Nonlocal interaction PDEs with nonlinear diffusion

Marco Di Francesco


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A discrete particle system

- $N$ particles, located at $X_1(t), \ldots, X_N(t) \in \mathbb{R}^d$ with masses $m_1, \ldots, m_N$.
- Subject to binary interaction forces depending on their position.
- Friction dominated regime: no inertia.

$$\frac{dX_j(t)}{dt} = - \sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t)), \quad j = 1, \ldots, N. \quad (1)$$

Typical assumptions for the interaction potential $G$

- $G \in C(\mathbb{R}^d)$, with $G(0) = 0$,
- Radial symmetry $G(x) = g(|x|)$,

Notation: $g$ increasing $\Rightarrow$ $G$ attractive, $g$ decreasing $\Rightarrow$ $G$ repulsive.

Stochastic version:

$$dX_j(t) = - \sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t))dt + \sigma_N dW^j(t)$$
An interdisciplinary model for interacting individuals

Figure: $N$ interacting particles
Main motivation: population dynamics

Animal swarming:
- Okubo (1980)
- Oelschläger (1989)
- Morale, Capasso, and Oelschläger (1998)
- Mogilner, Edelstein-Keshet (1999)
- Topaz, Bertozzi, and Lewis (2006)

Typical interaction potentials:
- attractive-repulsive *Morse* potentials \( G(x) = -C_a e^{-|x|/l_a} + C_r e^{-|x|/l_r} \)
- combination of *Gaussian* potentials \( G(x) = -C_a e^{-|x|^2/l_a} + C_r e^{-|x|^2/l_r} \)
- smoothed characteristic functions of a set \( G(x) = \alpha \delta_{\epsilon} \ast \chi_A(x) \).
Hydrodynamic $N \to +\infty$ limit

Empirical measure:

$$\mu_N(t) = \left( \sum_{j=1}^{N} m_j \right)^{-1} \sum_{k=1}^{N} m_k \delta X_k(t)$$

Formal limit of $\mu_N$ in the stochastic case

Assuming $\lim_{N \to +\infty} \sigma_N = \sigma > 0$, then

$$\frac{\partial \mu}{\partial t} = \frac{\sigma^2}{2} \Delta \mu + \text{div}(\mu \nabla G \ast \mu)$$

Distributional PDE for $\mu_N$ for $\sigma = 0$

$$\frac{\partial \mu}{\partial t} = \text{div}(\mu \nabla G \ast \mu)$$
More motivations: Interplay with physics

Mean-field limits of large particle systems in statistical mechanics:

- Onsager (1949) - Vortex dynamics
- Morrey (1955) - Derivation of hydrodynamics from statistical mechanics
- Dobrushin (1993) - Vlasov equation
- Golse (2003) - Review paper

In those contexts, the potential $G$ blows-up at the origin, which renders the rigorous analytical framework of the model a challenging issue.

Kinetic modeling for granular media:

- Benedetto, Caglioti, Pulvirenti (1997)
- Toscani (2004)

Here, $G$ is a convex attractive potential, typically $G(x) = |x|^{\alpha}$ with $\alpha > 1$. 
More motivations: chemotaxis

- In many problems in biology, such as the 2d Keller-Segel model

\[ \partial_t \rho = \Delta \rho + \frac{\chi}{2\pi} \text{div}(\rho \nabla \log | \cdot | \ast \rho), \]

the dichotomy between the repulsive linear diffusion term and the attractive log ‘chemotaxis’ term produces blow-up (concentration) of solutions in finite time. No one knows (up to now) how to define solutions in a measure sense after blow up.

- The large time behavior for models with ‘milder’ aggregation force and with nonlinear diffusion

\[ \partial_t \rho = \Delta \rho^m + \text{div}(\rho \nabla G \ast \rho) \]

\[ G(x) = g(|x|), \quad g'(r) > 0, \quad G \in W^{1,\infty}, \]

is a (most of the times) highly nontrivial question.

More motivations:


- Kinetic dithering

$$\partial_t \rho = -\text{div}(\rho \nabla (G \ast (\rho - \sigma)))$$

with $\sigma \in L^1_+$ being a given profile, and $\int \rho = \int \sigma$. Typically, $G(x) = |x|^\alpha$. Stationary solution $\rho = \sigma$. Stable for large times? PhD thesis of J.-C. Hütter. Ref: Fornasier, Haškovec, Steidl - 2012.


Mathematical motivation

- Models with nonlocal attractive-repulsive kernels

\[ \frac{\partial \rho}{\partial t} = \text{div}(\rho \nabla G \ast \rho) \]

with \( G \) being a \textit{double-well} potential, e.g. Lennard–Jones. Stationary solutions? How do they look like?

- Fetecau, Huang, Kolkolnikov - 2011: \( L^1 \) stationary states.
- von Brecht, Bertozzi - 2012: aggregation sheets.

- Similarities with 2\( d \) incompressible Euler.

- A repulsive nonlocal approximation for nonlinear diffusion

\[ \frac{\partial \rho}{\partial t} = \text{div}(\rho \nabla G_\epsilon \ast \rho) \]
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What is a gradient flow?

Given a smooth function $F : \mathbb{R}^d \to \mathbb{R}$, a differentiable curve $[0, +\infty) \ni t \mapsto X(t) = \mathbb{R}^d$ is a gradient flow of $F$ if $X(t)$ satisfies

$$\dot{X}(t) = -\nabla F(X(t)).$$

- Energy dissipation:
  $$\frac{d}{dt} F(X(t)) = -|\nabla F(X(t))|^2$$

- Implicit Euler variational derivation: time step $\tau > 0$, $X_\tau(t) = X_\tau^n$ for $t \in ((n - 1)\tau, n\tau]$, with
  $$X_\tau^n = \arg\min \left\{ \frac{1}{2\tau} |X - X_\tau^n|^2 + F(X), \ X \in \mathbb{R}^d \right\}$$

- $D^2 F \geq \lambda I$ implies stability
  $$\frac{d}{dt} |X_1(t) - X_2(t)|^2 = -2 < X_1(t) - X_2(t), \nabla F(X_1(t)) - \nabla F(X_1(t)) > \leq -2\lambda |X_1(t) - X_2(t)|^2.$$
Gradient flow structure of the ODE particle system

Consider

\[
\frac{dX_j(t)}{dt} = - \sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t)), \quad j = 1, \ldots, N.
\]

with \( G(-x) = G(x) \) and \( G \in C^2(\mathbb{R}^d) \).

Weighted metric structure

Denote \( \mathbf{m} = (m_1, \ldots, m_N) \). For \( \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{dN} \), let

\[
\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2_m} := \sum_{j=1}^{N} m_j X_j Y_j, \quad \| \mathbf{X} \|_{L^2_m}^2 = \langle \mathbf{X}, \mathbf{X} \rangle_{L^2_m}.
\]

Fréchet differential

Let \( \mathbf{F} \in C^1(\mathbb{R}^{dN}) \). The linear operator \( \text{grad}_X \mathbf{F}[\mathbf{X}] \) is defined by

\[
\lim_{\epsilon \to 0} \frac{\mathbf{F}[\mathbf{X} + \epsilon \mathbf{Y}] - \mathbf{F}[\mathbf{X}]}{\epsilon} =: \langle \text{grad}_X \mathbf{F}[\mathbf{X}], \mathbf{Y} \rangle_{L^2_m} = \sum_{j=1}^{N} m_j \nabla x_j \mathbf{F}[\mathbf{X}] \cdot Y_j.
\]
Gradient flow structure of the ODE particle system

Energy functional

Let $\mathbf{X} := (X_1, \ldots, X_N)^T.$

$$G[\mathbf{X}] := \frac{1}{2} \sum_{i,j} m_i m_j G(X_i - X_j)$$

Then

$$\dot{\mathbf{X}}(t) = -\nabla_{\mathbf{X}} G[\mathbf{X}(t)].$$

Problem (2) makes sense if $G \in C^1(\mathbb{R}^d).$

Regularity and collisions

If $G \in C^2(\mathbb{R}^d),$ then particles do not collide.
Mildly singular, locally attractive kernels

Assume

(K1) \( G(-x) = G(x) \)

(K2) \( G \in C^1(\mathbb{R}^d \setminus \{0\}) \)

(K3) \( G \) has a local minimum at \( x = 0 \)

(K4) \( G \) is \( \lambda \)-convex, i.e. \( G(x) - \frac{\lambda}{2} |x|^2 \) is convex on \( \mathbb{R}^d \).

Examples:

- Morse type potentials \( G(x) = -e^{-a|x|} \), with \( a > 0 \),
- Pointy potentials, i.e. with a Lipschitz point at the origin,
- Power laws \( G(x) = |x|^\alpha \) with \( \alpha \in (1, 2) \), cf. [Li, Toscani - 2004], [Burger, DF - 2008]

Kernels with above assumptions (K1)–(K4) possibly produce finite time collapse \( \mu = \delta_{x_c} \), with \( x_c \) center of mass of the particles (constant in time).
Gradient flow structure of the discrete model

Weaker gradient flow structure

Introduce the sub-differential of $G$

$$
\partial G(x) := \left\{ k \in \mathbb{R}^d : G(y) - G(x) \geq k \cdot (y - x) + o(|x - y|) \text{ for all } y \in \mathbb{R}^d \right\},
$$

and the minimal sub-differential of $G$

$$
\partial^0 G(x) = \arg\min_{k \in \partial G(x)} |k| = \begin{cases} 
\nabla G(x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
$$

Sub-differential structure of $L^2_m$

$$
\partial G[X] := \left\{ K \in L^2_m : G(Y) - G(X) \geq \langle K, (Y - X) \rangle_{L^2_m} + o(\|X - Y\|_{L^2_m}) \text{ for all } Y \in L^2_m \right\}.
$$
We replace our particle system with

\[
\frac{dX_j(t)}{dt} \in - \sum_{k \in C_j(t)} m_k \partial^0 G(X_j(t) - X_k(t)), \quad C_j(t) = \{ k : X_j(t) \neq X_k(t) \}. \quad (3)
\]

Then, it is easily checked that

\[
\dot{X}(t) \in -\partial^0 G[X(t)], \quad (4)
\]

with \( \partial^0 G = \arg\min_{K \in \partial G} \| K \|_{L^2_m} \).

Well posedness in the discrete case

- \( \lambda \)-convexity of the functional \( G \)
- Existence and uniqueness of gradient flows.
Finite time collapse for attractive potentials

Assume $G$ satisfies (K1)–(K4) and the additional conditions

$$G(x) = g(|x|), \quad g'(r) > 0 \text{ for } r > 0, \quad \frac{g'(r)}{r} \text{ non-increasing.} \quad (5)$$

Proposition (Finite time collapse)

Let $X_1, \ldots, X_N$ evolve according to (4), i. e.

$$\dot{X}_j(t) = - \sum_{X_k(t) \neq X_j(t)} m_k \nabla G(X_j(t) - X_k(t)).$$

Then, all the particles collapse in a finite time, i. e. $X_j(t) = \delta_{C_m}$ for all $t \geq t^*$ for some $t^*$, iff

$$\int_0^\varepsilon \frac{1}{g'(z)} dz < +\infty \quad (6)$$

for some $\varepsilon > 0$.

$^1$ $G$ is called attractive when $g'(r) > 0$ and repulsive when $g'(r) < 0$
Figure: The quantity $R(t) = \max\{|X_j(t) - C_m|, j = 1, \ldots, N\}$.
Proof

Assume $\sum_{j=1}^{N} m_j = 1$. Center of mass $C_m = \sum_{j=1}^{N} m_j X_j(t)$ is preserved. Assume for simplicity $C_m = 0$. 

$$\frac{d}{dt} R(t) = \frac{d}{dt} |X_1(t)| = - \frac{X_1(t)}{|X_1(t)|} \cdot \sum_{j \neq 1} m_j \nabla G(X_1(t) - X_j(t))$$

$$= - \sum_{j \neq 1} m_j X_1(t) \cdot (X_1(t) - X_j(t)) \frac{g'(|X_1(t) - X_j(t)|)}{|X_1(t)||X_1(t) - X_j(t)|}.$$ 

Since $X_1(t) \cdot X_j(t) \leq |X_1(t)|^2$, and since $g'(r)/r$ is non increasing, we use $|X_1(t) - X_j(t)| \leq 2|X_1(t)|$:

$$\frac{d}{dt} R(t) \leq - \frac{g'(2|X_1(t)|)}{2|X_1(t)|^2} \sum_{j \neq 1} m_j \left(|X_1(t)|^2 - X_1(t) \cdot X_j(t)\right)$$

$$= -(1 - m_1)g'(2|X_1(t)|) + \frac{g'(2|X_1(t)|)}{2|X_1(t)|^2} X_1(t) \cdot (-m_1 X_1(t)) = -g'(2R(t))$$

and the assertion is proven. Notice that the collapse time is independent of $N$. 

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Ingredients for the continuum theory\textsuperscript{2}

Aim: produce a unique notion of measure solution for

\[ \frac{\partial \mu}{\partial t} = \text{div}(\mu \nabla G * \mu). \]

The measure space

\[ \mu \in \mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d), \quad \int |x|^2 d\mu(x) < +\infty \right\} \]

endowed with the 2-\textit{Wasserstein} distance

\[ d_2(\mu, \nu)^2 = \inf \left\{ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma(x, y), \quad \gamma \in \Gamma(\mu, \nu) \right\} \]

\[ \Gamma(\mu, \nu) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \mu \text{ and } \mu \text{ are the marginals of } \gamma \right\} \]

The functional

\[ \mathcal{G}[\mu] = \frac{1}{2} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} G(x - y)d\mu(x)d\mu(y) \]

\textsuperscript{2}Ambrosio, Gigli, Savaré - Birkhäuser 2005
Why the Wasserstein distance?

Go back to the discrete case:

\[ \mu := \sum_{j=1}^{N} m_j X_j, \quad \nu := \sum_{j=1}^{N} m_j Y_j. \]

The natural distance is

\[ d(\mu, \nu)^2 = \inf \left\{ \int_0^1 \| \frac{d}{ds} x(\cdot) \|_{L^2_m}^2, \; X_j(0) = X_j, \; X_j(1) = Y_j \right\}. \]

The natural continuum version is:

\[ d(\mu, \nu)^2 = \inf \left\{ \int_0^1 \int |v_s(x)|^2 d\mu_s(x), \; \partial_s \mu_s + \text{div}(\mu_s v_s)0, \; \mu_0 = \mu, \; \mu_1 = \nu \right\}, \]

which coincides with the 2–Wasserstein distance according to the Benamou-Brenier formula.
Definition of Wasserstein gradient flow

An absolutely continuous curve \([0, +\infty) \ni t \mapsto \mu(t) \in \mathcal{P}(\mathbb{R}^d)\) is a Wasserstein gradient flow of the functional \(\mathcal{G}\) iff

\[
\frac{\partial \mu(t)}{\partial t} + \text{div}(\mu(t) \nu(t)) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^d \times [0, +\infty))
\]

\[
\nu(t) = -\partial^0 G \ast \mu(t) = -\int_{x \neq y} \nabla G(x - y) d\mu(y, t).
\]

Notice that \(\partial^0 G \ast \mu(t)\) coincides with the minimal sub-differential of \(\mathcal{G}\) on \(\mathcal{P}_2(\mathbb{R}^d)\), namely

\[
\partial^0 G \ast \mu(t) = \arg \min_{\nu \in \partial \mathcal{G}[\mu]} \|\nu\|_{L^2(d\mu: \mathbb{R}^d)}
\]

\[
\partial \mathcal{G}[\mu] = \{ \nu \in L^2(d\mu) : \mathcal{G}[\nu] - \mathcal{G}[\mu] \geq \inf_{\gamma_o \in \Gamma(\mu, \nu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nu(x) \cdot (y - x) d\gamma_o(x, y) + o(d_2(\mu, \nu)) \}
\]

\[
\gamma_o \in \Gamma(\mu, \nu) \text{ such that } d_2(\mu, \nu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma_o(x, y).
\]
Existence and uniqueness of solutions

Theorem (Existence and uniqueness\textsuperscript{a})

\textsuperscript{a}Carrillo, DF, Figalli, Laurent, Slepcev - Duke Math. J. - 2011

\begin{itemize}
  \item Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then, there exists a unique Wasserstein gradient flow solution for $\mathcal{G}$ with $\mu_0$ as initial datum. Moreover,
  \begin{align}
    \mathcal{G}[\mu(t)] + \int_0^t ds \int_{\mathbb{R}^2} \left| \partial^0 G \ast \mu(x, s) \right|^2 d\mu(x, s) &\leq \mathcal{G}[\mu_0], \\
    \text{for all } t \geq 0.
  \end{align}
  \end{itemize}

\begin{itemize}
  \item Let $\mu_1^0, \mu_2^0 \in \mathcal{P}_2(\mathbb{R}^d)$. Let $\mu_1(t)$ and $\mu_2(t)$ be Wasserstein gradient flows for $\mathcal{G}$ with $\mu_1^0$ and $\mu_2^0$ as initial data respectively. Then,
  \begin{align}
    d_2(\mu_1(t), \mu_2(t)) &\leq e^{\lambda t} d_2(\mu_1^0, \mu_2^0), \\
    \text{for all } t \geq 0.
  \end{align}
  \end{itemize}
The continuum theory

Finite time collapse for general solutions

Theorem (Finite total collapse$^a$)


Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ compactly supported. Let $\mu(t)$ the corresponding gradient flow of $G$. Let

$$C_m := \int_{\mathbb{R}^d} x d\mu(x, t).$$

Then, there exists a time $t^*$ depending only on the radius of $\text{sp}t(\mu_0)$ such that

$$\mu(t) = \delta_{C_m},$$

for all $t \geq t^*$. 

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Proof

Similar to an old idea of R. Dobrushin (1979).

1. **Atomization** of $\mu_0$: for a fixed arbitrary $\varepsilon > 0$, take $\mu_0^N = \sum_{j=1}^{N} m_j \delta_{X_j}$ such that
   \[ d_2(\mu_0, \mu_0^N) \leq \varepsilon. \]

2. Let the particles $X_1, \ldots, X_N$ evolve via the discrete particle system. Let $t^*$ be the collapse time,
   \[ X_1(t) = \ldots = X_N(t) = C_m, \quad \text{for all } t \geq t^*. \]

3. This means that $\mu^N(t) := \sum_{j=1}^{N} Nm_j \delta_{X_j(t)} = \delta_{C_m}$ for all $t \geq t^*$.

4. The stability property (8) implies
   \[ d_2(\mu(t^*), \mu^N(t^*)) \leq e^{-\lambda t^*} d_2(\mu_0, \mu_0^N) \leq \varepsilon e^{-\lambda t^*}, \]
   which is an arbitrary small quantity. Hence,

5. $\mu(t^*) = \mu^N(t^*) = \delta_{C_m}.$
Global confinement for attractive-repulsive potentials$^3$

Assume $G$ as in (K1)–(K4), plus

(K5) $G(x) = g(|x|)$, $g \in C^1((0, +\infty))$,

(K6) $g'(r) > 0$ for $r > R_a$ for some $R_a > 0$,

(K7) $g'(r) > -C_G$ for $r < R_a$ for some $C_G > 0$.

Moreover, assume either

(K8) there exists $\bar{R} > 0$ such that $g'(r) \geq 4C_G$ for all $r \geq \bar{R}$,

or

(K9) $\lim \inf_{r \to 0} g(r) > -\infty$, and $\lim_{r \to +\infty} g'(r) \sqrt{r} = +\infty$.

Then, there exists $R^* > 0$ depending only on $G$ and on $\mu_0$ such that

$$\text{spt}(\mu(t)) \subset B(0, R^*),$$

for all $t \geq 0$.

Remark: conditions (K5)–(K7) alone are not enough for global confinement (Theil, 2006).

$^3$Carrillo, DF, Figalli, Laurent, Slepcev - Nonlinear Anal. - 2012
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$N$–dependent repulsion range$^4$

$$
\frac{dX_j(t)}{dt} = - \sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t)) - \sum_{k \neq j} m_k \nabla V_N(X_j(t) - X_k(t)), \quad j = 1, \ldots, N
$$

$$
V_N(x) = N^{d\beta} V(N^\beta x), \quad \beta \in (0, 1)
$$

$$
V(x) = v(|x|), \quad v \in C^2((0, +\infty)), \quad v'(r) < 0, \quad \text{as } r > 0,
$$

$$
V \geq 0, \quad \int_{\mathbb{R}^d} V(x)dx = \varepsilon.
$$

- $V_N$ is a repulsive kernel, with a range of interaction $O(N^{-\beta})$ and strength of the interaction force $O(N^{d\beta})$ depending on the number of individuals $N$.
- Formally $V_N(x) \rightarrow \varepsilon \delta$ in $\mathcal{D}'$ as $N \rightarrow +\infty$.

Formal limit of the particle system

$$
\frac{\partial \mu}{\partial t} = \text{div}(\mu \nabla G \ast \mu) + \varepsilon \text{div}(\mu \nabla \mu).
$$

Hence... a quadratic porous medium type diffusion term appears.

Basic properties of the limiting equation

Assume

- $G(x) = g(|x|), g \in C^2([0, +\infty)),$
- $g'(r) > 0$ for all $r > 0,$
- $\text{spt } G = \mathbb{R}^d, G \leq 0, G \in L^1(\mathbb{R}^d).$

Regularizing effect

For all initial data $\mu_0 \in P_2(\mathbb{R}^d),$ the corresponding solutions are densities, $\mu(t) = \rho(t)d\mathcal{L}_d.$

Conservation of the center of mass

Let

$$CM[\rho(t)] := \int x\rho(x, t)dx,$$

then $CM[\rho(t)] = CM[\rho_0]$ for all $t \geq 0.$
Wasserstein gradient flow for the limiting equation

\[ \frac{\partial \rho}{\partial t} = \text{div}(\rho \nabla (\varepsilon \rho + G * \rho)). \]

Energy functional:

\[ E[\rho] := \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \rho^2(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x - y) \rho(y) \rho(x) \, dy \, dx. \quad (10) \]

Energy identity:

\[ E[\rho(t)] + \int_0^T \int_{\mathbb{R}^d} \rho \left| \nabla (\varepsilon \rho + G * \rho) \right|^2 \, dx \, dt = E[\rho_0]. \quad (11) \]

The identity (11) can be proven rigorously in the context of the Wasserstein gradient flow theory developed in [Ambrosio, Gigli, Savaré, Birkhäuser 2003].
A model with moderate repulsion

A key question: large time behavior

How does $\rho(t)$ behave as $t \to +\infty$? There are (basically) three possibilities:

(i) **Diffusion dominated case:** $\rho(t)$ decays to zero in some $L^p$ norm with $p > 1$. In this case, the repulsive effects dominates.

(ii) **Aggregation dominated case:** $\rho(t)$ concentrates to a singular measure (delta) in finite or infinite time. Here, the aggregation effect dominates.

(iii) **Balanced case:** $\rho(t)$ converges to some (stable) non trivial $L^1$ steady state for large times.

Unlike the Keller-Segel system, here no mass threshold phenomenon occurs, since the equation is quadratically homogeneous.
A model with moderate repulsion

A minimization problem

$$\argmin_{\rho \in L^1_+ (\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} \Phi(\rho(x)) \, dx - \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x) \rho(y) G(x - y) \, dx \, dy \right\}.$$ 

Existence of nontrivial minimizers\(^5\) under the assumptions

- Total mass sufficiently large,
- \(\Phi(tu) \leq t^\nu \Phi(u)\) with \(1 < \nu < 2\),
- \(G\) slow decaying at infinity, i.e. \(G(tx) \geq t^{-\alpha} G(x)\) with \(\alpha \in (0, d)\),
- \(\Phi(u) = o(u^{1+\frac{\alpha}{d}})\) as \(u \to 0\).

---

A critical exponent

Nontrivial minimizers exist\(^6\) if

- \( G \in L^1_+ \),
- \( \Phi(u) = cu^2 + o(u^2) \) as \( u \to 0 \) with \( c > 0 \),
- either \( c = 0 \) or \( 2c < \int G \).

Case \( \Phi(u) = u^m \): the exponent \( m = 2 \) is critical:

- \( m > 2 \) \( \Rightarrow \) aggregation dominates \( \Rightarrow \) nontrivial stationary patterns,
- \( m < 2 \) \( \Rightarrow \) diffusion dominates (large time decay expected),
- \( m = 2 \) \( \Rightarrow \) ??

---

\(^6\) [Bedrossian, 2012]
Stationary states in multiple dimensions.

\[ \frac{\partial \rho}{\partial t} = \text{div}(\rho \nabla (\varepsilon \rho + G \ast \rho)). \]

Threshold phenomenon\(^a\)

\(^a\)[Burger, DF, Franek - to appear on CMS], [Bedrossian, AML 2011]

- Let \( \varepsilon < \|G\|_{L^1} \). Then, there exists at least one non trivial \( L^1 \) steady state, which is also a minimizer for the energy \( E[\rho] \).
- Let \( \varepsilon \geq \|G\|_{L^1} \). Then, there exist no steady states except \( \rho \equiv 0 \).

Finite time concentration is not possible under the present smoothness assumptions on \( G \).

Stationary points of \( E[\rho] \) are steady states and vice-versa
Uniqueness of steady states in one space dimension

With \( d = 1 \) we can characterize all the steady states as follows.

**Theorem (Burger-DF-Franek - to appear on CMS)**

Let \( \varepsilon < \|G\|_{L^1} \). Then, there exists a unique \( \rho \in L^2 \cap \mathcal{P} \) with zero center of mass which solves

\[
\rho \partial_x (\varepsilon \rho + G \ast \rho) = 0.
\]

Moreover,

- \( \rho \) is symmetric and monotonically decreasing on \( x > 0 \),
- \( \rho \in C^2(\text{supp}[\rho]) \),
- \( \text{supp}[\rho] \) is a bounded interval in \( \mathbb{R} \),
- \( \rho \) has a global maximum at \( x = 0 \) and \( \rho''(0) < 0 \),
- \( \rho \) is the global minimizer of the energy \( E[\rho] = \frac{\varepsilon}{2} \int \rho^2 \, dx - \frac{1}{2} \int \rho G \ast \rho \, dx \).
Sketch of the proof

- Fix $L > 0$. Look for $\rho \in C(\mathbb{R})$ symmetric on $\text{spt}\rho = [-L, L]$, strictly decreasing on $(0, L]$:
  \[
  \varepsilon \rho(x) = -\int_0^L (G(x - y) + G(x + y)) \rho(y) dy + C \tag{12}
  \]

- Differentiate (12) w.r.t. $x$, set $u(x) = -\rho_x(x)$:
  \[
  \varepsilon u(x) = -\int_0^L (G(x - y) - G(x + y)) u(y) dy =: G_L[u](x) \tag{13}
  \]

- Solve the eigenvalue problem (13) with Krein-Rutman theorem. $G_L$ is a strictly positive operator, therefore $\varepsilon = \varepsilon(L)$ is a simple eigenvalue $\Rightarrow$ uniqueness of $\rho(x) = \int_x^L u(y) dy$ with $\int_0^L \rho(x) dx = 1$.

- Prove that the function $(0, +\infty) \ni L \mapsto \varepsilon(L) \in (0, 1)$ is continuous and $1 : 1 \Rightarrow$ uniqueness is proven provided all steady states are supported on a bounded interval, symmetric and decreasing on $x > 0$.

- Prove that all steady states are as above. Main tools: symmetric rearrangement and connected support.
Remarks and open problems:

- The uniqueness is surprising because the functional is neither geodesically convex in the Wasserstein space nor convex in the classical sense.
- Uniqueness in many dimensions? We believe it true in the radially symmetric case.
- Porous medium exponent $\gamma \neq 2$ (ongoing discussion with M. Burger, R. Fetecau, Y. Huang).
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The JKO scheme produces entropy solutions

- Nonlocal interaction equations with nonlinear diffusion

\[ \partial_t \rho = \Delta \rho^m + \text{div}(\rho \nabla G \ast \rho) = 0 \quad (14) \]

with \( m > 1 \) and \( G \in C^2 \) and \( G \) even. Here, both notions of entropy solutions and gradient flow solutions have been used (almost at the same time!) to prove uniqueness of solutions.

- Nonlinear diffusion equations with in-homogeneous term

\[ \partial_t \rho = \partial_x (\rho \partial_x (a(x) \rho^{m-1})) \]

with \( a(x) \geq c > 0 \). In [DF, Matthes - submitted 2012] we prove that the notions of gradient flow solution and entropy solutions coincide.

The results in [DF, Matthes] can be applied also for (14).
A one dimensional repulsive equation\(^7\)

Consider \( \rho \) gradient flow solution to

\[
\rho_t = \partial_x (\rho \partial_x (G * \rho)), \quad G(x) = -|x|.
\] (15)

Let

\[
F(x, t) = \int_{-\infty}^{x} \rho(y, t) dy,
\]

then \( F \) is an entropy solution to the Burgers’ type equation

\[
F_t + (F^2 - F)_x = 0.
\] (16)

Applications:

- Smoothing effect: initial deltas become densities,
- Wave front tracking approximation for (16) provide particle approximation for (15).

\(^7\)Work in preparation with G. Bonaschi and J. A. Carrillo
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A two species model\textsuperscript{8}

- $X_1, \ldots, X_N$ particles of the first species with masses $n_1, \ldots, n_N$,
- $Y_1, \ldots, Y_M$ are particles of the second species with masses $m_1, \ldots, m_M$.

Particle system:

\[
\begin{aligned}
\dot{X}_i(t) &= -\sum_{X_i \neq X_k} n_k \nabla H_1(X_i(t) - X_k(t)) - \sum_{X_i \neq Y_k} m_k \nabla K_1(X_i(t) - Y_k(t)) \\
\dot{Y}_j(t) &= -\sum_{Y_j \neq Y_k} m_k \nabla H_2(Y_j(t) - Y_k(t)) - \sum_{Y_j \neq X_k} n_k \nabla K_2(Y_j(t) - X_k(t))
\end{aligned}
\]

Continuum version:

\[
\begin{aligned}
\partial_t \mu_1 &= \text{div} \left( \mu_1 \nabla H_1 \ast \mu_1 + \mu_1 \nabla K_1 \ast \mu_2 \right) \\
\partial_t \mu_2 &= \text{div} \left( \mu_2 \nabla H_2 \ast \mu_2 + \mu_2 \nabla K_2 \ast \mu_1 \right).
\end{aligned}
\]

\textsuperscript{8}[DF, Fagioli - submitted]
Motivation

- Pedestrian movements, lane formation, segregation, cf. [Appert-Rolland, Degond, Motsch - 2011], [Colombo, Lécureux-Mercier - 2012].
- Two species chemotaxis, cf. [Horstmann - 2011], [Espejo, Stevens, Velázquez - 2009], [Conca, Espejo, Vilches - 2011].
Systems with many species

**Symmetrizable case**

\[
\begin{align*}
\partial_t \mu_1 &= \text{div} \left( \mu_1 \nabla K_{11} \ast \mu_1 + \mu_1 \nabla K_{12} \ast \mu_2 \right) \\
\partial_t \mu_2 &= \alpha \text{div} \left( \mu_2 \nabla K_{22} \ast \mu_2 + \mu_2 \nabla K_{12} \ast \mu_1 \right).
\end{align*}
\] (17)

System (17) has a gradient flow structure, with functional

\[
F(\mu_1, \mu_2) = \frac{1}{2} \int_{\mathbb{R}^d} K_{11} \ast \mu_1 d\mu_1 + \frac{1}{2} \int_{\mathbb{R}^d} K_{22} \ast \mu_2 d\mu_2 + \int_{\mathbb{R}^d} K_{12} \ast \mu_2 d\mu_1.
\]

The quantity

\[
c_{M, \alpha} := \alpha \int x d\mu_1(x) + \int x d\mu_2(x)
\]

is preserved.

**Metric product structure**

\[
\mu = (\mu_1, \mu_2) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d),
\]

\[
W^2_{2, \alpha}(\mu, \nu) = W^2_2(\mu_1, \nu_1) + \frac{1}{\alpha} W^2_2(\mu_2, \nu_2).
\]
Results in the symmetrizable case

Assumptions: all the kernels $K_{ij}$ are mildly singular and $\lambda_{ij}$–convex. We prove:

- $\lambda$ convexity of the interaction energy on a suitable sub-differential structure.
- Existence, uniqueness, and stability of gradient flow solutions, by generalizing the one-species theory.
- Finite time collapse if all the kernels are of Non–Osgood type.
- Partial intermediate collapse of each species if the cross interaction kernel decays at infinity.
General case: the strategy

No gradient flow structure in general, no variational formulation. Main idea: semi-implicit version of the JKO scheme.

For all $\mu \in \mathcal{P}(\mathbb{R}^d)^2$ we set

$$F[\mu|\nu] = \frac{1}{2} \int_{\mathbb{R}^d} H_1 \ast \mu_1 d\mu_1 + \int_{\mathbb{R}^d} K_1 \ast \nu_2 d\mu_1 + \frac{1}{2} \int_{\mathbb{R}^d} H_2 \ast \mu_2 d\mu_2 + \int_{\mathbb{R}^d} K_2 \ast \nu_1 d\mu_2.$$ 

Let $\tau > 0$ be a fixed time step, and let $\mu_0 = (\mu_{0,1}, \mu_{0,2}) \in \mathcal{P}(\mathbb{R}^d)^2$ be a fixed initial pair of probability measures. For a given $\mu_\tau^n \in \mathcal{P}(\mathbb{R}^d)^2$, we define the sequence $\mu_\tau^{n+1}$ as

$$\mu_\tau^{n+1} \in \arg\min_{\mu \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2\tau} \mathcal{W}_2^2(\mu_\tau^n, \mu) + F[\mu|\mu_\tau^n] \right\}.$$
General case: the results

- Existence of weak measure solutions

\[
\frac{d}{dt} \int \phi(x) d\mu_1(x, t) = -\frac{1}{2} \int \int \nabla H_1(x - y) \cdot (\nabla \phi(x) - \nabla \phi(y)) d\mu_1(x) d\mu_1(y) \\
- \int \int \nabla K_1(x - y) \cdot \nabla \phi(x) d\mu_1(x) d\mu_2(y)
\]

\[
\frac{d}{dt} \int \psi(x) d\mu_2(x, t) = -\frac{1}{2} \int \int \nabla H_2(x - y) \cdot (\nabla \psi(x) - \nabla \psi(y)) d\mu_2(x) d\mu_2(y) \\
- \int \int \nabla K_2(x - y) \cdot \psi(x) d\mu_2(x) d\mu_1(y).
\]

as limit of the semi-implicit JKO scheme.

- Uniqueness in case \(H_j\) and \(K_j\) are \(W^{2,\infty}\), via a variant of the characteristics method.
Open problems and future work

- Open problem: uniqueness in the two species system for less regular potentials.

- Many species with nonlocal aggregation and nonlinear cross-diffusion terms: segregation. Ongoing project with M. Burger and A. Stevens.

- Derivation of multi-species continuum second order models via particle methods.

- Derivation of first order systems as damping dominated limits of second order systems.
End of the talk

Thank you for your attention!