Different types of phase transitions for a simple model of alignment of oriented particles

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KI-Net workshop on “Kinetic description of social dynamics: From consensus to flocking”
CSCAMM, November 8th, 2012
Goal: macroscopic description of some animal societies

- Local interactions without leader
- Emergence of macroscopic structures, phase transitions

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Modeling of interacting self-propelled particles

- Vicsek et al. (1995).
  Discrete in time (interval $\Delta t$), alignment only, synchronous reorientation.

  \[
  \text{New direction} = \text{Mean direction of neighboring particles at previous step} + \text{Noise}
  \]

  Simulations: phase transition phenomenon, emergence of coherent structures.

  Time-continuous version: relaxation (with constant rate $\nu$) towards the local mean direction.
  Hydrodynamic limit without phase transition phenomenon.

- Model presented here: making $\nu$ (and the intensity of the noise) a function of the (norm of the) local mean momentum.
Outline

1. Time-continuous Vicsek model with phase transition
   - Presentation of the model
   - Kinetic model – Homogeneous setting
   - Stability issues

2. Examples of different types of phase transitions
   - Second order (or continuous) phase transition
   - First order (discontinuous) phase transition – Hysteresis
Individual dynamics

Particles at positions: $X_1, \ldots, X_N$ in $\mathbb{R}^n$.

Orientations $\omega_1, \ldots, \omega_N$ in $\mathbb{S}$ (unit sphere).

\[
\begin{cases}
  dX_k = \omega_k dt \\
  d\omega_k = \nu P_{\omega_k} \bar{\omega}_k dt + \sqrt{2\tau} P_{\omega_k} \circ dB^k_t
\end{cases}
\]

Target direction:

\[
\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \frac{1}{N} \sum_{j=1}^{N} K(X_j - X_k) \omega_j.
\]

Setting $\nu = \nu(|J_k|)$ and $\tau = \tau(|J_k|)$, no singularity if $\frac{\nu(|J|)}{|J|}$ is Lipschitz.
Kinetic description

Assumptions: $K$ with finite second moment, and $K, |J| \mapsto \frac{\nu(|J|)}{|J|}$ and $\tau$ bounded Lipschitz.

**Theorem (following Bolley, Cañizo, Carrillo, 2012)**

Probability density function $f(x, \omega, t)$, as $N \to \infty$:

$$\partial_t f + \omega \cdot \nabla_x f + \nu(|J_f|) \nabla_\omega \cdot (P_{\omega \perp} \bar{\omega}_f f) = \tau(|J_f|) \Delta_\omega f$$

$$\bar{\omega}_f = \frac{J_f}{|J_f|}, \quad J_f = K \ast J, \quad J = \int_{\omega \in S} \omega f(x, \omega, t) \, d\omega.$$

Tool: coupling process + estimations.

\[
\begin{cases}
  d\bar{X}_k = \bar{\omega}_k \, dt \\
  d\bar{\omega}_k = \nu(|J_{f_N}^t|) P_{\omega_k} \bar{\omega}_{f_N} \, dt + \sqrt{2\tau(|J_{f_N}^t|)} P_{\omega_k} \circ dB_t^k \\
  f_N^t = \text{law}(\bar{X}_1, \bar{\omega}_1) = \text{law}(\bar{X}_k, \bar{\omega}_k)
\end{cases}
\]
Space-homogeneous version

Reduced equation, for a function $f(\omega, t)$:

\[
\partial_t f = Q(f),
\]

\[
Q(f) = -\nu(|J_f|) \nabla_\omega \cdot (P_{\omega \perp} \Omega_f f) + \tau(|J_f|) \Delta_\omega f,
\]

\[
\Omega_f = \frac{J_f}{|J_f|}, \quad J_f(t) = \int_S f(\omega, t) \omega \, d\omega.
\]

Key parameter: the conserved quantity $\rho = \int_S f$.
Writing $h(|J|) = \frac{\nu(|J|)}{\tau(|J|)}$, we get

\[
Q(f) = \tau(|J_f|) \nabla_\omega \cdot (e^{h(|J_f|) \omega \cdot \Omega_f} \nabla_\omega (e^{-h(|J_f|) \omega \cdot \Omega_f} f)).
\]

Main assumption: $|J| \mapsto h(|J|)$ is increasing. Its inverse: $\sigma$. 
Equilibria

Definitions: Fisher–von Mises distribution

\[ M_{\kappa,\Omega}(\omega) = \frac{e^{\kappa \omega \cdot \Omega}}{\int_{S} e^{\kappa \nu \cdot \Omega} \, d\nu} . \]

Orientation \( \Omega \in S \), concentration \( \kappa \geq 0 \).

Order parameter: \( c(\kappa) = |J_{M_{\kappa,\Omega}}| = \frac{\int_{0}^{\pi} \cos \theta \, e^{\kappa \cos \theta \sin^{n-2} \theta} \, d\theta}{\int_{0}^{\pi} e^{\kappa \cos \theta \sin^{n-2} \theta} \, d\theta} . \)

For \( \kappa_f = h(|J_f|) \), we can write \( Q \) under the form:

\[ Q(f) = \tau(|J_f|) \nabla \omega \cdot \left[ M_{\kappa_f,\Omega_f} \nabla \omega \left( \frac{f}{M_{\kappa_f,\Omega_f}} \right) \right] . \]

Equilibria: \( f_{eq} = \rho M_{\kappa,\Omega} \), for some \( \Omega \in S \).

Then \( |J_{f_{eq}}| = \rho |J_{M_{\kappa,\Omega}}| = \rho c(\kappa) \), and \( \kappa = \kappa_{f_{eq}} = h(|J_{f_{eq}}|) \).

Compatibility condition: \( \kappa = h(\rho c(\kappa)) \), i.e. \( \sigma(\kappa) = \rho c(\kappa) \).
Solutions to the compatibility condition

Uniform distribution $f = \rho$: always equilibrium.

Possible shapes for function $\frac{c}{\sigma}(=\frac{1}{\rho})$: 2 informations: $\to \frac{1}{\rho_c}$ as $\kappa \to 0$, and $\to 0$ as $\kappa \to \kappa_{max}$ (the maximum of $h(|J|)$).
Existence, uniqueness, regularity, positivity, bounds

Theorem

For an initial probability measure $f_0 \in H^s(\mathbb{S})$, (for an arbitrary $s$):

- Existence and uniqueness of a weak solution $f$.
- Global solution, in $C^\infty(\mathbb{R}_+^* \times \mathbb{S})$, and $f > 0$ for $t > 0$.
- Instantaneous regularity estimates and uniform bounds:

$$
\|f(t)\|_{H^{s+m}}^2 \leq C \left( 1 + \frac{1}{t^m} \right) \|f_0\|_{H^s}^2.
$$

Tool: spherical harmonics decomposition.
Nonlinearity: finite number of coefficients.
Main tool: Onsager free energy

Free energy: \( \mathcal{F}(f) = \int_S f \ln f - \Phi(|J_f|) \), with \( \frac{d\Phi}{d|J|} = h(|J|) \).

Dissipation: \( \mathcal{D}(f) = \tau(|J_f|) \int_S f |\nabla_{\omega} (\ln f - h(|J_f|)\omega \cdot \Omega_f)|^2 \geq 0 \).

\[ \frac{d}{dt} \mathcal{F} + \mathcal{D} = 0 \implies \mathcal{F}(f) \text{ decreasing towards } \mathcal{F}_\infty. \]

LaSalle’s principle

Limit set: \( \mathcal{E}_\infty = \{ f \in C^\infty(S) | \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty \} \).

\[ \lim_{t \to \infty} \inf_{g \in \mathcal{E}_\infty} \| f(t) - g \|_H^s = 0. \]

Refined: if the roots of \( \sigma(\kappa) = \rho c(\kappa) \) are isolated, there exists a solution \( \kappa_\infty \) such that:

\[ \lim_{t \to \infty} |J_f(t)| = \rho c(\kappa_\infty) \quad \text{and} \quad \forall s \in \mathbb{R}, \lim_{t \to \infty} \| f(t) - \rho M_{\kappa_\infty} \Omega_f(t) \|_H^s = 0. \]
Local analysis near uniform equilibria

Critical value: \( \rho_c = \lim_{\kappa \to 0} \frac{\sigma(\kappa)}{c(\kappa)} \in (0, +\infty] \).

Theorem: Strong unstability – Exponential stability

- \( \rho > \rho_c \): if \( J_{f_0} \neq 0 \), then \( f \) cannot converge to the uniform equilibrium.
- \( \rho < \rho_c \): There exists a universal constant \( \delta \) such that if \( \| f_0 - \rho \|_{H^s} < \delta \), then we have for all \( t \geq 0 \)

\[
\| f(t) - \rho \|_{H^s} \leq \frac{\| f_0 - \rho \|_{H^s} e^{-\lambda t}}{1 - \frac{1}{\delta} \| f_0 - \rho \|_{H^s}}, \quad \text{with} \quad \lambda = (n-1)\tau_0 (1 - \frac{\rho}{\rho_c}).
\]

Tools: linearization for the evolution of \( J_f \), and then energy estimates for the whole equation.
Proposition: weak stability/unstability

We denote by $\mathcal{F}_\kappa$ the value of $\mathcal{F}(\rho M_{\kappa \Omega})$ (independent of $\Omega \in \mathbb{S}$).

- $(\frac{\sigma}{c})'(\kappa) < 0$ (unstable): in any neighborhood of $\rho M_{\kappa \Omega}$, there exists $f_0$ such that $\mathcal{F}(f_0) < \mathcal{F}_\kappa$.

- $(\frac{\sigma}{c})'(\kappa) > 0$ (stable): If $\|f_0\|_{H^s} \leq K$, with $s > \frac{n-1}{2}$, then there exists $\delta > 0$ and $C$ (depending only on $K$ and $s$), such that if $\|f_0 - \rho M_{\kappa \Omega}\|_{L^2} \leq \delta$ for some $\Omega \in \mathbb{S}$, then for all $t \geq 0$, we have

$$\mathcal{F}(f) \geq \mathcal{F}_\kappa, \text{ and } \|f - \rho M_{\kappa \Omega f}\|_{L^2} \leq C \|f_0 - \rho M_{\kappa \Omega f_0}\|_{L^2}.$$ 

Tools: expansion of $\mathcal{F} - \mathcal{F}_\kappa$, and Sobolev interpolation.
Stronger stability: exponential convergence to equilibrium

**Theorem:** exponential stability in case \((\frac{\sigma}{c})'(\kappa) > 0\)

For all \(s > \frac{n-1}{2}\), there exist universal constants \(\delta > 0\) and \(C > 0\), such that if \(\| f_0 - \rho M_{\kappa, \Omega} \|_{H^s} < \delta\) for some \(\Omega \in \mathbb{S}\), there exists \(\Omega_\infty \in \mathbb{S}\) such that

\[
\| f - \rho M_{\kappa, \Omega_\infty} \|_{H^s} \leq C \| f_0 - \rho M_{\kappa, \Omega_0} \|_{H^s} e^{-\lambda t},
\]

with

\[
\lambda = \frac{c \tau(\sigma)}{\sigma'} \Lambda_{\kappa}(\frac{\sigma}{c})',
\]

where \(\Lambda_{\kappa}\) is the best constant for the following weighted Poincaré inequality:

\[
\langle |\nabla g|^2 \rangle_{M_{\kappa, \Omega}} \geq \Lambda_{\kappa} \langle (g - \langle g \rangle_{M_{\kappa, \Omega}})^2 \rangle_{M_{\kappa, \Omega}}
\]
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General statements in the case \((\sigma / c)' > 0\) for all \(\kappa\)

- If \(\rho < \rho_c\), then the solution converges exponentially fast towards the uniform distribution \(f_\infty = \rho\).
- If \(\rho = \rho_c\), the solution converges to the uniform distribution.
- If \(\rho > \rho_c\) and \(J_{f_0} \neq 0\), then there exists \(\Omega_\infty\) such that \(f\) converges exponentially fast to the von Mises distribution \(f_\infty = \rho M_\kappa \Omega_\infty\), where \(\kappa > 0\) is the unique positive solution to the equation \(\rho c(\kappa) = \sigma(\kappa)\).

We can then define \(c\) (order parameter) as a function of \(\rho\), and this function is continuous.

Critical exponent \(\beta\): when \(c(\rho) \asymp (\rho - \rho_c)^\beta\). Can be any number in \((0, 1]\), as one can artificially choose \(\sigma(\kappa) = c(\kappa)(1 + \kappa^{1/\beta})\).
Practical criteria for continuous phase transition

Lemma

If \( h(\frac{|J|}{J}) \) is a nonincreasing function of \(|J|\), then we are in the previous case of a continuous phase transition. In that case, the critical exponent \( \beta \), if it exists, can only take values in \([\frac{1}{2}, 1]\).

In that case, we also have a special cancellation (related to the so-called conformal Laplacian), which gives global exponential decay when \( \rho < \rho_c \):

Proposition

If \( \rho < \rho_c \), there exists a universal constant \( C \) such that we have for all \( t \geq 0 \)

\[
\| f(t) - \rho \|_{H^s} \leq C \| f_0 - \rho \|_{H^s} e^{-\lambda t}, \quad \text{with} \quad \lambda = (n - 1) \tau_{\text{min}} (1 - \frac{\rho}{\rho_c}).
\]
A special example

Focus: $\nu(|J|) = |J|$ and $\tau(|J|) = \frac{1}{1 + |J|}$ (related to the so-called “extrinsic noise”).
In that case, we get $h(|J|) = |J| + |J|^2$, so we are not indeed in the previous case.
We obtain $\sigma(\kappa) = \frac{1}{2}(\sqrt{1 + 4\kappa} - 1)$. 
The phase diagram

![Graph showing phase transitions for different densities and order parameters.](image)
Comparison of Free energy levels

\[ \mathcal{F}_K - \mathcal{F}(\rho) \]

Density \( \rho \)

- \( n=2 \) (red line)
- \( n=3 \) (blue line)
Rates of convergence

![Graph showing rates of convergence for different densities and von Mises distributions for n=2 and n=3.](image-url)
Numerical illustration of the hysteresis phenomena

Change of scale \( \tilde{f} = \frac{f}{\rho} \). The parameter \( \rho \) can now be considered as a free parameter that we let evolve in time.