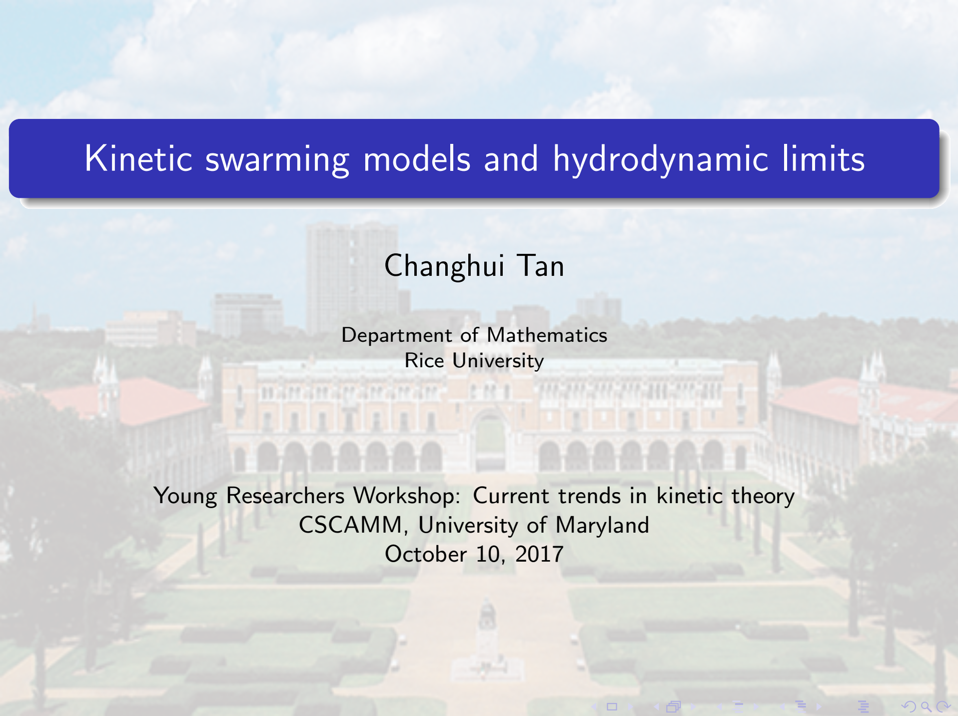

Kinetic swarming models and hydrodynamic limits

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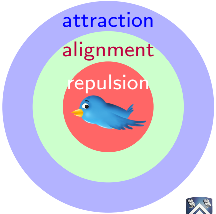
Swarming

Three-zone models for swarms: [Reynolds '87]

- Long range: Attraction
- Short range: Repulsion
- Middle range: Alignment



Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t)).$

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i).$

[**Cucker-Smale '07**, Motsch-Tadmor '11, Vicsek '95, ...]

Flocking: $|x_i(t) - x_j(t)| \leq D, \quad v_i(t) \overset{t\to\infty}{\longrightarrow} v_\infty.$

Unconditional flocking if $\int_1^\infty \phi(r)dr = \infty.$ [Ha-Liu '09]

Kinetic swarming models

- Mean-field limit: Vlasov-type kinetic equations

$$\boxed{\partial_t f + v \cdot \nabla_x f + \frac{1}{m}\nabla_v \cdot (F(f)f) = 0,}$$

where $f = f(t,x,v)$ is a probability measure in (x,v) space.

- Nonlocal interaction forces:

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v)f(t,y,v_*)dv_* dy$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y)f(t,y,v_*)dv_* dy.$$

Kinetic flocking models

$$\boxed{\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left(F^{CS}(f) f \right) = 0,}$$

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v) f(t,y,v_*) dv_* dy.$$

- Derivation, global wellposedness and flocking [Ha-Tadmor '08]
- Unconditional flocking: [Carrillo-Fornasier-Rosado-Toscani '10]

$$S(t) := \sup_{(x,v),(y,v^*) \in \text{supp} f(t)} |x-y| \le D < \infty,$$

$$V(t) := \sup_{(x,v),(y,v^*) \in \text{supp} f(t)} |v - v^*| \xrightarrow{t \to \infty} 0.$$

- Motsch-Tadmor alignment force [T. '17]
- Global wellposedness when ϕ is singular [Mucha-Peszek '17]

Numerical treatments on concentration of velocity

Flocking asymptotics: $\lim_{t\to\infty} f(t,x,v) = \rho_\infty(x)\delta_{v=\bar{v}}$.

Difficulty: solution becomes more and more singular as $t \to \infty$.

Numerical implementation:

1. Discontinuous Galerkin method. [T. '17]
   Efficient, stable, suitable for non-flocking asymptotics as well.
2. Velocity scaling method [Rey-T. '16]

$$f(t,x,v) = \omega(t,x)^n g(t,x,\xi) \quad \text{with } \xi = \omega(v-u).$$

u is the *macroscopic velocity*: $u(t,x) = \frac{\int vf(t,x,v)dv}{\int f(t,x,v)dv}$.
ω is the *scaling factor*.
g is the *rescaled profile*.

Main idea: choose ω wisely so that g is not singular.

Vlasov equation with attractive-repulsive potentials

$$\boxed{\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\left(F^{AR}(f)f\right) = 0,}$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*)dv_* dy.$$

- When K is the Newtonian potential, the system becomes Vlasov-Poission equations in plasma physics.
  Landau damping [Mouhot-Villani '11, Bedrossian-Masmoudi '15]
- For less singular potential, global wellposedness theory is standard.

Hydrodynamic limits

$$\boxed{\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = 0}$$

- Integrate f and vf in v, we obtain the macroscopic system.

$$\boxed{\begin{aligned}&\partial_t \rho + \nabla \cdot (\rho u) = 0,\\ &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P = \rho F.\end{aligned}}$$

where

$$\rho = \int f \ dv, \quad \rho u = \int vf \ dv, \quad P = \int (v-u) \otimes (v-u) f \ dv.$$

Euler-Alignment system

$$\boxed{\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho u)=0,\\&\partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla P=\int\phi(|x-y|)(u(y)-u(x))\rho(x)\rho(y)dy.\end{aligned}}$$

- Formal derivation [Ha-Tadmor '08]
- Rigorous derivation by imposing a closure on the pressure

$$P=\int(v-u)\otimes(v-u)f\ dv$$

1. Isothermal ansatz: $f(x,v)=\rho(x)\frac{1}{(2\pi)^{n/2}}e^{-\frac{|v-u(x)|^2}{2}}$.

$$\partial_t f+v\cdot\nabla_x f+\nabla_v\cdot(F(f)f)=\frac{1}{\epsilon}\left[\nabla_v\cdot((v-u)f)+\Delta_v f\right].$$

2. Mono-kinetic ansatz: $f(x,v)=\rho(x)\delta_{v=u(x)}$.

Euler-Alignment system

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   $P=\rho\mathbb{I}$. [Karper-Mellet-Trivisa '15]
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2. Mono-kinetic ansatz: $f(x,v) = \rho(x)\delta_{v=u(x)}$.

   $P = 0$. (Pressureless) [Figalli-Kang '17]

Pressureless Euler-Alignment system

$$\boxed{\begin{aligned}&\partial_t \rho + \nabla\cdot(\rho u) = 0,\\ &\partial_t u + u\cdot\nabla u = \int \phi(|x-y|)(u(y)-u(x))\rho(y)dy.\end{aligned}}$$

- If $\phi \equiv 0$, the system is known as *pressureless Euler equations*. Finite time formation of singular shocks.
- Alignment operator intends to regularize the system.

Question: Global regularity or finite time blowup?

#1 Nonlocal mean $\phi(r) = (1+r)^{-\alpha}, \alpha < 1.$

$$\boxed{\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0, \\ &\partial_t u + u \cdot \nabla u = \int \phi(|x-y|)(u(y) - u(x))\rho(y) dy. \end{aligned}}$$

- **Critical threshold phenomenon**: regularity depends on initial data.
- 1D: sharp critical threshold [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16]
  - If $\partial_x u_0 + \phi * \rho_0 \geq 0$ for all $x \in \mathbb{R}$, then the system is globally regular.
  - If there exists an $x \in \mathbb{R}$ such that $\partial_x u_0(x) + \phi * \rho_0(x) < 0$, then the system forms a singular shock in finite time.
- Extension to 2D and Motsch-Tadmor alignment operator [Tadmor-T. '14, He-Tadmor '17]
- Unconditional flocking when $\int_1^\infty \phi(r) dr = \infty$. [Tadmor-T. '14]

#2 Fractional dissipation $\phi(r) = c_\alpha r^{-(n+\alpha)}, \alpha > 0.$

Burgers equation with *density-dependent* fractional dissipation

$$\boxed{\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0, \\ &\partial_t u + u \cdot \nabla u = c_\alpha \int \frac{u(y) - u(x)}{|x-y|^{n+\alpha}} \rho(y) dy. \end{aligned}}$$

- Singular influence: enforcing strong alignment nearby.
- Relationship to Burgers equation with fractional dissipation

$$\boxed{\partial_t u + u \cdot \nabla u = -(-\Delta)^{\alpha/2} u = c_\alpha \int \frac{u(y) - u(x)}{|x-y|^{n+\alpha}} dy.}$$

1D fractional Burgers equation

$$\boxed{\partial_t u + u\partial_x u = -(-\Delta)^{\alpha/2} u.}$$

α		
$\alpha = 0$	Inviscid Burgers equation	Shock occurs in finite time for generic initial data
$\alpha < 1$	Supercritical regime	Shock occurs in finite time for some smooth initial data
$\alpha = 1$	Critical regime	Global strong solution exists for all smooth initial data
$\alpha > 1$	Subcritical regime	
$\alpha = 2$	Viscous Burgers equation	

c.f. [Kiselev-Nazarov-Shterenberg '08]

1D fractional Burgers equation

$$\boxed{\partial_t u + u\partial_x u = -(-\Delta)^{\alpha/2} u.}$$

α

$\alpha = 0$ Inviscid Burgers equation

Shock occurs in finite time
for generic initial data

$\alpha < 1$ Supercritical regime

Shock occurs in finite time
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$\alpha = 1$ Critical regime

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$\alpha = 2$ Viscous Burgers equation

2D critical quasi-geostrophic equation
[Kiselev-Nazarov-Volberg '07, Caffarelli-Vasseur '10,
Kiselev-Nazarov '10, Constantin-Vicol '12]

c.f. [Kiselev-Nazarov-Shterenberg '08]

The comparison

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t u + u\partial_x u = c_\alpha \int_{\mathbb{R}} \frac{u(y)-u(x)}{|x-y|^{1+\alpha}}dy.$$

$\alpha = 0$ Finite time blow up

$\alpha \in (0,1)$ Finite time blow up

$\alpha \in [1,2]$ Global wellposedness

$$\partial_t \rho + \partial_x(\rho u) = 0, \qquad \rho > 0$$
$$\partial_t u + u\partial_x u = c_\alpha \int_{\mathbb{R}} \frac{u(y)-u(x)}{|x-y|^{1+\alpha}}\rho(y)dy.$$

$\alpha = 0$ Finite time blow up

$\alpha \in (0,1)$ Global wellposedness

[Do-Kiselev-Ryzhik-T. '17]

$\alpha \in [1,2]$ Global wellposedness

[Shvydkoy-Tadmor '17]

- Blow up: singular shock $\partial_x u(x,t) \to -\infty$, $\rho(x,t) \to +\infty$.
- The growth of density enhances dissipation dynamically.

Extensions

$$\boxed{\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho u)=0,\\&\partial_t u+u\cdot\nabla u=\int\phi(|x-y|)(u(y)-u(x))\rho(y)dy.\end{aligned}}$$

- General singular kernels:

  Global regularity if $\int_0^1\phi(r)dr=\infty$. [Kiselev-T. '17]

- Existence of vacuum ($\rho_0\geq 0$, but $\rho_0 \not> 0$):

  Solution loses C^α regularity in finite time. [T. '17]

Euler-Poisson-Alignment system

$$\boxed{\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho u)=0,\\&\partial_t u+u\cdot\nabla u=k\nabla\Delta^{-1}\rho+\int\phi(|x-y|)(u(y)-u(x))\rho(y)dy.\end{aligned}}$$

	$k<0$ attractive	$k>0$ repulsive
$\phi=0$ Euler-Poisson	finite time blowup	critical threshold
		[Engelberg-Liu-Tadmor '01, $\cdots$]
ϕ bounded Lipshitz	finite time blowup	critical threshold
		[Carrillo-Choi-Tadmor-T. '16]
ϕ singular	global regularity	global regularity
		[Kiselev-T. '17]

Zero inertia limit

- Euler-Alignment system is the hydrodynamic limit of

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = \frac{1}{\epsilon}\left[\nabla_v \cdot ((v-u)f)\right].$$

- Zero inertia limit: total mass $m = \epsilon \to 0$. **No** extra terms involved.

$$\boxed{\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon)f_\epsilon) = 0,}$$

- Two systems that we concern:
  1. [ARR] Attraction-Repulsion-Relaxation: $F = F^{AR} - v$.
  2. [ARA] Attraction-Repulsion-Alignment(3 zones): $F = F^{AR} + F^{CS}$.

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v)f(t,y,v_*)dv_*dy$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y)f(t,y,v_*)dv_*dy.$$

Formal derivation

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\nabla_v \cdot (F(f)f) = 0$$

$\varphi(v) = 1$: $\quad \partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: $\quad$ [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: $\quad$ [ARR] $\int |v-u|^2 f(x,v)dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v)dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

Formal derivation

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Formal derivation

- A formal derivation of the $\epsilon \to 0$ limit $(f_\epsilon \to f)$:

$$\int \nabla_v \varphi(v) \cdot F(f) f \ dv = 0.$$

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$$\Rightarrow \quad f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

- ϕ is bounded Lipschitz [Fetecau-Sun-CT '16]
  - Wellposedness under additional momentum conservation assumption

$$\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$$

  - Rigorous passage to the limit

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
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$$\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$$

  - Rigorous passage to the limit

$$f_\epsilon \overset{*}{\rightharpoonup} f, \quad \text{in} \ \ \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n).$$

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t\rho + \nabla_x\cdot((-\nabla_x K * \rho)\rho) = 0.$$

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\begin{aligned}&\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\ &\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).\end{aligned}$$

- ϕ is bounded Lipschitz [Fetecau-Sun-CT '16]
  - Wellposedness under additional momentum conservation assumption
  - Rigorous passage to the limit
- ϕ is singular [Poyato-Soler '17] (See David's talk)

Numerical treatment: asymptotic-preserving schemes

Asymptotic-preserving schemes [Jin '99]:

$$\begin{tikzcd}
f_\epsilon^h \arrow[r, "h \to 0"] \arrow[d, "\epsilon \to 0"] & f_\epsilon \arrow[d, "\epsilon \to 0"] \\
f^h \arrow[r, "h \to 0"] & f
\end{tikzcd}$$

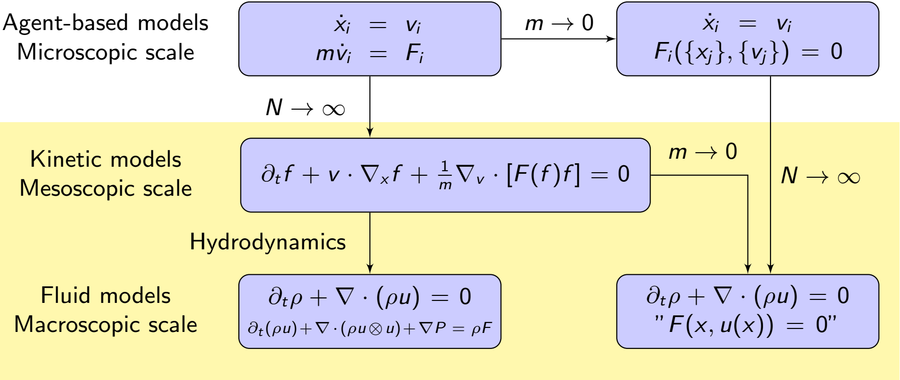
- Given $f_\epsilon \to f$, design a discretization f_ϵ^h for f_ϵ that converges to the discretization f^h for f.
- *Asymptotic-preserving property*: h does not depend on ϵ.
- Extremely powerful in solving kinetic systems with hydrodynamic limits.

When the limit is singular

$$\begin{tikzcd}
f^h_\epsilon \arrow[r, "h \to 0"] \arrow[dd, "\epsilon \to 0"] & f_\epsilon \arrow[dd, "\epsilon \to 0"] & & g^h_\epsilon \arrow[r, "h \to 0"] \arrow[dd, "\epsilon \to 0"] & g_\epsilon \arrow[dd, "\epsilon \to 0"] \\
& & {} \arrow[r, shift left, "\mathcal{T}_\epsilon"] & {} \arrow[l, shift left, "\mathcal{T}_\epsilon^{-1}"] & \\
f^h \arrow[r, "h \to 0"] & f & & g^h \arrow[r, "h \to 0"] & g
\end{tikzcd}$$

- In our case, f is singular: $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$.
  The discretization f^h can not be accurate. So f^h_ϵ is also not accurate when ϵ is small.
- **Idea:** Construct a family of invertible maps $\mathcal{T}_\epsilon$, so that $g_\epsilon = \mathcal{T}_\epsilon f_\epsilon$ converges to a non-singular profile g.
- **Main Difficulty:** Find $\mathcal{T}_\epsilon$ that correctly captures the singularity.
- Construction using velocity scaling method [Chertock-T.-Yan '17]

Summary



Agent-based models
Microscopic scale

Kinetic models
Mesoscopic scale

Fluid models
Macroscopic scale

$$\begin{tikzcd}
\begin{aligned} \dot{x}_i &= v_i \\ m\dot{v}_i &= F_i \end{aligned} \arrow[r, "m \to 0"] \arrow[d, "N \to \infty"'] & \begin{aligned} \dot{x}_i &= v_i \\ F_i(\{x_j\},\{v_j\}) &= 0 \end{aligned} \arrow[dd, "N \to \infty"] \\
\partial_t f + v \cdot \nabla_x f + \frac{1}{m}\nabla_v \cdot [F(f)f] = 0 \arrow[d, "\text{Hydrodynamics}"'] \arrow[dr, "m \to 0"] & \\
\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P &= \rho F \end{aligned} & \begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0 \\ "F(x, u(x)) &= 0" \end{aligned}
\end{tikzcd}$$

Thanks for your attention!