Bose-Einstein Condensation: Bound State of Periodic Microstructure

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Workshop on Quantum Systems: A Mathematical Journey...
CSCAMM
University of Maryland, College Park MD
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(Simplistic) Schematic of BEC concept in atomic gas

Non-interacting particles in a box ($T$: temperature) [Ketterle, ’99]:

High $T$:
“billiard balls”

Low $T$:
Evident wave-like behavior: “wave packets”

$T=T_c$: BEC onset
“Matter wave overlap”
$\Delta x \sim d$

$T=0$: BE condensate
“Giant matter wave”
Evolution of \(N\) Boson particles of repulsive interactions, \(N \gg 1\):

\[
H \Psi_N(t, \bar{x}) = i \partial_t \Psi_N(t, \bar{x}); \quad \Psi_N(t, \cdot) \in L^2_s(\mathbb{R}^{3N})
\]

\[
H = \sum_{j=1}^{N} \left[ -\nabla_j + V_e(x_j) \right] + \sum_{j<l}^{N} \mathcal{V}(x_j, x_l) : \text{Hamiltonian} \quad (\hbar = 2m = 1)
\]

**Usually:** \(\mathcal{V}(x_i, x_j) \approx 8\pi a \delta(x_i - x_j)\);

\(a\): scattering length; here \(a > 0\)

1. What macroscopic description, mean field limit, emerges?
2. What are plausible corrections to this limit, \(N \gg 1\)?

*Our focus: (2) formally; lowest bound state with microstructure*
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$N$ weakly interacting particles in periodic box ($N \gg 1$)

- Macroscopic 1-particle state: zero momentum ("condensate")
- Many-bound ground state: Atoms are primarily scattered from 0 momentum to pairs of opposite momenta ("pair excitation")

The condensate is partially depleted.
Review: Periodic case [Lee, Huang, Yang, 1957]

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Tensor product of 1-particle states (BEC signature)

Approximate $N$-body wave function for Boson gas (zeroth order):

$$\tilde{\Psi}_N(t, \vec{x}) \approx \tilde{\Psi}_N^0[\tilde{\Phi}](t, \vec{x}) = \prod_{j=1}^{N} \tilde{\Phi}(t, x_j) ; \quad \vec{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$$

- For constant sc. length and certain assumptions on interactions:
  $$i\partial_t \tilde{\Phi}(t, x) = [-\Delta + V_e(x) + 8\pi a|\tilde{\Phi}|^2] \tilde{\Phi}(t, x) \quad \text{(Gross-Pitaevskii Eq)}$$

- Lowest bound (ground) state:
  $$\tilde{\Psi}_N(t, \vec{x}) = e^{-iE_N t} \Psi_N(\vec{x}) ; \quad \tilde{\Phi}(t, x) = e^{-i\mu t} \Phi(x) \ (\Phi : \mathbb{R}^3 \to \mathbb{R})$$
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Review: Beyond GPE: Pair excitation [Wu, 1961]

Pair Excitation Hypothesis

(Uncontrolled) Ansatz:

\[ \tilde{\Psi}_N(t, \vec{x}) \propto e^{\mathcal{P}[\tilde{K}]} \tilde{\Psi}_N^0(\tilde{\Phi})(t, \vec{x}) \]

- \( \mathcal{P}[\tilde{K}] = \mathcal{P}_N \): operator that describes scattering of atoms \textit{in pairs};
  \( \tilde{K} = \tilde{K}(t, x, y) \) is \textbf{pair collision kernel} (“pair excitation function”)
- \( \tilde{K}(t, x, y) \) is not known a priori; obeys \textit{integro-PDE}.
- \( \mathcal{P} \) induces \textit{partial depletion} to condensate (\( \tilde{\Phi} \))
- \( \tilde{K}(t, x, y) = \tilde{K}(t, y, x) \) (without loss of generality)
- For bound states: \( \tilde{K}(t, x, y) = e^{-i2\mu t} K(x, y); \)
  \( K(x, y) = \mathcal{O}(1/|x - y|) \) as \( |x - y| \to 0 \).
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In this talk:

$H = \sum_{j=1}^{N} [-\Delta_j + V_e(x_j)] + \frac{1}{2} \sum_{i \neq j} V(x_i, x_j); \quad V_e > 0, \text{ smooth; } V_e(x) \to \infty \quad |x| \to \infty$

- Heuristically introduce spatially varying scattering length:

$$
V(x_i, x_j) = g^\epsilon(x_i) \delta(x_i - x_j); \quad g^\epsilon(x) := 8\pi a^\epsilon(x) = g_0[1 + A(x/\epsilon)] > 0
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- For lowest bound state: derive PDEs for $\Phi, K$.
- Apply: classical homogenization up to two orders in $\epsilon$;
- (singular) perturbations for slowly varying trap, $V_e(x) = U(\tilde{\epsilon} x)$.
- Describe depletion of $\Phi$. Will show:

$$(\text{Fraction at } \Phi) \xi \sim 1 - c \int_{\mathbb{R}_0} dx \left[ \mu_0 \left\| U(x) \right\|^{3/2} + \epsilon^2 f[U] \right] A^{-1}$$

$\mu_0$: lowest chem. pot./particle; $H^{-1}_{av} = \{ f \in H^{-1}(\mathbb{T}^3) | \langle f \rangle = 0 \}$
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Pair excitation *and* varying scattering length. Why?

- Experimental efforts to study quantum depletion in atomic gases [Cornell, Ensher, Wieman, 1999; Ketterle, Durfee, Stamper-Kurn, 1999; Xu et al., 2006].

- Modification of interactions in atomic gas, e.g., by controlling scattering length via external fields [Claussen et al., 2003; Cornish et al., 2000; Inouye et al., 1998; Stenger et al., 1998; Xu et al., 2006]

- Related theoretical work on bound states for *focusing* (attractive interactions) NLS by Fibich, Sivan and Weinstein [2006] via classical homogenization
Motivation: Condensate depletion

Quantum depletion of $^{23}\text{Na} \text{BECondensate}$ [Xu et al, 2006]

Depletion seems to be enhanced by manipulation of ext. potential
Results: I. Consistency of pair excitation hypothesis with many-body dynamics

Proposition 1 [DM, 2012] (Lowest bound state; varying sc. length)

The condensate wave function obeys:

\[ \mathcal{L}[\Phi] \Phi(x) := [-\nabla_x + V_e(x) + g(x)\Phi^2 - \mu] \Phi(x) = 0 ; \quad N^{-1}\|\Phi\|^2_{L^2(\mathbb{R}^3)} = 1 \]

The pair collision kernel \( K(x, y) \) satisfies

\[
\{ \mathcal{L}[\Phi](x) + \mathcal{L}[\Phi](y) + [g(x)\Phi(x)^2 + g(y)\Phi(y)^2] \} K(x, y) \\
+ \int_{\mathbb{R}^3} \text{d}z \ g(z)\Phi(z)^2 \ K(x, z) \ K(y, z) = -g(x)\Phi(x)^2 \delta(x - y)
\]
Addendum: Elements of bosonic Fock space, $\mathbb{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \left( L^2(\mathbb{R}^3) \right)^{\otimes_s n}$

- Elements of $\mathbb{F}$: $\nu = \{ \nu^{(n)} \}_{n \geq 0}$ where $\nu^{(0)} \in \mathbb{C}$, $\nu^{(n)} \in L^2_s(\mathbb{R}^3^n)$ are symm. in $x_1, \ldots, x_n$. Hilbert space structure: $\langle \nu, \chi \rangle_{\mathbb{F}} = \sum_{n \geq 0} \int_{\mathbb{R}^3^n} \nu^{(n)}(x) \chi^{(n)*}(x) \, dx$.

Creation (annihilation) operator $a_f^*$ ($a_f$): creates (destroys) particle at state $f$:

$$(a^*_f \nu)^{(n)}(\vec{x}_n) = n^{-1/2} \sum_{j=1}^{n} f(x_j) \nu^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

$$(a_f \nu)^{(n)}(\vec{x}_n) = \sqrt{n + 1} \int_{\mathbb{R}^3} dx_0 f^*(x_0) \nu^{(n+1)}(x_0, \vec{x}_n), \quad \vec{x}_n := (x_1, \ldots, x_n)$$

$\Rightarrow [a_f, a^*_f] = a_f a^*_f - a^*_f a_f = 1$

- Operator-valued distributions, $\psi^*(x)$ and $\psi(x), x \in \mathbb{R}^3$:

$$(a^*_f \psi)^{(n)} = \int \! dx f(x) \psi^*(x), \quad (a_f \psi)^{(n)} = \int \! dx f^*(x) \psi(x)$$

$\Rightarrow [\psi(x), \psi^*(y)] = \delta(x - y)1, \ [\psi^*(x), \psi^*(y)] = [\psi(x), \psi(y)] = 0$
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- Operator-valued distributions, \( \psi^* (x) \) and \( \psi (x) \), \( x \in \mathbb{R}^3 \):

\[
a_f^* =: \int \text{d}x f(x) \psi^* (x) , \quad a_f =: \int \text{d}x f^*(x) \psi (x)
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Sketch of (formal) proof of Proposition 1

- Hamiltonian, $\mathcal{H} : \mathbb{F} \to \mathbb{F}$:

$$\mathcal{H} = \int dx \psi^*(x)[-\Delta_x + V_e(x)]\psi(x) + \frac{1}{2} \int dx dy \psi^*(x)\psi^*(y) \underbrace{\mathcal{N}(x, y)}_{\mathcal{V}(x, y)} \psi(y)\psi(x)$$

- Perturbation scheme: Field operator **splitting**:

$$\psi(x) = N^{-1/2} \Phi(x)a_{\Phi} + \psi_1(x) ; (\Phi, \psi_1) = 0, N_1 = \int \psi_1^*(x)\psi_1(x) \, dx$$

- $N$-body Schrödinger eq. and **pair excitation ansatz** [Wu, 1961]:

$$\mathcal{H}\Psi_N = E_N\Psi_N ; \Psi_N \propto e^{\mathcal{P}[K]} \underbrace{\Psi_N^0[\Phi]}_{\text{tensor prod. of } \Phi} \in \mathbb{F}; \quad N = \langle \Psi_N, \int \psi^*(x)\psi(x)\Psi_N \rangle$$

- **Pair excitation operator**:

$$\mathcal{P}[K] = (2N)^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi_1^*(x)\psi_1^*(y) K(x, y) a_{\Phi}^2$$

- **Scheme**: keep up to terms **quadratic** in $\psi_1, \psi_1^*$ in $\mathcal{H}$. Enforce Schr. eq. □
Sketch of (formal) proof of Proposition 1

- Hamiltonian, $\mathcal{H} : \mathbb{F} \to \mathbb{F}$:

$$\mathcal{H} = \int dx \psi^\ast(x)[-\Delta_x + V_e(x)]\psi(x) + \frac{1}{2} \int dx dy \psi^\ast(x)\psi^\ast(y) \overbrace{\text{g}(x)\delta(x-y)}^{\overbrace{N(x,y)}^{\psi(y)\psi(x)}}$$

- Perturbation scheme: Field operator splitting:

$$\psi(x) = N^{-1/2} \underbrace{\Phi(x)a_\Phi}_{\text{condensate}} + \underbrace{\psi_1(x)}_{\langle \Psi_N, N_1 \Psi_N \rangle_{\mathbb{F}} \ll N} ; (\Phi, \psi_1) = 0, \quad N_1 = \int \psi_1^\ast(x)\psi_1(x) \, dx$$

- $N$-body Schrödinger eq. and pair excitation ansatz [Wu, 1961]:

$$\mathcal{H} \Psi_N = E_N \Psi_N ; \quad \Psi_N \propto e^{\mathcal{P}[K]} \underbrace{\Psi_N^0[\Phi]}_{\text{tensor prod. of } \Phi} \in \mathbb{F} ; \quad N = \langle \Psi_N, \int \psi^\ast(x)\psi(x)\Psi_N \rangle$$

- Pair excitation operator:

$$\mathcal{P}[K] = (2N)^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dx dy \underbrace{\psi_1^\ast(x)\psi_1^\ast(y)}_{\text{creates 2 part. states at } \Phi} \underbrace{K(x,y)}_{\text{annih. 2 part. at } \Phi} a_\Phi^2$$

- Scheme: keep up to terms quadratic in $\psi_1, \psi_1^\ast$ in $\mathcal{H}$. Enforce Schr. eq.
Sketch of (formal) proof of Proposition 1

- Hamiltonian, $\mathcal{H} : \mathcal{F} \rightarrow \mathcal{F}$:

$$\mathcal{H} = \int dx \psi^*(x)[-\Delta_x + V_e(x)]\psi(x) + \frac{1}{2} \int dx \, dy \, \psi^*(x)\psi^*(y) \, g(x)\delta(x-y) \, \mathcal{V}(x, y) \, \psi(y)\psi(x)$$

- Perturbation scheme: Field operator splitting:

$$\psi(x) = N^{-1/2} \Phi(x)a_\Phi + \psi_1(x) \quad ; (\Phi, \psi_1) = 0, \, N_1 = \int \psi_1^*(x)\psi_1(x) \, dx$$

- Condensate

- $N$-body Schrödinger eq. and pair excitation ansatz [Wu, 1961]:

$$\mathcal{H} \Psi_N = E_N \Psi_N ; \Psi_N \propto e^{\mathcal{P}[K]} \Psi^0_N[\Psi] \in \mathcal{F}; \, N = \langle \Psi_N, \int \psi^*(x)\psi(x)\Psi_N \rangle$$

- Pair excitation operator:

$$\mathcal{P}[K] = (2N)^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dx \, dy \, \psi_1^*(x)\psi_1^*(y) \, K(x, y) \, a_\Phi^2$$

- Creates 2 part. @ states \perp \Phi

- Scheme: keep up to terms quadratic in $\psi$, $\psi^*$ in $\mathcal{H}$, Enforce Schr. eq.
Sketch of (formal) proof of Proposition 1

- Hamiltonian, $\mathcal{H} : F \rightarrow F$:

$$\mathcal{H} = \int dx \psi^*(x) [-\Delta_x + V_e(x)] \psi(x) + \frac{1}{2} \int dx dy \psi^*(x) \psi^*(y) \left\{ \nabla(x, y) \psi(y) \psi(x) \right\}$$

- Perturbation scheme: Field operator splitting:

$$\psi(x) = N^{-1/2} \Phi(x) a_\Phi + \psi_1(x); (\Phi, \psi_1) = 0, N_1 = \int \psi_1^*(x) \psi_1(x) \, dx$$

- $N$-body Schrödinger eq. and pair excitation ansatz [Wu, 1961] :

$$\mathcal{H} \Psi_N = E_N \Psi_N; \Psi_N \propto e^{\mathcal{P}[K]} \Psi_0^N[\Phi] \in F; \quad N = \langle \Psi_N, \int \psi^*(x) \psi(x) \Psi_N \rangle$$

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- Scheme: keep up to terms quadratic in $\psi_1, \psi_1^*$ in $\mathcal{H}$, Enforce Schr. eq. \square
Results: II. Homogenization

Governing (elliptic) PDEs:

\[ \mathcal{L}_x[\Phi^\varepsilon]\Phi^\varepsilon := [-\triangle_x + V_\varepsilon(x) + g^\varepsilon(x) (\Phi^\varepsilon)^2 - \mu^\varepsilon] \Phi^\varepsilon(x) = 0, \quad N^{-1}\|\Phi^\varepsilon\|^2_{L^2} = 1; \]

\[ \{ \mathcal{L}_x[\Phi^\varepsilon] + \mathcal{L}_y[\Phi^\varepsilon] + [g^\varepsilon(x) \Phi^\varepsilon(x)^2 + g^\varepsilon(y) \Phi^\varepsilon(y)^2]\} K^\varepsilon(x, y) \]
\[ + \int \text{d}z \, g^\varepsilon(z) \Phi^\varepsilon(z)^2 K^\varepsilon(x, z) K^\varepsilon(y, z) = -g^\varepsilon(x) \Phi^\varepsilon(x)^2 \delta(x - y). \]

Periodic microstructure:

\[ g^\varepsilon(x) = g_0[1 + A(x/\varepsilon)]. \]

Seek (formally) two-scale expansions for \( \Phi^\varepsilon(x), K^\varepsilon(x, y) \):

\[ \Phi^\varepsilon(x) = \Phi_0(x, \tilde{x}) + \varepsilon \Phi_1(x, \tilde{x}) + \varepsilon^2 \Phi_2(x, \tilde{x}) + \ldots, \quad \tilde{x} = x/\varepsilon; \]

\[ K^\varepsilon(x, y) = K_0(x, y, \tilde{x}, \tilde{y}) + \varepsilon K_1(x, y, \tilde{x}, \tilde{y}) + \varepsilon^2 K_2(x, y, \tilde{x}, \tilde{y}) + \ldots \]

Accordingly, write \( \mu^\varepsilon = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \)
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\{ \mathcal{L}_x[\Phi^\epsilon] + \mathcal{L}_y[\Phi^\epsilon] + [g^\epsilon(x)\Phi^\epsilon(x)^2 + g^\epsilon(y)\Phi^\epsilon(y)^2] \} K^\epsilon(x, y) \\
+ \int dz \, g^\epsilon(z) \Phi^\epsilon(z)^2 K^\epsilon(x, z) K^\epsilon(y, z) = -g^\epsilon(x)\Phi^\epsilon(x)^2 \delta(x - y).
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Accordingly, write \(\mu^\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \ldots\)
Proposition 2.1 [DM, 2012] (Classical period. homogen. for $\Phi^\varepsilon$)

The coefficients of two-scale expansion for $\Phi^\varepsilon$ read

$$
\Phi_0(x, \tilde{x}) = f_0(x), \quad \Phi_1(x, \tilde{x}) = 0,
$$

$$
\Phi_2(x, \tilde{x}) = g_0 f_0(x)^3 [\triangle_{\tilde{x}}^{-1} A(\tilde{x})] + f_2(x);
$$

$$
\mathcal{L}_{0,x}[f_0]f_0 := [-\triangle_x + V_e(x) + g_0 f_0(x)^2 - \mu_0] f_0(x) = 0, \quad N^{-1} ||f_0||_{L^2}^2 = 1,
$$

$$
\mathcal{L}_{2,x} f_2 := [\mathcal{L}_{0,x}[f_0] + 2g_0 f_0(x)^2] f_2(x) = 3g_0^2 f_0^5 ||A||_{H_{av}^{-1}}^2 + \mu_2 f_0, \quad \langle f_0, f_2 \rangle = 0;
$$

$$
\mu_0 = \zeta_0 + \zeta_{\Delta_0} + \zeta_{e_0}, \quad \mu_1 = 0, \quad \mu_2 = -3g_0^2 ||A||_{H_{av}^{-1}}^2 \frac{\langle f_0, \mathcal{L}_{2}^{-1} f_0^5 \rangle}{\langle f_0, \mathcal{L}_{2}^{-1} f_0 \rangle};
$$

where $\zeta_0 = g_0 N^{-1} ||f_0^2||_{L^2}^2, \quad \zeta_{\Delta_0} = N^{-1} ||\nabla f_0||_{L^2}^2, \quad \zeta_{e_0} = N^{-1} \langle f_0, V_e f_0 \rangle$. 

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$$\mathcal{L}_{2,x} f_2 := [\mathcal{L}_{0,x}[f_0] + 2 g_0 f_0(x)^2] f_2(x) = 3 g_0^2 f_0^5 \|A\|_{H^{-1}_{av}}^2 + \mu_2 f_0, \quad \langle f_0, f_2 \rangle = 0;$$

$$\mu_0 = \zeta_0 + \zeta_{\Delta 0} + \zeta_{\varepsilon 0}, \quad \mu_1 = 0, \quad \mu_2 = -3 g_0^2 \|A\|_{H^{-1}_{av}}^2 \frac{\langle f_0, \mathcal{L}_{2}^{-1} f_0^5 \rangle}{\langle f_0, \mathcal{L}_{2}^{-1} f_0 \rangle};$$

where $\zeta_0 = g_0 N^{-1} \|f_0^2\|_{L^2}^2$, $\zeta_{\Delta 0} = N^{-1} \|\nabla f_0\|_{L^2}^2$, $\zeta_{\varepsilon 0} = N^{-1} \langle f_0, V_\varepsilon f_0 \rangle$. 
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$$

$$
L_{0,x}[f_0]f_0 := [-\triangle_x + V_e(x) + g_0 f_0(x)^2 - \mu_0]f_0(x) = 0, \quad N^{-1}||f_0||_{L^2}^2 = 1,
$$

$$
L_{2,x}f_2 := [L_{0,x}[f_0] + 2g_0 f_0(x)^2]f_2(x) = 3g_0^2 f_0^5 ||A||^2_{H^{-1}_{av}} + \mu_2 f_0, \quad \langle f_0, f_2 \rangle = 0;
$$

$$
\mu_0 = \zeta_0 + \zeta_{\Delta 0} + \zeta_{e 0}, \quad \mu_1 = 0, \quad \mu_2 = -3g_0^2 ||A||^2_{H^{-1}_{av}} \frac{\langle f_0, L_{2}^{-1} f_0^5 \rangle}{\langle f_0, L_{2}^{-1} f_0 \rangle};
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Proposition 2.2 [DM, 2012] (Classical periodic homogenization for $K^\varepsilon$)

Coefficients in two-scale expansion for $K^\varepsilon$:

$$K_0(x, y, \tilde{x}, \tilde{y}) = \kappa_0(x, y), \quad K_1(x, y, \tilde{x}, \tilde{y}) = 0,$$
$$K_2(x, y, \tilde{x}, \tilde{y}) = 2g_0[(\nabla_{\tilde{x}}^{-1} A(\tilde{x}))f_0(x)^2 + (\nabla_{\tilde{y}}^{-1} A(\tilde{y}))f_0(y)^2] \kappa_0 + \kappa_2(x, y);$$

\[\mathcal{L}_{(xy)} \kappa_0 := \{ \mathcal{L}_{0,x}[f_0] + \mathcal{L}_{0,y}[f_0] + g_0[f_0(x)^2 + f_0(y)^2] \} \kappa_0(x, y)\]
\[= -C[f_0^2; \kappa_0] \kappa_0(x, y) + B_0(x, y); \quad B_0(x, y) = -g_0f_0(x)^2 \delta(x - y);\]

\[\mathcal{L}_{(xy)} \kappa_2 = -2C[f_0^2; \kappa_0] \kappa_2(x, y) + B_2[f_0, f_2](x, y);\]

\[C[f; \kappa] \ell(x, y) := \frac{1}{2}g_0 \int dz f(z) [\kappa(x, z) \ell(y, z) + \ell(x, z) \kappa(y, z)],\]

\[B_2(x, y) = 2g_0 \left[ 3g_0 \|A\|_{H^{-1}_{av}}^2 f_0(x)^4 - f_0(x)f_2(x) \right] \delta(x - y) + \{ 2Z_2 \]
\[+ 9g_0^2 \|A\|_{H^{-1}_{av}}^2 [f_0(x)^4 + f_0(y)^4] - 4g_0[f_0(x)f_2(x) + f_0(y)f_2(y)] \} \kappa_0 \]
\[- 2C[f_0f_2, \kappa_0] \kappa_0 + 6g_0 \|A\|_{H^{-1}_{av}}^2 C[f_0^4, \kappa_0] \kappa_0,\]

\[Z_2 = N^{-1}g_0 \left[ 2\langle f_0^3, f_2 \rangle - 3g_0 \|A\|_{H^{-1}_{av}}^2 \|f_0^3\|_{L^2}^2 \right].\]
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$$\mathcal{L}_{(xy)}\kappa_0 \equiv \{\mathcal{L}_{0,x}[f_0] + \mathcal{L}_{0,y}[f_0] + g_0[f_0(x)^2 + f_0(y)^2]\} \kappa_0(x, y)$$

$$= -C[f_0^2; \kappa_0] \kappa_0(x, y) + B_0(x, y); \quad B_0(x, y) = -g_0f_0(x)^2\delta(x-y);$$

$$\mathcal{L}_{(xy)}\kappa_2 = -2C[f_0^2; \kappa_0] \kappa_2(x, y) + B_2[f_0, f_2](x, y);$$

$$C[f; \kappa]l(x, y) := \frac{1}{2}g_0 \int \mathrm{d}z f(z) [\kappa(x, z)l(y, z) + l(x, z)\kappa(y, z)];$$

$$B_2(x, y) = 2g_0[3g_0||A||_{H_{av}^{-1}}f_0(x)^4 - f_0(x)f_2(x)]\delta(x-y) + \{2Z_2$$

$$+ 9g_0^2||A||_{H_{av}^{-1}}^2[f_0(x)^4 + f_0(y)^4] - 4g_0[f_0(x)f_2(x) + f_0(y)f_2(y)]\} \kappa_0$$

$$- 2C[f_0f_2, \kappa_0]\kappa_0 + 6g_0 ||A||_{H_{av}^{-1}}^2 C[f_0^4, \kappa_0] \kappa_0,$$

$$Z_2 = N^{-1}g_0[2\langle f_0^3, f_2 \rangle - 3g_0 ||A||_{H_{av}^{-1}}^2 ||f_0^3||_{L^2}^2].$$
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= -\mathcal{C}[f_0^2; \kappa_0] \kappa_0(x, y) + B_0(x, y) ; \quad B_0(x, y) = -g_0f_0(x)^2 \delta(x - y) ;
\]

\[
\mathcal{L}_{(xy)} \kappa_2 = -2\mathcal{C}[f_0^2; \kappa_0] \kappa_2(x, y) + B_2[f_0,f_2](x, y) ;
\]

\[
\mathcal{C}[f; \kappa] \ell(x, y) := \frac{1}{2}g_0 \int dz f(z) \left[ \kappa(x, z) \ell(y, z) + \ell(x, z) \kappa(y, z) \right],
\]

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B_2(x, y) = 2g_0\left[3g_0\|A\|_{H_{av}}^{-1}f_0(x)^4 - f_0(x)f_2(x)\right] \delta(x - y) + \{2Z_2
+ 9g_0^2\|A\|^2_{H_{av}}[f_0(x)^4 + f_0(y)^4] - 4g_0[f_0(x)f_2(x) + f_0(y)f_2(y)]\} \kappa_0
- 2\mathcal{C}[f_0f_2, \kappa_0] \kappa_0 + 6g_0 \|A\|^2_{H_{av}} \mathcal{C}[f_0^4, \kappa_0] \kappa_0,
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\[
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= -C[f_0^2; \kappa_0] \kappa_0(x, y) + B_0(x, y) \quad B_0(x, y) = -g_0f_0(x)^2 \delta(x - y);
\]

\[
\mathcal{L}_{(xy)} \kappa_2 = -2C[f_0^2; \kappa_0] \kappa_2(x, y) + B_2[f_0,f_2](x, y);
\]

\[
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\]

\[
B_2(x, y) = 2g_0 \left[ 3g_0 \|A\|_{H_{av}^{-1}}^2 f_0(x)^4 - f_0(x)f_2(x) \right] \delta(x - y) + \left\{ 2Z_2 + 9g_0^2 \|A\|_{H_{av}^{-1}}^2 [f_0(x)^4 + f_0(y)^4] - 4g_0[f_0(x)f_2(x) + f_0(y)f_2(y)] \right\} \kappa_0 \\
- 2C[f_0 f_2, \kappa_0] \kappa_0 + 6g_0 \|A\|_{H_{av}^{-1}}^2 C[f_0^4, \kappa_0] \kappa_0,
\]

\[
Z_2 = N^{-1}g_0 \left[ 2\langle f_0^3, f_2 \rangle - 3g_0 \|A\|_{H_{av}^{-1}}^2 \|f_0^3\|_{L_2}^2 \right].
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\end{align*}

\begin{align*}
\mathcal{L}_{(xy)} \kappa_0 &:= \{\mathcal{L}_{0,x}[f_0] + \mathcal{L}_{0,y}[f_0] + g_0[f_0(x)^2 + f_0(y)^2]\} \kappa_0(x, y) \\
&= -C[f_0^2; \kappa_0] \kappa_0(x, y) + B_0(x, y) ; \quad B_0(x, y) = -g_0f_0(x)^2 \delta(x - y) ; \\
\mathcal{L}_{(xy)} \kappa_2 &= -2C[f_0^2; \kappa_0] \kappa_2(x, y) + B_2[f_0, f_2](x, y) ;
\end{align*}

\begin{align*}
C[f; \kappa] \ell(x, y) &:= \frac{1}{2}g_0 \int dz f(z) \left[\kappa(x, z) \ell(y, z) + \ell(x, z) \kappa(y, z)\right] , \\
B_2(x, y) &= 2g_0 \left[3g_0 \|A\|_{H_{av}^{-1}}^2 f_0(x)^4 - f_0(x)f_2(x)\right] \delta(x - y) + \{2Z_2 \\
&+ 9g_0^2 \|A\|_{H_{av}^{-1}}^2 [f_0(x)^4 + f_0(y)^4] - 4g_0[f_0(x)f_2(x) + f_0(y)f_2(y)]\} \kappa_0 \\
&- 2C[f_0f_2, \kappa_0] \kappa_0 + 6g_0 \|A\|_{H_{av}^{-1}}^2 C[f_0^4, \kappa_0] \kappa_0, \\
Z_2 &= N^{-1}g_0 \left[2\langle f_0^3, f_2 \rangle - 3g_0 \|A\|_{H_{av}^{-1}}^2 \|f_0^3\|_{L^2}^2\right] .
\end{align*}
Remarks on (formal) proof of Proposition 2

- Need "compatibility condition" on terms up to $\mathcal{O}(\varepsilon^4)$ (see Lemma 1 below) [Bensoussan, Lions, Papanicolaou, 1978].

- **Difficulty**: Nonlocal term in PDE for $K$ (see Lemmas 2, 3).

- Two-scale convergence is **not** addressed.
(Formal) Proof of Proposition 2: Useful lemmas

By substitution of expansions in PDEs, obtain cascade of equations:

$$ -\Delta \tilde{x} u = S(\tilde{x}) $$

Lemma 1 (Implication of Fredholm alternative)

The equation $$ -\Delta \tilde{x} u = S(\tilde{x}) $$, where $$ S(\tilde{x}) $$ is (1-)periodic, admits a (1-)periodic solution $$ u(\tilde{x}) $$ only if $$ \langle S \rangle = 0 $$ (compatibility condition). Then, $$ u(\tilde{x}) = ( -\Delta \tilde{x} )^{-1} S(\tilde{x}) + c. $$

In nonlocal term for $$ K $$, some averaging is needed:

Lemma 2 (Asymptotics for nonlocal term. Part I.)

If $$ P(\tilde{x}) $$ is 1-periodic with $$ P \in L^\infty(\mathbb{R}^d) $$ and $$ \langle P \rangle = 0 $$, and $$ h \in W^{m,1}(\mathbb{R}^d) $$ with vanishing derivatives at $$ \infty $$, then

$$ \int_{\mathbb{R}^d} P \left( \frac{x}{\epsilon} \right) h(x) \, dx = \mathcal{O}(\epsilon^m) ; \ m = 1, 2, \ldots \ (\epsilon \downarrow 0) $$
(Formal) Proof of Proposition 2: Useful lemmas

By substitution of expansions in PDEs, obtain cascade of equations:

$$-\Delta \tilde{x} u = S(\tilde{x})$$

**Lemma 1 (Implication of Fredholm alternative)**

The equation $$-\Delta \tilde{x} u = S(\tilde{x})$$, where $$S(\tilde{x})$$ is (1-)periodic, admits a (1-)periodic solution $$u(\tilde{x})$$ only if $$\langle S \rangle = 0$$ (compatibility condition). Then, $$u(\tilde{x}) = (-\Delta \tilde{x})^{-1} S(\tilde{x}) + c$$.

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$$\int_{\mathbb{R}^d} P\left(\frac{x}{\epsilon}\right) h(x) \, dx = O(\epsilon^m) ; \ m = 1, 2, \ldots \ (\epsilon \downarrow 0)$$
Lemma 3 (Refinement of Lemma 2 via Fourier Transform)

Consider the 1-periodic $P$ where $P \in L^2(\mathbb{T}^d)$ and $\langle P \rangle = 0$, and $h \in L^2(\mathbb{R}^d)$. Suppose $e^{i\lambda \cdot x_0} \hat{h}(\lambda) = c_1 \lambda^{-2s} + o(|\lambda|^{-2s})$ as $|\lambda| \to \infty$, $\lambda \in \mathbb{R}^d$, for some $s > d/4$, $x_0 \neq 0$. Then,

$$\int_{\mathbb{R}^d} P\left(\frac{x}{\epsilon}\right) h(x) \, dx = c_1 \epsilon^{2s} \left[(-\Delta)^{-s} P\right](x_0/\epsilon) + o(\epsilon^{2s}) \quad \text{as } \epsilon \downarrow 0.$$ 

In the above, $P(\tilde{x}) \equiv \partial_{\tilde{x}}^{-\alpha} A(\tilde{x})$. 
Lemma 3 (Refinement of Lemma 2 via Fourier Transform)

Consider the 1-periodic $P$ where $P \in L^2(\mathbb{T}^d)$ and $\langle P \rangle = 0$, and $h \in L^2(\mathbb{R}^d)$. Suppose $e^{i \lambda \cdot x_0} \hat{h}(\lambda) = c_1 \lambda^{-2s} + o(|\lambda|^{-2s})$ as $|\lambda| \to \infty$, $\lambda \in \mathbb{R}^d$, for some $s > d/4$, $x_0 \neq 0$. Then,

$$\int_{\mathbb{R}^d} P\left(\frac{x}{\epsilon}\right) h(x) \, dx = c_1 \epsilon^{2s} \left[(-\Delta)^{-s} P\right](x_0/\epsilon) + o(\epsilon^{2s}) \quad \text{as } \epsilon \downarrow 0.$$ 

In the above, $P(\tilde{x}) \equiv \partial_{\tilde{x}}^{-\alpha} A(\tilde{x})$. 

Slow-varying trap: Classical solution for $\Phi_n$

Assume $V_\epsilon(x) = U(\epsilon x)$, $\epsilon \ll \epsilon$.

Apply heuristics for $\Phi_n$ via boundary layer theory

I. Zeroth-order homog. soln., $f_0(x)$. $x \mapsto \bar{x} = \epsilon x$, $\phi_0(\bar{x}) := f_0(\bar{x}/\epsilon)$,

$$[-\epsilon^2 \nabla^2_x + U(x) + g_0 \phi^2_0 - \mu_0]\phi_0(x) = 0; \quad \int \phi_0(x)^2 \, dx = \epsilon^3 N = 1$$

- Outer solution (for $\epsilon = 0$), $\phi_0(x) \sim \phi^0_0(x)$:

$$\phi^0_0(x) = \begin{cases} g_0^{-1/2} \sqrt{\mu^0_0 - U(x)} & x \in \mathcal{R}^\delta_0 \\ 0 & x \in \mathcal{R}^c_0 \mathcal{R}_0 \delta \end{cases}$$

$$\mathcal{R}^\delta_0 := \mathcal{R}_0 \setminus \mathcal{B}(\partial \mathcal{R}_0, \delta), \mathcal{R}^c_0 = \mathbb{R}^3 \setminus \mathcal{R}_0; \quad \mathcal{R}_0 := \{ x \in \mathbb{R}^3 \mid U(x) < \mu^0_0 \}$$

$$\mu_0 \sim \mu^0_0 = |\mathcal{R}_0|^{-1} g_0 + \langle U \rangle_0; \quad \langle U \rangle_0 := |\mathcal{R}_0|^{-1} \int_{\mathcal{R}_0} U(x) \, dx$$
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Assume $V_e(x) = U(\tilde{\epsilon}x)$, $\tilde{\epsilon} \ll \epsilon$.
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$$[-\tilde{\epsilon}^2 \Delta^2_x + U(x) + g_0 \phi_0^2 - \mu_0] \phi_0(x) = 0; \quad \int \phi_0(x)^2 \, dx = \tilde{\epsilon}^3 N = 1$$

- Outer solution (for $\tilde{\epsilon} = 0$), $\phi_0(x) \sim \phi_0^0(x)$:

$$\phi_0^0(x) = \begin{cases} g_0^{-1/2} \sqrt{\mu_0^0 - U(x)} & x \in \mathcal{R}_0^\delta, \\ 0 & x \in \mathcal{R}_0^{c,\delta}. \end{cases}$$

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Apply heuristics for $\Phi_n$ via boundary layer theory

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Slowly-varying trap: Classical solutions for $\Phi_n$ (cont.)

$\phi_0^0$ is not $H^1_{loc}$ near $\partial \mathcal{K}_0$: Boundary layer

- Inner solution (near $\partial \mathcal{K}_0$), $\phi_0^{in}(\eta)$:

By $U(x) = U(x_{bd}) + \Upsilon \nu \cdot (x - x_{bd}) + o(|x - x_{bd}|)$, fixed $x_{bd} \in \partial \mathcal{K}_0$:

$$[-\partial^2_\eta + \eta + (\phi_0^{in})^2] \phi_0^{in} = 0; \quad \eta := \left(\frac{\Upsilon}{\varepsilon^2}\right)^{1/3} \nu \cdot (x - x_{bd}) , \quad \phi_0^{in} := \frac{g_0^{1/2}}{(\varepsilon \Upsilon)^{1/3}} \phi_0$$

Apply matching $\phi_0^{in} \to 0$ as $\eta \to \infty$; $\phi_0^{in} \sim \sqrt{-\eta}$ as $\eta \to -\infty$

$\Rightarrow \phi_0^{in}(\eta) = P_{II}(\eta)$: case of 2nd Painlevé transcendent [DM, '00]
II. Next-order homogenized soln.,
\( \Phi_2(x, \tilde{x}) = g_0 f_0(x)^3 [\triangle_{\tilde{x}}^{-1} A(\tilde{x})] + f_2(x); \phi_2(x) := f_2(x/\varepsilon) \)

- **Outer solution, \( \phi_2^0(x) \):**

\[
\phi_2^0(x) = g_0^{-1/2} \left\{ \frac{3}{2} [\mu_0^0 - U(x)]^{3/2} ||A||^2_{H_{av}^{-1}} + \frac{1}{2} \mu_2^0 [\mu_0^0 - U(x)]^{-1/2} \right\}
\]

if \( x \in \mathcal{R}_0^\delta; \phi_2^0(x) = 0 \) if \( x \in \mathcal{R}_0^{c, \delta} \).

\[
\mu_2 \sim \mu_2^0 = -3 ||A||^2_{H_{av}^{-1}} |\mathcal{R}_0|^{-1} \int_{\mathcal{R}_0} [\mu_0^0 - U(x)]^2 \, dx
\]
II. Next-order homogenized soln.,
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\[ \phi_2^0(x) = g_0^{-1/2} \left\{ \frac{3}{2} [\mu_0^0 - U(x)]^{3/2} ||A||_{H^{-1}_{av}}^2 + \frac{1}{2} \mu_2^0 [\mu_0^0 - U(x)]^{-1/2} \right\} \]

if \( x \in \mathcal{R}_0^\delta \); \( \phi_2^0(x) = 0 \) if \( x \in \mathcal{R}_0^c, \delta \).

\[ \mu_2 \sim \mu_2^0 = -3 ||A||_{H^{-1}_{av}}^2 |\mathcal{R}_0|^{-1} \int_{\mathcal{R}_0} [\mu_0^0 - U(x)]^2 \, dx \]
Boundary layer near \( \partial \mathcal{R}_0 \)

- **Inner solution (near \( \partial \mathcal{R}_0 \)), \( \phi_2^{in}(\eta) \)**

\[
\left[ \partial^2_{\eta} - \eta - 3P_{II}(\eta)^2 \right] \phi_2^{in}(\eta) = P_{II}(\eta) ; \quad \phi_2^{in} := -\left( \mu_2^0 \right)^{-1} g_0^{1/2} (U_o \varepsilon)^{1/3} \phi_2 ,
\]

where by **matching** with outer solution:

\( \phi_2^{in}(\eta) \to 0 \) as \( \eta \to \infty \), \( \phi_2^{in}(\eta) \sim -\frac{1}{2} (-\eta)^{-1/2} \) as \( \eta \to -\infty \).

\[
\Rightarrow \phi_2^{in}(\eta) = P_{II}'(\eta)
\]
Slowly varying trap: Classical solutions for $\Phi_n$ (cont.)

Boundary layer near $\partial R_0$

- **Inner solution (near $\partial R_0$), $\phi_2^{in}(\eta)$**

\[
[\partial_\eta^2 - \eta - 3P_{II}(\eta)^2] \phi_2^{in}(\eta) = P_{II}(\eta); \quad \phi_2^{in} := -(\mu_2^0)^{-1} g_0^{1/2} (U_0 \bar{\epsilon})^{1/3} \phi_2,
\]

where by **matching** with outer solution:

$\phi_2^{in}(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$, \quad $\phi_2^{in}(\eta) \sim -\frac{1}{2} (-\eta)^{-1/2}$ as $\eta \rightarrow -\infty$.

$\Rightarrow \phi_2^{in}(\eta) = P'_{II}(\eta)$
Slowly-varying trap: Classical solutions for $K_n$

Because of $\delta(x - y)$-forcing, $K$ depends on $x - y$ if $V_e \approx \text{const.}$

Transform to center of mass: $(x, y) \mapsto (x\#, X) = (x - y, \frac{x+y}{2})$

Apply FT in $x\#$; boundary-layer theory in $X$.

**I. Zeroth-order homg. kernel, $\kappa_0(x, y)$**: Let $X \mapsto \tilde{X} = \tilde{\epsilon}X$: slow; $x\# = \mathcal{O}(1)$; define $\tilde{\kappa}_0(x\#, \tilde{X}) := \kappa_0(\tilde{X}/\tilde{\epsilon} + x\#/2, \tilde{X}/\tilde{\epsilon} - x\#/2)$.

Apply FT in $x\#$; dual variable is $\lambda \in \mathbb{R}^3$.

**Outer solution, $\tilde{\kappa}_0^0(x\#, X)$**.

$$
\tilde{\kappa}_0^0(\lambda, X) = \frac{-\lambda^2 - \lambda_0(X)^2 + \sqrt{[\lambda^2 + \lambda_0(X)^2]^2 - g_0^2\phi_0^0(X)^4}}{g_0\phi_0^0(X)^2};
$$

$$
\lambda_0(X)^2 := U(X) + 2g_0\phi_0^0(X)^2 - \mu_0^0; X \in \mathbb{R}^3 \setminus \mathbb{B}(\partial\mathcal{K}_0, \delta),
$$

$\delta = \mathcal{O}(\tilde{\epsilon}^{2/3})$.

**Inversion**: Lommel’s fcns.

**Inner solution, $X \in \mathbb{B}(\partial\mathcal{K}_0, \delta)$. Obtain ODE near $\partial\mathcal{K}_0$; $\lambda$ is parameter [DM, 2012]
Slowly-varying trap: Classical solutions for $K_n$

Because of $\delta(x - y)$-forcing, $K$ depends on $x - y$ if $V_e \approx \text{const.}$
Transform to center of mass: $(x, y) \mapsto (x\#, X) = (x - y, \frac{x+y}{2})$
Apply FT in $x\#$; boundary-layer theory in $X$.

I. Zeroth-order homg. kernel, $\kappa_0(x, y)$: Let $X \mapsto \tilde{X} = \varepsilon X$: slow;
$x\# = \mathcal{O}(1)$; define $\mathcal{K}_0(x\#, \tilde{X}) := \kappa_0(\tilde{X}/\varepsilon + x\#/2, \tilde{X}/\varepsilon - x\#/2)$.
Apply FT in $x\#$; dual variable is $\lambda \in \mathbb{R}^3$.

Outer solution, $\mathcal{K}_0^0(x\#, X)$.

$$\mathcal{K}_0^0(\lambda, X) = \frac{-\lambda^2 - \lambda_0(X)^2 + \sqrt{[\lambda^2 + \lambda_0(X)^2]^2 - g_0^2\phi_0^0(X)^4}}{g_0\phi_0^0(X)^2};$$

$$\lambda_0(X)^2 := U(X) + 2g_0\phi_0^0(X)^2 - \mu_0^0; \quad X \in \mathbb{R}^3 \setminus \mathcal{B}(\partial\mathcal{R}_0, \delta),$$
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Inner solution, $X \in \mathcal{B}(\partial\mathcal{R}_0, \delta)$. Obtain ODE near $\partial\mathcal{R}_0$; $\lambda$ is parameter [DM, 2012]
Because of $\delta(x - y)$-forcing, $K$ depends on $x - y$ if $V_e \approx \text{const}$. Transform to center of mass: $(x, y) \mapsto (x^#, X) = (x - y, \frac{x+y}2)$.

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**Outer solution, $\tilde{\kappa}_0^0(x^#, X)$**.

$$
\tilde{\kappa}_0^0(\lambda, X) = \frac{-\lambda^2 - \lambda_0(X)^2 + \sqrt{[\lambda^2 + \lambda_0(X)^2]^2 - g_0^2 \phi_0^0(X)^4}}{g_0 \phi_0^0(X)^2};
$$

$
\lambda_0(X)^2 := U(X) + 2g_0 \phi_0^0(X)^2 - \mu_0^0; \quad X \in \mathbb{R}^3 \setminus \mathcal{B}(\partial \mathcal{R}_0, \delta),$

$\delta = \mathcal{O}(\epsilon^{2/3})$.

**Inversion:** Lommel’s fcns. 

**Inner solution, $X \in \mathcal{B}(\partial \mathcal{R}_0, \delta)$.** Obtain ODE near $\partial \mathcal{R}_0$; $\lambda$ is parameter [DM, 2012]
Application: Partial depletion of $\Phi$

Fraction of particles **out** of $\Phi$ (**depletion fraction**) [Wu, 1961; DM, 2011]:

$$\xi^e_{sc} = \langle \Psi^e_N, (\psi_1^*/\psi_1/N) \Psi^e_N \rangle_F = N^{-1} \text{tr} \mathcal{W}^e ; \quad 0 < \xi^e_{sc} \ll 1;$$

$$\mathcal{W}^e := \mathcal{W}^e_1 (1 - \mathcal{W}^e_1)^{-1}, \quad \mathcal{W}^e_1 := \mathcal{K}^e \ast \mathcal{K}^e,$$ and $\mathcal{K}^e$ has repr. $K^e(x, y)$.

Proposition 3 (Depletion fraction under slowly varying trap) [DM, 2012]

If $g(x) = g_0[1 + A(x/\epsilon)]$ and $V_\epsilon(x) = U(\tilde{\epsilon}x)$, the depletion fraction is

$$\xi_{sc} \sim \frac{\sqrt{2}}{12\pi^2} \int_{\mathcal{R}_0} dx \left[ \mu_0^0 - U(x) \right]^{3/2}$$

$$- \epsilon^2 \frac{3\sqrt{2}}{8\pi^2} \|A\|^2 H_{av}^{-1} \int_{\mathcal{R}_0} \left\{ g_0^2 \phi_0^0(x)^4 \right\}$$

$$+ |\mathcal{R}_0|^{-1} \|g_0(\phi_0^0)^2\|^2_{L^2} \left[ g_0\phi_0^0(x)^2 \right]^{1/2} dx \quad \text{as } \epsilon, \tilde{\epsilon} \downarrow 0, \tilde{\epsilon}/\epsilon \downarrow 0.$$
Application: Partial depletion of $\Phi$

Fraction of particles out of $\Phi$ (depletion fraction) [Wu, 1961; DM, 2011]:

$$\xi_{sc}^\varepsilon = \langle \Psi_N^\varepsilon, (\psi_1^* \psi_1/N) \Psi_N^\varepsilon \rangle_F = N^{-1} \text{tr} \mathcal{W}^\varepsilon; \quad 0 < \xi_{sc}^\varepsilon \ll 1;$$

$$\mathcal{W}^\varepsilon := \mathcal{W}_1^\varepsilon (1 - \mathcal{W}_1^\varepsilon)^{-1}, \quad \mathcal{W}_1^\varepsilon := \mathcal{K}_1^\varepsilon \mathcal{K}_1^\varepsilon, \quad \text{and} \quad \mathcal{K}_1^\varepsilon \text{ has repr. } K^\varepsilon(x,y).$$

**Proposition 3 (Depletion fraction under slowly varying trap) [DM, 2012]**

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$$- \varepsilon^2 \frac{3\sqrt{2}}{8\pi^2} \|A\|_{H^{-1}_\text{av}}^2 \int_{\mathcal{R}_0} \{g_0^2 \phi_0^0(x)^4$$

$$+ |\mathcal{R}_0|^{-1} \|g_0(\phi_0^0)^2\|_{L^2}^2 \} \left[ g_0^2 \phi_0^0(x)^2 \right]^{1/2} dx \quad \text{as } \varepsilon, \varepsilon \downarrow 0, \varepsilon/\varepsilon \downarrow 0.$$
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Fraction of particles $\textbf{out}$ of $\Phi$ (depletion fraction) [Wu, 1961; DM, 2011]:

$$\xi^e_{sc} = \langle \Psi^e_N, (\psi_1^* \psi_1 / N) \Psi^e_N \rangle_F = N^{-1} \text{tr} \mathcal{W}^e ; \quad 0 < \xi^e_{sc} \ll 1;$$

$$\mathcal{W}^e := \mathcal{W}_1^e (1 - \mathcal{W}_1^e)^{-1}, \mathcal{W}_1^e := \mathcal{K}^e \cdot \mathcal{K}^e , \text{ and } \mathcal{K}^e \text{ has repr. } K^e(x, y).$$

**Proposition 3 (Depletion fraction under slowly varying trap) [DM, 2012]**

If $g(x) = g_0 [1 + A(x / \epsilon)]$ and $V(x) = U(\tilde{c} x)$, the depletion fraction is

$$\xi_{sc} \sim \frac{\sqrt{2}}{12 \pi^2} \int_{\mathcal{R}_0} dx [\mu_0^0 - U(x)]^{3/2}$$

$$- \epsilon^2 3\sqrt{2} \frac{||A||^2_{Hav}}{8 \pi^2} \int_{\mathcal{R}_0} \{ g_0^2 \phi_0^0(x)^4 \}$$

$$+ |\mathcal{R}_0|^{-1} || g_0 (\phi_0^0)^2 ||_{L^2}^2 \} [g_0 \phi_0^0(x)^2]^{1/2} dx \quad \text{as } \epsilon, \tilde{c} \downarrow 0, \tilde{c}/\epsilon \downarrow 0.$$
Application: Partial depletion of $\Phi$

Fraction of particles out of $\Phi$ (depletion fraction) [Wu, 1961; DM, 2011]:

$$\xi_{sc} = \langle \Psi_N^\epsilon, (\psi_1/N) \Psi_N^\epsilon \rangle_F = N^{-1} \text{tr} \mathcal{W}^\epsilon; \quad 0 < \xi_{sc} \ll 1;$$

$$\mathcal{W}^\epsilon := \mathcal{W}_1^\epsilon (1 - \mathcal{W}_1^\epsilon)^{-1}, \quad \mathcal{W}_1^\epsilon := \mathcal{K}^\epsilon * \mathcal{K}^\epsilon, \text{ and } \mathcal{K}^\epsilon \text{ has repr. } K^\epsilon(x, y).$$

Proposition 3 (Depletion fraction under slowly varying trap) [DM, 2012]

If $g(x) = g_0[1 + A(x/\epsilon)]$ and $V_e(x) = U(\bar{\epsilon} x)$, the depletion fraction is

$$\xi_{sc} \sim \frac{\sqrt{2}}{12\pi^2} \int_{\mathcal{R}_0} dx \left[ \mu_0^0 - U(x) \right]^{3/2}$$

$$- \epsilon^2 \frac{3\sqrt{2}}{8\pi^2} \|A\|^2_{Hav^{-1}} \int_{\mathcal{R}_0} \left\{ g_0^2 \phi_0^0(x)^4 \right\}$$

$$+ |\mathcal{R}_0|^{-1} \| g_0(\phi_0^0)^2 \|_{L^2}^2 \right\} \left[ g_0 \phi_0^0(x)^2 \right]^{1/2} dx \quad \text{as } \epsilon, \bar{\epsilon} \downarrow 0, \frac{\bar{\epsilon}}{\epsilon} \downarrow 0.$$
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Remarks on formula for depletion fraction, $\xi_{sc}$

- Interplay of external potential and spatial variation of scattering length.
- Depletion fraction, $\xi_{sc}$, can be enhanced via external potential.
- For fixed $\epsilon$: Spatial (periodic) variation of scattering length causes reduction of the $\xi_{sc}$ solely due to pair excitation.
- Decreasing oscillations of scattering length (i.e., increasing $\epsilon$) can cause decrease of $\xi_{sc}$. 
Rigorous analysis/justification for many-body wave function of pair excitation?
On the basis of recent work [Grillakis, Machedon, DM, 2010] for $V_e = 0$, one may expect (with a trap):
$$\|\Psi_{N,\text{ex}} - \Psi_{N,\text{pair}}\|_{L^2(\mathbb{R}^{3N})} \leq C(t)N^{-1/2},$$
$C(t)$: bounded locally in time.

In our program, subscale $\epsilon$ of scattering length is assumed. What may be the physical origin of such $\epsilon$?
Derivation of spatial variation of $a$, as an emergent concept when $N \to \infty$?

Within our approximation scheme, pair excitation does not act back on NLS for $\Phi$.
Modified equation of motion for $\Phi$ via pair excitation?
### Epilogue: Pending issues

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What is the appropriate macroscopic description for finite but “small” temperatures (below the phase transition point)?

**Complication:** Particles are distributed over thermally excited states. In addition to $\Phi$ and $K$, one must use $\{\phi_j\}_{j=1}^{\infty}$, 1-particle excitation wave functions.

Coupled PDEs for $\Phi(x)$, $\phi_j(x)$ ($j = 1, 2, \ldots$):

\[
\mu \Phi(x) = [-\nabla^2 + V_e(x) + \nu g(x)\Phi(x)^2 + 2g(x)\varrho_n(x)]\Phi(x),
\]

\[
\mu_j \phi_j(x) = [-\nabla^2 + V_e(x) + 2\nu g(x)\Phi(x)^2 + 2g(x)\varrho_n(x)]\phi_j(x) - \Phi(x)N^{-1} \int dy \Phi(y)\nu g(y)\Phi(y)^2 \phi_j(y);
\]

where $\varrho_n(x) = N^{-1} \sum_j |\phi_j(x)|^2 n_j^0$, and $\nu$: fraction at condensate
Epilogue: Pending issues (cont.)

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