Sports Leagues and Competitive Societies

Eli Ben-Naim
Los Alamos National Laboratory

Randomness in Competitions
EB, N. Hengartner, S. Redner, and F. Vazquez

Modeling and Control in Social Dynamics, Camden NJ, October 8, 2014

Talk, papers available from: http://cnls.lanl.gov/~ebn
Thanks

• Sidney Redner and Federico Vazquez
  Los Alamos & Boston University
• Jin Sup Kim and Byugnam Kahng
  Los Alamos & Seoul National University
• Nicholas Hengartner
  Los Alamos National Laboratory
• Micha Ben-Naim
  Massachusetts Institute of Technology
What is the most competitive sport?

- Soccer
- Baseball
- Hockey
- Basketball
- Football
What is the most competitive sport?

- Soccer
- Baseball
- Hockey
- Basketball
- Football

Can competitiveness be quantified?
How can competitiveness be quantified?
I. Modeling competitions
Parity of a sports league

- Teams ranked by win-loss record
- Win percentage
  \[ x = \frac{\text{Number of wins}}{\text{Number of games}} \]
- Standard deviation in win-percentage
  \[ \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \]
- Cumulative distribution = Fraction of teams with winning percentage < x
  \[ F(x) \]

Major League Baseball
American League
2014 Season-end Standings

<table>
<thead>
<tr>
<th>East</th>
<th>W</th>
<th>L</th>
<th>PCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-Baltimore</td>
<td>96</td>
<td>66</td>
<td>.593</td>
</tr>
<tr>
<td>NY Yankees</td>
<td>84</td>
<td>78</td>
<td>.519</td>
</tr>
<tr>
<td>Toronto</td>
<td>83</td>
<td>79</td>
<td>.512</td>
</tr>
<tr>
<td>Tampa Bay</td>
<td>77</td>
<td>85</td>
<td>.475</td>
</tr>
<tr>
<td>Boston</td>
<td>71</td>
<td>91</td>
<td>.438</td>
</tr>
</tbody>
</table>

In baseball
\[ 0.400 < x < 0.600 \]
\[ \sigma = 0.08 \]
Data

- 300,000 Regular season games (all games ever played)
- 5 Major sports leagues in United States & England

<table>
<thead>
<tr>
<th>sport</th>
<th>league</th>
<th>full name</th>
<th>country</th>
<th>years</th>
<th>games</th>
</tr>
</thead>
<tbody>
<tr>
<td>soccer</td>
<td>FA</td>
<td>Football Association</td>
<td>🇬🇧</td>
<td>1888-2005</td>
<td>43,350</td>
</tr>
<tr>
<td>baseball</td>
<td>MLB</td>
<td>Major League Baseball</td>
<td>🇺🇸</td>
<td>1901-2005</td>
<td>163,720</td>
</tr>
<tr>
<td>hockey</td>
<td>NHL</td>
<td>National Hockey League</td>
<td>🇨🇦</td>
<td>1917-2005</td>
<td>39,563</td>
</tr>
<tr>
<td>basketball</td>
<td>NBA</td>
<td>National Basketball Association</td>
<td>🇺🇸</td>
<td>1946-2005</td>
<td>43,254</td>
</tr>
<tr>
<td>football</td>
<td>NFL</td>
<td>National Football League</td>
<td>🇺🇸</td>
<td>1922-2004</td>
<td>11,770</td>
</tr>
</tbody>
</table>

Standard deviation in winning percentage

Distribution of winning percentage clearly distinguishes sports

- Baseball most competitive?
- Football least competitive?

Fort and Quirk, 1995
The competition model

• Two, randomly selected, teams play

• Outcome of game depends on team record
  - Weaker team wins with probability $q < 1/2$
  - Stronger team wins with probability $p > 1/2$
  - When two equal teams play, winner picked randomly

\[
\begin{align*}
(i, j) &\rightarrow \begin{cases} 
(i + 1, j) & \text{probability } p \\
(i, j + 1) & \text{probability } 1 - p
\end{cases} \\
\end{align*}
\]

\[p + q = 1\]

• Initially, all teams are equal (0 wins, 0 losses)

• Teams play once per unit time $\langle x \rangle = \frac{1}{2}$
Rate equation approach

• **Probability distribution functions**
  
  \[ g_k = \text{fraction of teams with } k \text{ wins} \]
  \[ G_k = \sum_{j=0}^{k-1} g_j = \text{fraction of teams with less than } k \text{ wins} \]
  \[ H_k = 1 - G_{k+1} = \sum_{j=k+1}^{\infty} g_j \]

• **Evolution of the probability distribution**

  \[
  \frac{d g_k}{d t} = (1 - q)(g_{k-1}G_{k-1} - g_kG_k) + q(g_{k-1}H_{k-1} - g_kH_k) + \frac{1}{2} (g_{k-1}^2 - g_k^2)
  \]

  better team wins    worse team wins    equal teams play

• **Closed equations for the cumulative distribution**

  \[
  \frac{dG_k}{dt} = q(G_{k-1} - G_k) + (1/2 - q) (G_{k-1}^2 - G_k^2)
  \]

  Boundary Conditions  \[ G_0 = 0 \]  \[ G_\infty = 1 \]  Initial Conditions  \[ G_k(t = 0) = 1 \]

Nonlinear Difference-Differential Equations
An exact solution

- Stronger always wins (q=0)
  \[
  \frac{dG_k}{dt} = G_k (G_k - G_{k-1})
  \]

- Transformation into a ratio
  \[
  G_k = \frac{P_k}{P_{k+1}}
  \]

- Nonlinear equations reduce to linear recursion
  \[
  \frac{dP_k}{dt} = P_{k-1}
  \]

- Integrable (discrete) Burgers equation!
  \[
  G_k = \frac{1 + t + \frac{1}{2!} t^2 + \cdots + \frac{1}{k!} t^k}{1 + t + \frac{1}{2!} t^2 + \cdots + \frac{1}{(k+1)!} t^{k+1}}
  \]

EB, Krapivsky
J Phys A 2012
Long-time asymptotics

- **Long-time limit**
  \[ G_k \rightarrow \frac{k + 1}{t} \]

- **Scaling form**
  \[ G_k \rightarrow F \left( \frac{k}{t} \right) \]

- **Scaling function**
  \[ F(x) = x \]

Seek similarity solutions
Use winning percentage as scaling variable
Scaling analysis

• Rate equation
\[
\frac{dG_k}{dt} = q(G_{k-1} - G_k) + \left(\frac{1}{2} - q\right)\left(G_{k-1}^2 - G_k^2\right)
\]

• Treat number of wins as continuous
\[G_{k+1} - G_k \to \frac{\partial G}{\partial k}\]

Inviscid Burgers equation
\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0
\]

\[
\frac{\partial G}{\partial t} + [q + (1 - 2q)G] \frac{\partial G}{\partial k} = 0
\]

• Stationary distribution of winning percentage
\[G_k(t) \to F(x) \quad x = \frac{k}{t}\]

• Scaling equation
\[
\left[(x - q) - (1 - 2q)F(x)\right] \frac{dF}{dx} = 0
\]
Scaling solution

- **Stationary distribution of winning percentage**
  \[ F(x) = \begin{cases} 
  0 & 0 < x < q \\
  \frac{x - q}{1 - 2q} & q < x < 1 - q \\
  1 & 1 - q < x. 
\end{cases} \]

- **Distribution of winning percentage is uniform**
  \[ f(x) = F'(x) = \begin{cases} 
  0 & 0 < x < q \\
  \frac{1}{1 - 2q} & q < x < 1 - q \\
  0 & 1 - q < x. 
\end{cases} \]

- **Variance in winning percentage**
  \[ \sigma = \frac{1/2 - q}{\sqrt{3}} \]
  \[ \rightarrow \begin{cases} 
  q = 1/2 & \text{perfect parity} \\
  q = 0 & \text{maximum disparity} 
\end{cases} \]
Approach to scaling

Numerical integration of the rate equations, $q=1/4$

- Winning percentage distribution approaches scaling solution
- Correction to scaling is very large for realistic number of games
- Large variance may be due to small number of games

$$\sigma(t) = \frac{1/2 - q}{\sqrt{3}} + f(t)$$

Large!

Variance inadequate to characterize competitiveness!
The distribution of win percentage

- Treat q as a fitting parameter, time=number of games
- Allows to estimate $q_{\text{model}}$ for different leagues
The upset frequency

- Upset frequency as a measure of predictability

\[ q = \frac{\text{Number of upsets}}{\text{Number of games}} \]

- Addresses the variability in the number of games

- Measure directly from game-by-game results
  - Ties: count as 1/2 of an upset (small effect)
  - Ignore games by teams with equal records
  - Ignore games by teams with no record
Soccer, baseball most competitive
Basketball, football least competitive

q differentiates the different sport leagues!
Evolution with time

- Parity, predictability mirror each other
- Football, baseball increasing competitiveness
- Soccer decreasing competitiveness (past 60 years)

\[ \sigma = \frac{1/2 - q}{\sqrt{3}} \]

I. Discussion

• Model limitation: it does not incorporate
  - Game location: home field advantage
  - Game score
  - Upset frequency dependent on relative team strength
  - Unbalanced schedule

• Model advantages:
  - Simple, involves only 1 parameter
  - Enables quantitative analysis
I. Conclusions

- Parity characterized by variance in winning percentage
  - Parity measure requires standings data
  - Parity measure depends on season length
- Predictability characterized by upset frequency
  - Predictability measure requires game results data
  - Predictability measure independent of season length
- Two-team competition model allows quantitative modeling of sports competitions
2. Tournaments
(post-season)
Single-elimination Tournaments

2006 NCAA Division I Men's Basketball Championship


Winner plays Villanova in First Round.

First Round: Connecticut 1 vs. Albany 16, Kentucky 8 vs. UAB 9, Washington 5 vs. Utah St. 12, Illinois 4 vs. Air Force 11, George Mason 3 vs. North Carolina 14, Murray St. 7 vs. Wichita St. 16, Seton Hall 2 vs. Tennessee 11, Winthrop 1 vs. Villanova 8, Arizona 9 vs. Wisconsin 5, Nevada 12 vs. Montana 1, Boston College 13 vs. Pacific 4, Oklahoma St. 6 vs. Wis.-Milwaukee 15, Florida 3 vs. South Ala. 14, Georgetown 12 vs. Northern Iowa 5, Ohio St. 2 vs. Davidson 11.

Final Four: Indianapolis, April 1.

National Champion: Indianapolis, April 3.

Final Four: Washington, D.C.

All times are local.

©2006 National Collegiate Athletic Association. No commercial use without the NCAA's written permission. The NCAA reserves all rights and no part of this material may be reproduced without written permission.
The competition model

- Two teams play, loser is eliminated
  \[ N \rightarrow N/2 \rightarrow N/4 \rightarrow \cdots \rightarrow 1 \]
- Teams have inherent strength (or fitness) \( x \)
- Outcome of game depends on team strength

\[ (x_1, x_2) \rightarrow \begin{cases} 
  x_1 & \text{probability } 1 - q \\
  x_2 & \text{probability } q 
\end{cases} \quad x_1 < x_2 \]
Recursive approach

- Number of teams
  \[ N = 2^k = 1, 2, 4, 8, \ldots \]

- \( G_N(x) \) = Cumulative probability distribution function for teams with fitness less than \( x \) to win an \( N \)-team tournament

- Closed equations for the cumulative distribution
  \[
  G_{2N}(x) = 2p G_N(x) + (1 - 2p) [G_N(x)]^2
  \]

Nonlinear Recursion Equation
1. Scale of Winner

\[ x_* \sim N^{-\ln 2p / \ln 2} \]

2. Scaling Function

\[ G_N(x) \rightarrow \Psi \left( \frac{x}{x_*} \right) \]

3. Algebraic Tail

\[ 1 - \Psi(z) \sim z^{\ln 2p / \ln 2q} \]

1. Large tournaments produce strong winners
3. High probability for an upset
The scaling function

Universal shape

\[ \Psi(2pz) = 2p\Psi(z) + (1 - 2p)\Psi^2(z) \]

Broad tail

\[ \Psi'(z) \sim z^{\ln 2p / \ln 2q - 1} \]
College Basketball

• Teams ranked 1-16
  Well defined favorite
  Well defined underdog
• 4 winners each year
• Theory: q=0.18
• Simulation: q=0.22
• Data: q=0.27
• Data: 1978-2006
• 1600 games

2008: all four top seed advance; 1 in 150 chance!
Evolution, Men vs Women

![Graph showing the evolution of men and women over time, with blue and magenta lines representing men and women respectively.](image-url)
2. Conclusions

- Strong teams fare better in large tournaments
- Tournaments can produce major upsets
- Distribution of winner relates parity with predictability
- Tournaments are efficient but not fair
3. Leagues
(regular season)
League champions

- N teams with fixed ranking
- In each game, favorite and underdog are well defined
- Favorite wins with probability \( p > \frac{1}{2} \)
  Underdog wins with probability \( q < \frac{1}{2} \)
- Each team plays \( t \) games against random opponents
  - Regular random graph
- Team with most wins is the champion

How many games are needed for best team to win?
Random walk approach

- Probability team ranked $n$ wins a game
  \[ P_n = p \frac{n - 1}{N - 1} + q \frac{N - n}{N - 1} \]
- Number of wins performs a biased random walk
  \[ w_n = P_n t \pm \sqrt{D_n t} \]
- Team $n$ can finish first at early times as long as
  \[ (2p - 1) \frac{n}{N} t \sim \sqrt{t} \]
- Rank of champion as function of $N$ and $t$
  \[ n^* \sim \frac{N}{\sqrt{t}} \]
Length of season

- For best team to finish first
  \[ 1 \sim \frac{N}{\sqrt{t}} \]

- Each team must play
  \[ t \sim N^2 \]

- Total number of games
  \[ T \sim N^3 \]

1. Normal leagues are too short
2. Normal leagues: rank of winner \( \sim \sqrt{N} \)
3. League champions are a transient!
Distribution of outcomes

• Scaling distribution for the rank of champion
  \[ Q_n(t) \sim \frac{1}{n_*} \psi \left( \frac{n}{n_*} \right) \]
  \[ n_* \sim \frac{N}{\sqrt{t}} \]

• Probability worse team wins decays exponentially
  \[ Q_N(t) \sim \exp(-\text{const} \times t) \]

• Gaussian tail because
  \[ \psi \left( t^{1/2} \right) \sim \exp(-t) \]
  \[ \psi(z) \sim \exp \left( -\text{const} \times z^2 \right) \]

• Normal league: Prob. (weakest team wins) \( \sim \exp(-N) \)

Leagues are fair: upset champions extremely unlikely
Leagues versus Tournaments

16 teams, $q=0.4$

$n_\ast \sim \sqrt{N}$

<table>
<thead>
<tr>
<th>n</th>
<th>league</th>
<th>tournament</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.5</td>
<td>12.9</td>
</tr>
<tr>
<td>2</td>
<td>18.2</td>
<td>11.4</td>
</tr>
<tr>
<td>3</td>
<td>13.6</td>
<td>10.1</td>
</tr>
<tr>
<td>4</td>
<td>10.3</td>
<td>8.9</td>
</tr>
<tr>
<td>5</td>
<td>7.9</td>
<td>7.9</td>
</tr>
<tr>
<td>6</td>
<td>6.1</td>
<td>7.1</td>
</tr>
<tr>
<td>7</td>
<td>4.7</td>
<td>6.3</td>
</tr>
<tr>
<td>8</td>
<td>3.7</td>
<td>5.7</td>
</tr>
<tr>
<td>9</td>
<td>2.9</td>
<td>5.1</td>
</tr>
<tr>
<td>10</td>
<td>2.2</td>
<td>4.6</td>
</tr>
<tr>
<td>11</td>
<td>1.7</td>
<td>4.2</td>
</tr>
<tr>
<td>12</td>
<td>1.3</td>
<td>3.8</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>3.4</td>
</tr>
<tr>
<td>14</td>
<td>0.81</td>
<td>3.1</td>
</tr>
<tr>
<td>15</td>
<td>0.63</td>
<td>2.8</td>
</tr>
<tr>
<td>16</td>
<td>0.49</td>
<td>2.6</td>
</tr>
</tbody>
</table>
What is the likelihood the best team has best record?

<table>
<thead>
<tr>
<th>league</th>
<th>season</th>
<th>games</th>
<th>likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFL</td>
<td>short</td>
<td>predictable</td>
<td>30%</td>
</tr>
<tr>
<td>MLB*</td>
<td>long</td>
<td>random</td>
<td>31%</td>
</tr>
<tr>
<td>NHL</td>
<td>moderate</td>
<td>moderate</td>
<td>32%</td>
</tr>
<tr>
<td>NBA</td>
<td>moderate</td>
<td>predictable</td>
<td>45%</td>
</tr>
</tbody>
</table>

*90% likelihood requires 15000 games/team!!!

Interplay between length of season and predictability of games
3. Conclusions

- Leagues are fair but inefficient
- Leagues do not produce major upsets
4. Ranking Algorithm
One preliminary round

- Preliminary round
  - Teams play a small number of games \( T \sim N_t \)
  - Top \( M \) teams advance to championship round \( M \sim N^\alpha \)
  - Bottom \( N-M \) teams eliminated
  - Best team must finish no worse than \( M \) place \( t \sim \frac{N^2}{M^2} \)

- Championship round: plenty of games \( T \sim M^3 \)

- Total number of games
  \[
  T \sim N^{3-2\alpha} + N^{3\alpha}
  \]

- Minimal when
  \[
  M \sim N^{3/5} \quad T \sim N^{9/5}
  \]
Two preliminary rounds

- Two stage elimination
  \[ N \rightarrow N^{\alpha_2} \rightarrow N^{\alpha_2 \alpha_1} \rightarrow 1 \]

- Second round
  \[ T_2 \sim N^{3-2\alpha_2} + N^{\alpha_2(3-2\alpha_1)} + N^{3\alpha_1 \alpha_2} \]

- Minimize number of games
  \[ 3 - 2\alpha_2 = \alpha_2(3 - 2\alpha_1) \quad \rightarrow \quad \alpha_2 = \frac{15}{19} \]

- Further improvement in efficiency
  \[ T \sim N^{27/19} \]
Multiple preliminary rounds

- Each additional round further reduces $T$
  \[ T_k \sim N^{\gamma_k} \]
  \[ \gamma_k = \frac{1}{1 - (2/3)^{k+1}} \]

- Gradual elimination
  \[ \gamma_k = 3, \frac{9}{5}, \frac{27}{19}, \frac{81}{65}, \cdots \]
  \[ N \rightarrow N^{\frac{57}{65}} \rightarrow N^{\frac{57}{65} \frac{15}{19}} \rightarrow N^{\frac{57}{65} \frac{15}{19} \frac{3}{5}} \rightarrow 1 \]

- Teams play a small number of games initially

Optimal linear scaling achieved using many rounds

\[ T_\infty \sim N \quad M_\infty \sim N^{1/3} \]

Preliminary elimination is very efficient!
4. Conclusions

- Gradual elimination is fair and efficient
- Preliminary rounds reduce the number of games
- In preliminary round, teams play a small number of games and almost all teams advance to next round
5. Social Dynamics
Competition and social dynamics

- Teams are agents
- Number of wins represents fitness or wealth
- Agents advance by competing against each other
- Competition is a mechanism for social differentiation
The social diversity model

- Agents advance by competition
  \[(i, j) \rightarrow \begin{cases} (i + 1, j) & \text{probability } p \\ (i, j + 1) & \text{probability } 1 - p \end{cases} \quad i > j\]

- Agent decline due to inactivity
  \[k \rightarrow k - 1 \quad \text{with rate } r\]

- Rate equations
  \[
  \frac{dG_k}{dt} = r(G_{k+1} - G_k) + pG_{k-1}(G_{k-1} - G_k) + (1 - p)(1 - G_k)(G_{k-1} - G_k) - \frac{1}{2}(G_k - G_{k-1})^2
  \]

- Scaling equations
  \[
  [(p + r - 1 + x) - (2p - 1)F(x)] \frac{dF}{dx} = 0
  \]
Social structures

1. Middle class
   Agents advance at different rates

2. Middle+lower class
   Some agents advance at different rates
   Some agents do not advance

3. Lower class
   Agents do not advance

4. Egalitarian class
   All agents advance at equal rates

Bonabeau 96
Sports
Concluding remarks

• Mathematical modeling of competitions sensible
• Minimalist models are a starting point
• Randomness a crucial ingredient
• Validation against data is necessary for predictive modeling
Publications

- Randomness in Competitions
  E. Ben-Naim, N.W. Hengartner

- Efficiency of Competitions
  E. Ben-Naim, N.W. Hengartner

- Scaling in Tournaments
  E. Ben-Naim, S. Redner, F. Vazquez
  Europhysics Letters 77, 30005 (2007)

- What is the Most Competitive Sport?
  E. Ben-Naim, F. Vazquez, S. Redner

- Dynamics of Multi-Player Games
  E. Ben-Naim, B. Kahng, and J.S. Kim

- On the Structure of Competitive Societies
  E. Ben-Naim, F. Vazquez, S. Redner

- Dynamics of Social Diversity
  E. Ben-Naim and S. Redner