J^\mu_{edge} = e \sum_{\alpha=1}^N Q_\alpha \, \partial^\mu \varphi^\alpha, \quad Q = (Q_1, \dots, Q_N), \quad \sigma_H = \frac{e^2}{h} Q \cdot Q^T$$

2. Multi-electron/hole states loc. along edge created by vertex ops.

$$: \exp i \left(\sum_{\alpha=1}^N q^j_\alpha \varphi^\alpha \right) :, \;\; q^j = \begin{pmatrix} q^j_1 \\ \vdots \\ q^j_N \end{pmatrix} \in \Gamma, \;\; j = 1, \dots, N. \qquad (12)$$

Charge $\leftrightarrow$ Statistics $\Rightarrow$ Γ an odd-integral lattice of rank N. Hence:

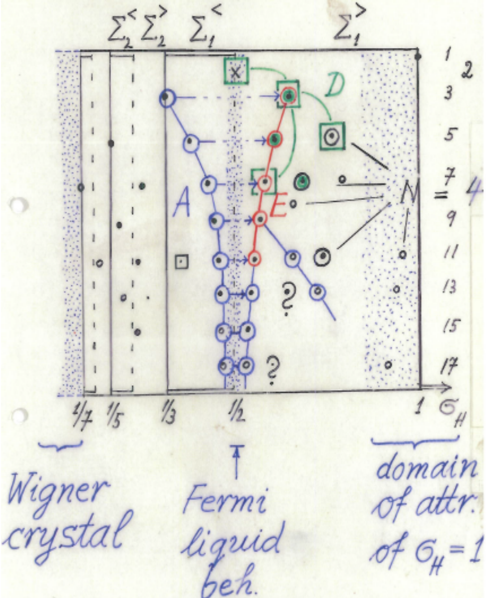
3. Classifying data are

$$\{\, \Gamma\,;\; Q \in \Gamma^* : \text{"visible"};\; (q^j_\alpha)^N_{j,\alpha=1} : \sim \text{CKM matrix}\;;\;\; v = (v_\alpha)^N_{\alpha=1} \,\}$$

$\rightarrow$ quasi-particles w. abelian braid statistics!

Success of classification – comparison with data

Γ = odd-integral lattice, $Q \in \Gamma^* \quad \Rightarrow \quad \left(\frac{e^2}{h}\right)^{-1}\sigma_H \in \mathbb{Q}\,(!)\,, \ldots$



2. Chiral Spin Currents in Planar Topological Insulators

So far, we have not paid attention to electron spin, although there are 2D EG exhibiting the fractional quantum Hall effect where spin plays an important role. Won't study these systems, today. Instead, we consider time-reversal-invariant 2d topological insulators (2D TI) exhibiting chiral spin currents.

Pauli Eq. for a spinning electron:

$$i\hbar D_o \Psi_t = -\frac{\hbar^2}{2m} g^{-1/2} D_k \left(g^{1/2} g^{kl}\right) D_l \Psi_t, \tag{13}$$

where m is the mass of an electron, (g_{kl}) = metric of sample,

$$\Psi_t(x) = \begin{pmatrix} \psi_t^{\uparrow}(x) \\ \psi_t^{\downarrow}(x) \end{pmatrix} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 : \qquad \text{2-component Pauli spinor}$$

$$i\hbar D_0 = i\hbar \partial_t + e\varphi - \underbrace{\vec{W}_0 \cdot \vec{\sigma}}_{\text{Zeeman coupling}}, \quad \vec{W}_0 = \mu c^2 \vec{B} + \frac{\hbar}{4} \vec{\nabla} \wedge \vec{V} \tag{14}$$

$U(1)_{em} \times SU(2)_{spin}$-gauge invariance

$$\frac{\hbar}{i} D_k = \frac{\hbar}{i} \partial_k + eA_k - m_0 V_k - \vec{W}_k \cdot \vec{\sigma}, \tag{15}$$

where $\vec{A}$ is em vector potential, $\vec{V}$ is velocity field describing mean motion (flow) of sample, $(\vec{\nabla} \cdot \vec{V} = 0)$,

$$\vec{W}_k \cdot \vec{\sigma} := \underbrace{[(-\tilde{\mu}\vec{E} + \frac{\hbar}{c^2}\dot{\vec{V}}) \wedge \vec{\sigma}]_k,}_{\text{spin-orbit interactions}}$$

and $\tilde{\mu} = \mu + \frac{e\hbar}{4mc^2}$ $(\leftarrow$ Thomas precession).
Note that the Pauli equation (13) respects $U(1)_{em} \times SU(2)_{spin}$ - gauge invariance.
We now consider an interacting 2D gas of electrons confined to a region Ω of the xy- plane, with $\vec{B} \perp \Omega$ and $\vec{E}, \vec{V} \| \Omega$. Then the $SU(2)$ - conn., $\vec{W}_\mu$, is given by $W_\mu^3 \cdot \sigma_3$, $(W^M = 0,$ for $M = 1, 2)$.

Effective action of a 2D TI

Thus the connection for parallel transport of the component $\psi^{\uparrow}$ of Ψ is given by $a + w$, while parallel transport of $\psi^{\downarrow}$ is determined by $a - w$, where $a_\mu = -eA_\mu + mV_\mu, w_\mu = W^3_\mu$. These connections are abelian, (phase transformations). Under time reversal,

$$a_0 \to a_0,\ a_k \to -a_k,\ \text{but } w_0 \to -w_0,\ w_k \to w_k. \tag{16}$$

The dominant term in the effective action of a 2D insulator is a Chern-Simons term. If there were only the gauge field a, with $w \equiv 0$, or only the gauge field w, with $a \equiv 0$, a Chern-Simons term would *not* be invariant under time reversal, and the dominant term would be given by

$$S(a) = \int \mathrm{d}t\mathrm{d}^2x\ \{\varepsilon \underline{E}^2 - \mu^{-1} B^2\} \tag{17}$$

But, in the presence of two gauge fields, a and w, satisfying (16):

Effective action of a 2D TI, ctd.

Combination of two Chern-Simons terms is time-reversal invariant:

$$\begin{aligned} S(a,w) &= \frac{\sigma}{2}\int \{(a+w)\wedge\, \mathrm{d}(a+w) - (a-w)\wedge\, \mathrm{d}(a-w)\} \\ &= \sigma \int \{a\wedge\, \mathrm{d}w + w\wedge\, \mathrm{d}a\} \end{aligned}$$

This reproduces (17) for phys. choice of w! ($\nearrow$ J.F., Les Houches '94!) – The gauge fields a and w transform independently under gauge transformations, and the Chern-Simons action is anomalous under these gauge trsfs. on a 2D sample space-time $\Lambda = \Omega \times \mathbb{R}$ with a non-empty boundary, $\partial\Lambda$. The anomalous chiral boundary actions,

$$\pm\sigma\Gamma\big((a \pm w)|_{\|}\big),$$

cancel anomaly of bulk action! Are generating functionals of conn. Green functions of two counter-propagating chiral edge currents:

Edge degrees of freedom: Spin currents

One of the two counter propagating edge currents has "spin-up" (in $+3$-direction, $\perp \Omega$), the other one has "spin down". Thus, a net chiral spin current, s^3_{edge}, can be excited to propagate along the edge; but there is no net electric edge current!

Response Equations, (2 oppositely (spin-)polarized bands):

$$\underline{j}(x) = 2\sigma(\underline{\nabla} B)^*, \qquad \text{and}$$

$$s_3^{\mu}(x) = \frac{\delta S(a, w)}{\delta w_{\mu}(x)} = 2\sigma \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x) \tag{18}$$

$\Rightarrow$ edge spin current – as in (7)!

We should ask what kinds of quasi-particles may produce the (bulk) Chern-Simons terms

$$S_{\pm}(a \pm w) = \pm \frac{\sigma}{2} \int \{(a \pm w) \wedge \, \mathrm{d}(a \pm w),$$

where, apparently $+$ stands for "spin-up" and $-$ stands for "spin-down". Well, it has been known ever since the seventies [1] that a two-component relativistic Dirac fermion with mass $M > 0$ ($M < 0$), coupled to an abelian gauge field A, breaks parity and time-reversal invariance and induces a Chern-Simons term

$$\underset{(-)}{+}\frac{1}{2\pi}\int A \wedge \mathrm{d}A$$

We thus argue that a 2D time-reversal invariant topological insulator with chiral edge spin-current exhibits two species of charged quasi-particles in the bulk, with one species (spin-up) related to the other one (spin-down) by time reversal, and each species has two degenerate states per wave vector mimicking a 2-component Dirac fermion (at small wave vectors).

[1] the first published account of this observation – originally due to Magnen, Sénéor and myself – appears in a paper by Deser, Jackiw and Templeton of 1982

3. 3D Topological Insulators, Axions

Encouraged by the findings of the last section, we propose to consider insulators in 3D with two filled bands "communicating" with each other, confined to a region

$$\Lambda := \Omega \times \mathbb{R}, \quad \partial\Lambda \neq \emptyset$$

of space-time. Again, we are interested in the general form of the effective action describing the response of such materials to turning on an external em field. Until the mid nineties:

$$S_\Lambda(A) = \frac{1}{2}\int_\Lambda \mathrm{d}t\ \mathrm{d}^3x\ \{\vec{E}\cdot\varepsilon\vec{E} - \vec{B}\cdot\mu^{-1}\vec{B}\}, \tag{19}$$

where ε is the tensor of dielectric constants of the material and μ is the magnetic permeability tensor. The action in (19) is dimensionless. Particle theorists taught us in the seventies that one could add another dimensionless term:

An effective action with a topological term

$$S_\Lambda(A) \to S_\Lambda^{(\theta)}(A) := S_\Lambda(A) + \theta\ I_\Lambda(A), \tag{20}$$

where I_Λ is a topological term ("instanton number") given by

$$\boxed{(2\pi^2)\, I_\Lambda(A) = \frac{1}{2}\int_\Lambda \mathrm{d}t\ \mathrm{d}^3x\ \vec{E}\cdot\vec{B} = \int_\Lambda F_A \wedge F_A \underset{Stokes}{=} \int_{\partial\Lambda} A \wedge \mathrm{d}A} \tag{21}$$

The partition function of an insulator is given by

$$\mathcal{Z}_\Lambda^{(\theta)}(A) = \exp\bigl(iS_\Lambda^{(\theta)}(A)\bigr),$$

with $S_\Lambda^{(\theta)}$ as in (20). In the thermodynamic limit, $\Omega \nearrow \mathbb{R}^3$, it is *periodic* in θ with period 2π and *invariant under time reversal* iff

$$\boxed{\theta = 0, \pi}$$

Surface degrees of freedom

Conventional insulator $\leftrightarrow$ $\theta = 0$. For $\theta = \pi$, $\mathcal{Z}_{\Lambda}^{(\theta)}(A)$ displays a boundary term given by

$$\exp\Big(\frac{i}{2\pi}\int_{\partial\Lambda} A \wedge \mathrm{d}A\Big), \tag{22}$$

see (21). This is the partition function of $(2+1)$D two-component charged, "relativistic" Dirac fermions on $\partial\Lambda$, coupled to external em field; ('76, '82, '84,...). Two species of charged quasi-particles ("spin-up" and "spin-down") with a conical Fermi surface that propagate along the surface of an insulator may appear in certain insulators with two bands communicating with each other. (Further possible surface degrees of freedom will be considered elsewhere.)

One may wonder whether it might make sense to view θ as the (ground-state) expectation value of a dynamical field, φ, and replace the term $\theta\ I_{\Lambda}(A)$ by

$$I_{\Lambda}(\varphi, A) := \frac{1}{2\pi^2}\int_{\Lambda} \varphi\ F_A \wedge F_A + S_0(\varphi), \tag{23}$$

Axionic topological insulators

where $S_0(\varphi)$ is invariant under shifts

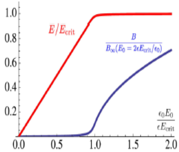
$$\varphi \mapsto \varphi + n\pi, \quad n \in \mathbb{Z} \tag{24}$$

with minima at $\varphi = n\pi, n \in \mathbb{Z}$. The field φ is a pseudo-scalar field, called "axion field". As a speculation, one may argue that axions may emerge in certain topological insulators with anti-ferromagn. short-range order and with two bands with conical Fermi surfaces and communicating with each other $\rightarrow$ anomalous axial vector current, j_5^μ! The time derivative of the axion then would have the interpretation of a chemical potential conjugate to axial-vector charge, ($\nearrow$ F-Pedrini, 2000(!), ... , Hehl et al., 2008, S.-C. Zhang et al., 2010):

$$\int \varphi \; F_A \wedge F_A = -\text{const.} \int \mathrm{d}\varphi \wedge j_5^* \quad (\textit{chiral anomaly}!) \tag{25}$$

Instabilities of axionic topological insulators

Assuming that (23) might be term in the effective action of a new kind of 3D topological insulator, one may want to study its properties:
By (24), the bulk of such a material will exhibit domain walls across which φ changes by π. Applying insight described after (22), we predict that domain walls carry gapless two-component charged Dirac fermions, which give rise to a non-vanishing conductivity, (i.e., to a break-down of insulating nature of the material, $\nearrow$ F-Werner, 2014).
Axion electrodynamics exhibits interesting instabilities: Time Reversal Invariance and Parity can be spontaneously broken inside bulk of axionic TI, (F-Pedrini, 2000). A related instability has been pointed out by Ooguri and Oshikawa (2012) on the basis of simple calculations:
Above a certain critical field strength, E_c, an external electric field applied to an axionic topological insulator is screened, with excess field converted into a magnetic field!



Courtesy Ooguri & Oshikawa

Conclusions

• Physics in 2D is surprisingly rich and has considerable potential for important technological applications. Interesting mathematical techniques – ranging from abstract algebra over the topology of fibre bundles all the way to hard analysis – find applications in the solution of problems of 2D Physics.

• 2D electron gases, Bose gases and magnetic materials are fascinating play grounds for experimentalists and theorists alike, not least because general principles of quantum theory, such as braid statistics, fractional spin & fractional electric charges, anomalies and their cancellation, current algebra, holography, two-comp. Dirac-like and Majorana fermions, etc., appear to manifest themselves in the physics of various 2D systems.

• It is interesting to consider higher-dimensional cousins of the QHE and of time-reversal invariant topological insulators. Some of them are likely to be relevant, e.g., in cosmology – in connection with the generation of primordial magnetic fields in the Universe, Dark Matter & Dark Energy. These matters are discussed on other occasions.

I thank you for your attention!

"Survivre et Vivre" – 47 years later

For those of you who understand some written French:

... depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *'Survivre'*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l'espèce humaine, et même de la vie tout court, menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d'armements. ...

Alexandre Grothendieck

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