Large-time rescaling behaviors of some rational type solutions to the Polubarinova-Galin equation with injection

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Abstract

The main goal of this paper is to give a precise description of rescaling behaviors of rational type global strong solutions to the Polubarinova-Galin equation. The Polubarinova-Galin equation is the reformulation of the zero surface tension Hele-Shaw problem with a single source at the origin by considering the moving domain as the Riemann mapping of the unit disk centered at the origin. The coefficients $\{a_k(t)\}_{k\geq 2}$ of the polynomial strong solution $f_{k_0}(\xi,t)=\sum_{i=1}^{k_0} a_i(t)\xi^i$ decay to zero algebraically as $t^{-\lambda_k}$ ($\lambda_k = k/2$) and the decay is even faster if the low Richardson moments vanish. The dynamics for global solutions are discussed as well.

Keywords: Hele-Shaw flow, Rescaling behavior.

1 Introduction

The present paper is mainly devoted to the following differential equation which arises from the reformulation of zero surface tension (ZST) Hele-Shaw flows with injection strength 2π at the origin, as in Richardson [10], that is:

$$Re\big[f_t(\xi,t)\overline{f'(\xi,t)\xi}\big]=1, \xi\in\partial B_1(0) \tag{1.1}$$

where $f(\xi,t): B_1(0)\to\Omega(t)$ is univalent and analytic in $\overline{B_1(0)}$, $f(0,t)=0$, $f'(0,t)>0$ and $\{\Omega(t)\}$ are the domains of the moving fluid. Equation (1.1)

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is called the Polubarinova-Galin equation since Galin and Polubarinova-Kochina first derived it and investigated the Riemann mapping method along these lines. A solution to equation (1.1) is said to be a strong solution for $t \in [0, b)$ if $f(\xi, t)$ is univalent and analytic in $\overline{B_1(0)}$, $f(0, t) = 0$, $f^{'}(\xi, 0) > 0$ and continously differentiable in t, $t \in [0, b)$. Equivalently, we obtain a strong solution $\Omega(t) = f(B_1(0), t)$ to the ZST Hele-Shaw problem with injection, where $\Omega(t)$ has a real analytic boundary and is simply connected. The existence and uniqueness of the P-G equation (locally in time) are proven in several different ways. Note that we put the restriction that $f(0, t) = 0$ and $f^{'}(0, t) > 0$ since we require the uniqueness of the conformal representation $f(\xi, t)$ for the domain $\Omega(t)$ and that the injection is always at the origin.

Define

$$O(E) = \left\{ f \mid f \text{ is univalent and analytic in } E, f(0) = 0, f^{'}(0) > 0 \right\}.$$

The interesting feature of equation (1.1) is that starting with a function $f(\xi, 0) \in O(\overline{B_1(0)})$, there exists a unique strong solution $f(\xi, t)$ at least for a short time. In Reissig and von Wolfersdorf [9], the solvability of a short time strong solution may be proven using the nonlinear abstract Cauchy-Kovalevskaya Theorem. In [2], the author reformualtes (1.1) to be

$$f_t(\xi, t) = \frac{f^{'}(\xi, t)\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{\mid f^{'}(z, t) \mid^2} \frac{z + \xi}{z - \xi} \frac{dz}{z}, f(\xi, t) \in O(\overline{B_1(0)})$$

and gives (1.1) an easier proof of its uniqueness and existence in the case that the initial functions are rational functions in $O(\overline{B_1(0)})$. Furthermore, the author shows that the pole structure of the strong solution is same as that of the initial rational function, but all poles except the one at infinity may move around. Particularly, starting with a degree n polynomial mapping $f_n(\xi, 0) \in O(\overline{B_1(0)})$, the solution to (1.1) $f_n(\xi, t)$ is also a polynomial of the same degree. This reformulation provides a new treatment for equation (1.1).

In Huntingford [5], it is shown that not any given function $f(\xi, 0) \in O(\overline{B_1(0)})$ can produce a strong global solution. Some solutions blow up in finite time due to the formation of cusp or double points. However, it is proven in Gustafsson, Prokhorov and Vasil'ev [3] that starting with a starlike mapping $f(\xi, 0) \in O(\overline{B_1(0)})$, the strong solution to (1.1) $f(\xi, t)$ is global. We show in section 2 that the condition of starlikeness is not a necessary condition for the initial functions of global strong solutions to (1.1). For a global solution $f(\xi, t)$, the initial domain $\Omega(0) = f(B_1(0), 0)$

can be as irregular as a nonstarlike domain. Note that convex domains must be strongly starlike domains as stated in Pommerenke [8].

In this paper, we focus on the global strong solutions of the rational type. We give rescaling behaviors of global strong polynomial solutions and demonstrate that of two rational type solutions which are not polynomial. There is a large class of polynomial functions in $O(\overline{B_1(0)})$ which give rise to global strong solutions to (1.1); for example, the subset of strongly starlike functions of order < 1, $\{\sum_{i=1}^{k} a_i \xi^i | \sum_{i=2}^{k} i \mid a_i \mid < \mid a_1 \mid, a_1 > 0, k \in N\}$ as shown in Pommerenke [8]. As shown in Gluchoff and Hartmann [1], the subset $\{\sum_{i=1}^{k} a_i \xi^i \mid \sum_{i=2}^{k} i \mid a_i \mid < \mid a_1 \mid, a_1 > 0, k \in N\}$ is too restrictive compared with the set of starlike functions in $O(\overline{B_1(0)})$. For example, $P_4 = \{\frac{\xi}{r} - \frac{15}{14}(\frac{\xi}{r})^2 + \frac{4}{7}(\frac{\xi}{r})^3 - \frac{1}{7}(\frac{\xi}{r})^4 \mid r > 1\} \subset O(\overline{B_1(0)})$ is a subset of strongly starlike functions of order < 1 and hence every function in the set P_4 is the initial function of a global polynomial solution to (1.1). Note that the curvature of those domains $\{f_4(B_1(0)) \mid f_4 \in P_4\}$ has no upper bound.

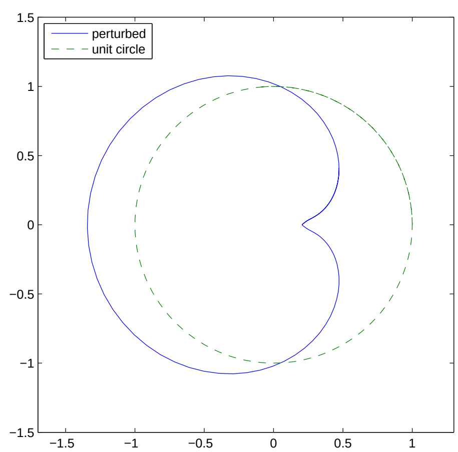

Figure 1.1: The perturbed domain is obtained by dividing $f(B_1(0))$ by the square root of $\frac{1}{\pi} \mid f(B_1(0)) \mid$ where $f(\xi) = \frac{\xi}{1.1} - \frac{15}{14}(\frac{\xi}{1.1})^2 + \frac{4}{7}(\frac{\xi}{1.1})^3 - \frac{1}{7}(\frac{\xi}{1.1})^4$, and the domain has area π.

This current paper is organized as follows. In section 2, we show that starlikeness is not a necessary condition for the initial functions of global strong polynomial solutions to (1.1). In section 3, we show a rescaling behavior of the global strong solution $f(\xi, t)$. In section 4, there are more precise rescaling results in the case that $f(\xi, t)$ is a global strong polynomial solution. In particular, $a_k(t)(\sqrt{2t})^k$ approaches the constant moment $\overline{M_{k-1}}$ algebraically $(\frac{1}{t^2})$ as time t goes to ∞ for $k \geq 2$ if $f(\xi,t) = \sum_{i=1}^{k_0} a_i(t)\xi^i$. Furthermore, in section 5, we also demonstrate the rescaling behavior of two types of rational solutions.

2 The dynamics of global solutions

Recall the definition of a starlike function as in Gustafsson, Prokhorov and Vasil'ev [3] and Pommerenke [8]. A function $f \in O(B_1(0))$, is said to be from S^*_α where $\alpha \in (0,1]$ if for all $\xi \in B_1(0)$,

$$\Big|\arg \frac{\xi f'(\xi)}{f(\xi)}\Big| < \alpha\frac{\pi}{2}.$$

Such a function is also called a strongly starlike function of order α. Furthermore, we define $f(\xi)$ to be a **strongly starlike function of exact order** α if

$$\sup_{\xi\in B_1(0)} \Big|\arg \frac{\xi f'(\xi)}{f(\xi)}\Big| = \alpha\frac{\pi}{2}.$$

As stated in [8], the domain $f(B_1(0))$ must satisfy

$$\tau x \in f(B_1(0)) \quad \text{for all} \quad x \in f(B_1(0)), \tau \in (0,1).$$

In this section, we aim to show that starlikeness is not a necessary condition for the initial function $f(\xi,0) \in O(\overline{B_r(0)})$ of a global strong solution to equation (1.1) $f(\xi,t)$ by finding some implicit counterexamples. In section 2.1, we prove the existence of a function in $O(\overline{B_1(0)})$ which is a strongly starlike function of exact order 1 in order to construct the counterexample in section 2.2.

2.1 The existence of a strongly starlike function of exact order 1

We first observe some properties of the solutions to the ZST Hele-Shaw problem with suction at the origin. The P-G type equation for the case of suction with strength 2π is

$$Re\big[f_t(\xi,t)\overline{f'(\xi,t)\xi}\big] = -1, \xi \in \partial B_1(0). \tag{2.1}$$

The time reversal of a solution to the ZST Hele-Shaw problem with injection at the origin is a solution to the ZST Hele-Shaw problem with suction at the origin. It is known that the solutions to the ZST Hele-Shaw problem with suction at the origin always blow up before all the fluid is sucked out except in the special case that the initial domain is a disk centered at the origin. The types of blow-ups are also discussed in Howison [4].

Also, in the suction case, there exists a short time strong solution if the initial function $f(\xi, 0) \in O(\overline{B_1(0)})$. The existence and uniqueness proof of (2.1) can be found in Reissig and von Wolfersdorf [9] and equation (2.1) is explained as the abstract Cauchy-Kovalevskaya type equation.

First, we quote Theorem 2.1 in Gustafsson, Prokhorov and Vasil'ev [2].

Lemma 2.1. *([2]) Let $f_0 \in S^*_\alpha, \alpha \in (0,1]$ be analytic and univalent in a neighborhood of $\overline{B_1(0)}$. Then the classical solution to the Polubarinova-Galin equation forms a subordination chain of strongly starlike functions of order $\alpha(t)$ with a strictly decreasing $\alpha(t)$ during the time of existence.*

Lemma 2.2. *There exists $f(\xi) \in O(\overline{B_1(0)})$ which is a strongly starlike function of exact order* 1*.*

Proof. Let $F(\xi) \in O(B_1(0))$ be a nonstarlike function. Note that

$$\lim_{\xi \to 0} Re \frac{\xi F^{'}(\xi)}{F(\xi)} = 1.$$

Therefore, there exists $0 < r_1 < 1$ such that

$$Re \frac{\xi F^{'}(\xi)}{F(\xi)} > 0, \xi \in B_{r_1}(0).$$

Define

$$r_0 = \max \Big\{ r \Big| \min_{\xi \in \overline{B_r(0)}} Re \frac{F^{'}(\xi)\xi}{F(\xi)} \geq 0 \Big\}.$$

Since $F(\xi)$ is a nonstarlike function, $r_0 < 1$. Also r_0 satisfies

$$\begin{cases} Re \frac{F^{'}(\xi)\xi}{F(\xi)} > 0, & \text{for} \quad \xi \in B_{r_0(0)} \\ \min_{\xi \in \overline{B_{r_0}(0)}} Re \frac{F^{'}(\xi)\xi}{F(\xi)} = 0, & \\ \min_{\xi \in \overline{B_r(0)}} Re \frac{F^{'}(\xi)\xi}{F(\xi)} < 0, & r > r_0. \end{cases}$$

Define $f(\xi) = F(r_0 \xi)$, then $f(\xi)$ satisfies

$$\begin{cases} Re \frac{f^{'}(\xi)\xi}{f(\xi)} > 0, & \text{for} \quad \xi \in B_1(0), \\ \min_{\xi \in \overline{B_1(0)}} Re \frac{f^{'}(\xi)\xi}{f(\xi)} = 0, & \\ f(\xi) \in O(B_{\frac{1}{r_0}}). & \end{cases}$$

This is equivalent to the following statement that

$$\begin{cases} \left|\arg \frac{f'(\xi)\xi}{f(\xi)}\right| < \frac{\pi}{2}, & \text{for} \quad \xi \in B_1(0) \\ \max_{\xi\in\overline{B_1(0)}} \left|\arg \frac{f'(\xi)\xi}{f(\xi)}\right| = \frac{\pi}{2}, \\ f(\xi) \in O(B_{\frac{1}{r_0}}(0)). \end{cases}$$

Therefore, $f(\xi)$ is exactly what we want. □

Remark 2.1. The value $r_0 \geq \tanh\frac{\pi}{4} \approx 0.656$. The constant $\tanh\frac{\pi}{4}$ is called the radius of starlikeness in Pommerenke [8].

Theorem 2.3. *Given $f(\xi,0) \in O(\overline{B_1(0)})$ which is a strongly starlike function of exact order* 1*, then the solution to*

$$\begin{cases} Re\left[f_t\overline{f'\xi}\right] & = -1, \\ f(\xi,t)\mid_{t=0} & = f(\xi,0) \end{cases}$$

is not strongly starlike as long as the solution exists.

Proof. The proof follows from Lemma 2.1 directly since the time reversal of a ZST Hele-Shaw problem solution with suction at the origin is a solution to the ZST Hele-Shaw problem with injection at the origin. □

2.2 Examples of nonstarlikeness functions which produce global solutions

Now, given that $f(\xi,0)$ satisfies Theorem 2.3, then there exists $s > 0$ such that the solution to (2.1) $f(\xi,s)$ is not a strongly starlike function, but is in $O(\overline{B_1(0)})$. However, the solution to equation (1.1) has global existence if $F(\xi,0) = f(\xi,s)$. Therefore, the following theorem is proven.

Theorem 2.4. *There exists a nonstarlike mapping $F(\xi,0) \in O(\overline{B_1(0)})$ which produces a global strong solution $\{F(\xi,t)\}_{t\geq 0}$ to* (1.1).

Remark 2.2. In Sakai [11], it is proven that any global weak solution eventually becomes strongly starlike. Since the weak solutions are exactly the strong solutions as long as strong solutions exist, we can conclude that any global strong solution eventually becomes strongly starlike no matter whether the initial function is a starlike function or not.

Theorem 2.5. *There exists $F(\xi,0) \in O(\overline{B_1(0)})$ which is* **polynomial** *and nonstarlike such that the strong solution to* (1.1) *$\{F(\xi,t)\}$ is global.*

Proof. If we can find a polynomial function in $O(B_1(0))$ which is nonstarlike, then by following the proofs of Lemma 2.2, Theorem 2.3 and Theorem 2.4, we can obtain a nonstarlike polynomial function $F(\xi,0) \in O(\overline{B_1(0)})$ such that the strong solution to (1.1) $F(\xi,t)$ is global.
Claim: There exist polynomial functions in $O(B_1(0))$ which are nonstarlike.

Proof. (of claim)
(1) There exists $g(\xi) \in O(B_1(0))$ which is nonstarlike.
(2) There exists $r > 1$ such that $g(\frac{\xi}{r})$ is a nonstarlike function but $g(\frac{\xi}{r}) \in O(\overline{B_{r_0}(0)})$ for some $r_0 > 1$.
(3) Denote $h(\xi) = g(\frac{\xi}{r}) = \sum_{i=1}^{\infty} a_i \xi^i \in O(\overline{B_{r_0}(0)})$. There exists $M > 0$ such that $\mid a_i \mid \leq M r_0^{-i}$.
(4) Define $g_k(\xi) = \sum_{i=1}^{k} a_i \xi^i$. Since for $r' < r$,

$$\max_{\xi\in\overline{B_{r'}(0)}} \left| g_k^{'}(\xi) - h^{'}(\xi) \right| \leq \sum_{i=k+1}^{\infty} i M r_0^{-i},$$

there exists $k_0 \in N$ such that $\{g_k(\xi)\}_{k\geq k_0} \subset O(\overline{B_1(0)})$.
(5)Since

$$\max_{\xi\in\overline{B_1(0)}} \left| g_k^{'}(\xi) - h^{'}(\xi) \right| \leq \sum_{i=k+1}^{\infty} i M r_0^{-i}$$

and

$$\max_{\xi\in\overline{B_1(0)}} \mid g_k(\xi) - h(\xi) \mid \leq \sum_{i=k+1}^{\infty} M r_0^{-i},$$

therefore

$$\lim_{k\to\infty} \max_{\xi\in\overline{B_1(0)}} \left| Re\frac{h^{'}\xi}{h} - Re\frac{g_k^{'}\xi}{g_k} \right| = 0. \tag{2.2}$$

Since h is not starlike, there exists $\xi_0 \in B_1(0)$ such that $Re\frac{h^{'}(\xi_0)\xi_0}{h(\xi_0)} < 0$. Then there exists $s_0 \in N$ such that $Re\frac{g_k^{'}(\xi_0)\xi_0}{g_k(\xi_0)} < 0$ for $k \geq s_0$ by (2.2). Hence $\{g_k(\xi)\}_{k\geq s_0}$ are nonstarlike functions. Let $M_0 = \max\{k_0, s_0\}$, then $\{g_k(\xi)\}_{k\geq M_0}$ are exactly those functions we are looking for in the claim. □

□

3 Rescaling behavior

Assume $f(\xi,t) = \sum_{i=1}^{\infty} a_i \xi^i$ is a global strong solution to equation (1.1). First, we quote two lemmas in Kuznetsova [6] which state the properties of those coefficients $\{a_i(t)\}_{i\geq 1}$ to help us to see rescaling behaviors of $f(\xi,t)$.

Lemma 3.1. *([6]) The function $a_1^2(t) - 2t$ is nondecreasing for every strong solution $f(\xi,t)$. Moreover,*

$$a_1^2(t) - 2t \leq \frac{1}{\pi} \mid \Omega(0) \mid$$

where $\mid \Omega(0) \mid$ is the initial area.

Lemma 3.2. *([6]) Suppose that $f(\xi,t)$ is a strong solution to* (1.1) *on* $[0,b)$*. Then the function*

$$g(t) \equiv \sum_{k=2}^{\infty} k \mid a_k^2(t) \mid$$

is nonincreasing for every $t \in [0,b)$; moreover, $g(t) \leq \frac{|\Omega(0)|}{\pi} - a_1^2(0)$.

Remark 3.1. If $f(\xi,t)$ is a global strong solution to (1.1), then

$$2t + a_1^2(0) \leq a_1^2(t) \leq 2t + \frac{1}{\pi} \mid \Omega(0) \mid$$

which implies

$$\lim_{t\to\infty} \frac{a_1(t)}{\sqrt{2t}} = 1.$$

Applying Lemma 3.1 and Lemma 3.2, the following rescaling behavior holds.

Theorem 3.3. *For $\xi \in B_1(0)$, $\lambda \in [0, \frac{1}{2})$*

$$\lim_{t\to\infty} \left| \frac{1}{\sqrt{2t}} f(\xi,t) - \xi \right| t^{\lambda} = 0.$$

Proof.

$$
\begin{aligned}
&\Big|\frac{1}{\sqrt{2t}}\big[a_1(t)(\xi)+a_2(t)(\xi)^2+\cdots\big]-\xi\Big|t^\lambda\\
=&\Big|\Big(\frac{a_1(t)}{\sqrt{2t}}-1\Big)\xi+\frac{1}{\sqrt{2t}}\big(a_2(t)\xi^2+a_3(t)\xi^3+\cdots\big)\Big|t^\lambda\\
\le&\Big|\frac{a_1(t)-\sqrt{2t}}{\sqrt{2t}}\Big|+\frac{1}{\sqrt{2t}}\sum_{i=2}^{\infty}\mid a_i(t)\mid\mid\xi\mid^i t^\lambda\\
\le&\frac{a_1^2(t)-2t}{\sqrt{2t}(a_1(t)+\sqrt{2t})}+\frac{1}{\sqrt{2t}}\Big(\sum_{i=2}^{\infty}\mid a_i(t)\mid^2 i\Big)^{1/2}\Big(\sum_{i=2}^{\infty}\mid\xi\mid^{2i}\frac{1}{i}\Big)^{1/2}t^\lambda\\
=&\frac{g(0)+a_1^2(0)-g(t)}{\sqrt{2t}(a_1(t)+\sqrt{2t})}+\frac{1}{\sqrt{2t}}\sqrt{g(t)}\Big(\sum_{i=2}^{\infty}\mid\xi\mid^{2i}\frac{1}{i}\Big)^{1/2}t^\lambda\\
\le&\frac{g(0)+a_1^2(0)}{\sqrt{2t}(a_1(t)+\sqrt{2t})}+\frac{1}{\sqrt{2t}}\sqrt{g(0)}\Big(\sum_{i=2}^{\infty}\mid\xi\mid^{2i}\frac{1}{i}\Big)^{1/2}t^\lambda
\end{aligned}
$$

which goes to 0 as t goes to ∞. □

4 The rescaling behavior of global strong polynomial solutions and moments

Given a family of domains $\{\Omega(t)\}_{t\geq 0}$ which is a solution to the zero surface tension Hele-Shaw problem with injection 2π, then the Richardson complex moments are defined as

$$M_k(t)=\frac{1}{\pi}\int_{\Omega(t)}z^n dxdy, z=x+iy.$$

These moments satisfy

$$\frac{d}{dt}M_k(t)=2\delta_k(0).$$

Here, we assume that $f(\xi,t)=\sum_{i=1}^n a_i\xi^i$ is a global strong degree n polynomial solution to the P-G equation (1.1), and that $\Omega(t)=f(B_1(0),t)$. By the result in Richardson [10], the moments $\{M_k(f(\xi,t))\}_{0\leq k\leq n-1}$ can be represented by these coefficients $\{a_k\}_{k\geq 1}$ as

$$M_k(f(\xi,t))=\sum_{i_1,\cdots,i_{k+1}}i_1a_{i_1}a_{i_2}\cdots a_{i_{k+1}}\overline{a_{i_1+\cdots+i_{k+1}}}. \tag{4.1}$$

The zero moment $M_0(t) = \sum_{i=1}^{n} i \mid a_i(t) \mid^2 = 2t + M_0(0) = a_1^2(t) + g(t)$, and is $\frac{1}{\pi} \mid \Omega(t) \mid$ where $\mid \Omega(t) \mid$ is the area of the domain $\Omega(t)$.

In Huntingford [5], it is observed that given $a_1(0) > 0$ and $\{a_i(0)\}_{2\leq i\leq 3} \subset R$ such that $\frac{a_2(0)}{a_1(0)}$ and $\frac{a_3(0)}{a_1(0)}$ satisfy some constraint stated in Huntingford [5], then the strong solution $a_1(t)\xi + a_2(t)\xi^2 + a_3(t)\xi^3$ to (1.1) is global. Coefficients $a_2(t)$ and $a_3(t)$ are expressed in terms of $a_1(t)$ and $\{M_k\}_{1\leq k\leq 2}$ as follows:

$$a_2(t) = M_1 \frac{a_1^2(t)}{3M_2 + a_1^4(t)}$$

and

$$a_3(t) = \frac{M_2}{a_1^3(t)}.$$

The moments in this case are real-valued as well since $\{a_i(t)\}_{1\leq i\leq 3}$ are real-valued. We can see that the coefficients $\{a_k\}_{2\leq k\leq 3}$ in this example satisfy

$$\lim_{t\to\infty} a_2(t)(a_1(t))^2 = M_1 = \overline{M_1}$$

and

$$\lim_{t\to\infty} a_3(t)(a_1(t))^3 = M_2 = \overline{M_2}.$$

Moreover,

$$a_2(t)a_1^2(t) - \overline{M_1} = a_2(t)a_1^2(t) - M_1 = -\frac{3M_1M_2}{3M_2 + a_1^4(t)} = O\Big(\frac{1}{a_1^4}\Big);$$

$$a_3(t)a_1^3(t) - \overline{M_2} = a_3(t)a_1^3(t) - M_2 = 0.$$

We discuss the general form of this kind of decay of coefficients in subsection 4.1, and give rescaling behaviors of global strong polynomial solutions to (1.1) in subsection 4.2. Furthermore, if the lower Richardson moments vanish, the decay rate of coefficients is much faster and the global polynomial solution to (1.1) has better rescaling behaviors.

4.1 Decay of $\{a_i(t)\}_{i\geq 2}$ and the Richardson complex moments

Lemma 4.1. *If $f(\xi,t)$ is a global strong polynomial solution of degree $n \geq 2$ to the P-G equation (1.1), then for $k \geq 2$*

$$\lim_{t\to\infty} a_1^k \overline{a_k} = M_{k-1}. \tag{4.2}$$

Furthermore, for $k \geq 2$,

$$M_{k-1} - a_1^k \overline{a_k} = O\Big(\frac{1}{a_1^4(t)}\Big).$$

Proof.
Step1: Since $a_n = \overline{M_{n-1}} a_1^{-n}$, it is clear that the result holds for $k = n$.
Step2: Assume that the results hold for $n \geq k \geq n - s_0$ where $s_0 + 1 \leq n - 2$. Prove by induction.
Claim: The result (4.2) holds for $k = n - (s_0 + 1)$.

Proof. (of claim) For $k = n - (s_0 + 1)$,

$$\begin{aligned}
&M_{k-1}(f(\xi,t)) \\
=&\sum_{i_1,\cdots,i_k} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}} \\
=&a_1^k \overline{a_k} + \sum_{i_1,\cdots,i_k; \sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}} \\
=&a_1^k \overline{a_k} + O\left(\frac{1}{a_1^2}\right)
\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} a_1^k \overline{a_k} = M_{k-1}.$$

□

Furthermore, since (4.2),

$$M_{k-1} - a_1^k \overline{a_k} = \sum_{i_1,\cdots,i_k; \sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}} = O\left(\frac{1}{a_1^4(t)}\right).$$

□

For the case that the lower Richardson moments disappear, better qualitative properties for the coefficients $\{a_i(t)\}_{i\geq 2}$ are given as follows.

Lemma 4.2. *Let $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$ and assume $n_0 \geq 2$, then*

$$\lim_{t\to\infty} a_1^{n_0+1} a_{n_0+1} = \overline{M_{n_0}}; \tag{4.3}$$

$$\lim_{t\to\infty} a_1^k a_k = \overline{M_{k-1}}, k > n_0 + 1; \tag{4.4}$$

and

$$\lim_{t\to\infty} a_1^{n_0+1} a_k = 0, 2 \leq k \leq n_0. \tag{4.5}$$

Proof. We split the proof into two parts.

claim1:

$$\lim_{t\to\infty} a_1^{n_0+1} a_{n_0+1} = \overline{M_{n_0}};$$

and

$$\lim_{t\to\infty} a_1^k a_k = \overline{M_{k-1}}, k > n_0 + 1.$$

Proof. (of claim1) As shown in (4.2). ◻

claim2:

$$\lim_{t\to\infty} a_1^{n_0+1} a_k = 0, 2 \le k \le n_0.$$

Proof. (of claim2) In (4.1), the Richardson moments have the following representations:

$$\begin{aligned} & M_k(f(\xi,t)) \\ =& \sum_{i_1,\cdots,i_{k+1}} i_1 a_{i_1} a_{i_2} \cdots a_{i_{k+1}} \overline{a_{i_1+\cdots+i_{k+1}}} \\ =& a_1^{k+1}\overline{a_{k+1}} + \sum_{i_1,\cdots,i_{k+1};\sqcap_{j=1}^{k+1} i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_{k+1}} \overline{a_{i_1+\cdots+i_{k+1}}}. \end{aligned}$$

This means

$$\overline{a_{k+1}} = \frac{1}{a_1^{k+1}} \Big[M_k - \sum_{i_1,\cdots,i_{k+1};\sqcap_{j=1}^{k+1} i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_{k+1}} \overline{a_{i_1+\cdots+i_{k+1}}} \Big]. \quad (4.6)$$

Taking complex conjugate of (4.6), the above becomes

$$a_{k+1} = \frac{1}{a_1^{k+1}} \Big[\overline{M_k} - \sum_{i_1,\cdots,i_{k+1};\sqcap_{j=1}^{k+1} i_j \neq 1} i_1 \overline{a_{i_1} a_{i_2} \cdots a_{i_{k+1}}} a_{i_1+\cdots+i_{k+1}} \Big]. \quad (4.7)$$

Hence

$$a_{k+1} = \frac{1}{a_1^{k+1}} \Big[\overline{M_k} - O\Big(\frac{1}{a_1^4}\Big) \Big]. \quad (4.8)$$

By substituting $n_0 - 1$ for k in (4.8) and applying the fact that $M_{n_0-1} = 0$, we can first show that

$$\lim_{t\to\infty} a_1^{n_0+1} a_{n_0} = 0.$$

If $n_0 = 2$, we are done. Assume $n_0 \ge 3$ now and finish the proof by induction.

Assume for some s_0 where $0 \leq s_0 \leq n_0 - 3$,

$$\lim_{t\to\infty} a_1^{n_0+1} a_{n_0-s} = 0, 0 \leq s \leq s_0. \tag{4.9}$$

We need to show that (4.9) holds for $s = s_0 + 1$. In (4.7), by substituting $n_0 - (s_0+1)$ for $k+1$, we have

$$a_{n_0-(s_0+1)} = \frac{-1}{a_1^{n_0-(s_0+1)}} \Big[\sum_{i_1,\cdots,i_{n_0-(s_0+1)};\sqcap_{j=1}^{n_0-(s_0+1)} i_j \neq 1} i_1 \overline{a_{i_1} a_{i_2} \cdots a_{i_{n_0-(s_0+1)}}} a_{i_1+\cdots+i_{n_0-(s_0+1)}} \Big].$$

However, if $\sqcap_{j=1}^{n_0-(s_0+1)} i_j \neq 1$, then $i_1 + \cdots + i_{n_0-(s_0+1)} \geq n_0 - s_0$. Therefore, $a_{i_1+\cdots+i_{n_0-(s_0+1)}} = O(\frac{1}{a_1^{n_0+1}})$ due to the assumption in (4.9), results (4.3) and (4.4). Hence

$$i_1 \overline{a_{i_1} a_{i_2} \cdots a_{i_{n_0-(s_0+1)}}} a_{i_1+\cdots+i_{n_0-(s_0+1)}} = a_1^{n_0-s_0-2} O\Big(\frac{1}{a_1^{n_0+1}}\Big).$$

Finally, $a_{n_0-(s_0+1)}$ equals to

$$\frac{-1}{a_1^{n_0-(s_0+1)}} \Big[\sum_{i_1,\cdots,i_{n_0-(s_0+1)};\sqcap_{j=1}^{n_0-(s_0+1)} i_j \neq 1} i_1 \overline{a_{i_1} a_{i_2} \cdots a_{i_{n_0-(s_0+1)}}} a_{i_1+\cdots+i_{n_0-(s_0+1)}} \Big] = O\Big(\frac{1}{a_1^{n_0+2}}\Big)$$

and this implies

$$\lim_{t\to\infty} a_{n_0-(s_0+1)} a_1^{n_0+1} = 0.$$

This says (4.9) also holds for $s = s_0 + 1$.

Hence, claim2 is proven by induction. □

By claim1 and claim2, the proof for Lemma 4.2 is done. □

4.2 The rescaling behavior of global polynomial solutions

Theorem 4.3. *Let $f(\xi, t)$ be a strong polynomial global solution to* (1.1). *We can see the following rescaling behaviors:*
(i)

$$\lim_{t\to\infty} \Big[f(\xi,t) - \sqrt{2t}\xi - \frac{M_0(0)}{2\sqrt{2t}}\xi \Big](2t) = \overline{M_1}\xi^2.$$

(ii)Let $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$. Then

$$\lim_{t\to\infty} \Big[f(\xi,t) - \sqrt{M_0(0) + 2t}\xi \Big](2t)^{\frac{n_0+1}{2}} = \overline{M_{n_0}}\xi^{n_0+1}.$$

(iii)

$$\lim_{t\to\infty}\Big[f(\sqrt{2t}\xi,t)-\Big(2t+\frac{M_0(0)}{2}\Big)\xi-\sum_{k=2}^{n}\overline{M_{k-1}}\xi^k\Big](\sqrt{2t})^2=\sum_{k=2}^{n}\frac{-M_0(0)}{2}(k)\overline{M_{k-1}}\xi^k.$$

Proof. (i)

$$\begin{aligned}
&f(\xi,t)-\sqrt{2t}\xi\\
=&\big(a_1(t)\xi-\sqrt{2t}\xi\big)+\big(f(\xi,t)-a_1(t)\xi\big)\\
=&\frac{a_1^2(t)-2t}{a_1(t)+\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big)\\
=&\frac{a_1^2(0)+g(0)-g(t)}{a_1(t)+\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big)\\
=&\frac{a_1^2(0)+g(0)}{a_1(t)+\sqrt{2t}}\xi-\frac{g(t)}{a_1(t)+\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big)\\
=&\frac{a_1^2(0)+g(0)}{2\sqrt{2t}}\xi+\Big[\frac{a_1^2(0)+g(0)}{a_1(t)+\sqrt{2t}}-\frac{a_1^2(0)+g(0)}{2\sqrt{2t}}\Big]\xi-\frac{g(t)}{a_1(t)+\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big).
\end{aligned}$$

Here

$$g(t)=O\Big(\frac{1}{t^2}\Big)\quad\text{and}\quad\Big[\frac{a_1^2(0)+g(0)}{a_1(t)+\sqrt{2t}}-\frac{a_1^2(0)+g(0)}{2\sqrt{2t}}\Big]=O\Big(\frac{1}{t^{\frac{3}{2}}}\Big).$$

Hence

$$\begin{aligned}
&f(\xi,t)-\sqrt{2t}\xi\\
=&\frac{a_1^2(0)+g(0)}{2\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big)+O\Big(\frac{1}{t^{\frac{3}{2}}}\Big)\\
=&\frac{M_0(0)}{2\sqrt{2t}}\xi+\big(f(\xi,t)-a_1(t)\xi\big)+O\Big(\frac{1}{t^{\frac{3}{2}}}\Big).
\end{aligned}$$

Aslo

$$\lim_{t\to\infty}(2t)\big(f(\xi,t)-a_1(t)\xi\big)=\overline{M_1}\xi^2,$$

therefore,

$$\lim_{t\to\infty}\Big[f(\xi,t)-\sqrt{2t}\xi-\frac{M_0(0)}{2\sqrt{2t}}\xi\Big](2t)=\overline{M_1}\xi^2.$$

(ii)

$$\begin{aligned} & f(\xi,t)-\sqrt{M_0(0)+2t}\xi \\ =& \big(a_1(t)-\sqrt{M_0(0)+2t}\big)\xi+\sum_{i=2}^{n} a_i\xi^i \\ =& I+II \end{aligned}$$

where $I=(a_1(t)-\sqrt{M_0(0)+2t})\xi$ and $II=\sum_{i=2}^{n} a_i\xi^i$.
(a)**Claim1:**

$$\lim_{t\to\infty}(2t)^{\frac{1+n_0}{2}}(II)=\overline{M_{n_0}}\xi^{n_0+1}. \tag{4.10}$$

Proof. (of claim1)
(1)If $n_0=1$, it is trivial since $\lim_{t\to\infty} a_1^k a_k=\overline{M_{k-1}}$ for $k\geq 2$ as (4.2) stated.
(2)If $n_0\geq 2$, by equation (4.3), (4.4) and (4.5) in Lemma 4.2,

$$\lim_{t\to\infty}(2t)^{\frac{1+n_0}{2}}(II)=\overline{M_{n_0}}\xi^{n_0+1}.$$

Therefore, claim1 is proven. □

(b)**Claim2:**

$$\lim_{t\to\infty}(2t)^{n_0+1}(I)=0. \tag{4.11}$$

Proof. (of claim2) Since

$$\begin{aligned} I &= a_1(t)-\sqrt{M_0(0)+2t} \\ &= \frac{a_1^2(t)-M_0(0)-2t}{a_1(t)+\sqrt{M_0(0)+2t}} \\ &= \frac{-g(t)}{a_1(t)+\sqrt{M_0(0)+2t}} \end{aligned}$$

where $g(t)=O(\frac{1}{a_1^{2n_0+2}})$ by (4.5), (4.3) and (4.4) in Lemma 4.2, we have

$$\lim_{t\to\infty}(2t)^{n_0+1}\big(a_1(t)-\sqrt{M_0(0)+2t}\big)=0.$$

□

By Claim1 and Claim2, (ii) is proven. □

(iii)

$$f(\sqrt{2t}\xi,t)=\sum_{i=1}^{n}a_i(t)\big(\sqrt{2t}\big)^i\xi^i.$$

$$\Big[f(\sqrt{2t}\xi,t)-\Big(2t+\frac{M_0(0)}{2}\Big)\xi-\sum_{k=2}^{n}\overline{M_{k-1}}\xi^k\Big]$$

$$=\Big(a_1(t)\sqrt{2t}-\big(\sqrt{2t}\big)^2-\frac{M_0(0)}{2}\Big)\xi+\sum_{k=2}^{n}\big(a_k(t)\big(\sqrt{2t}\big)^k-\overline{M_{k-1}}\big)\xi^k.$$

In order to prove (iii), it is enough to show the following claim.
Claim: For $k\geq 2$

$$\lim_{t\to\infty}\big(a_k\big(\sqrt{2t}\big)^k-\overline{M_{k-1}}\big)\big(\sqrt{2t}\big)^2=-\frac{M_0(0)}{2}(k)\overline{M_{k-1}} \tag{4.12}$$

and

$$\lim_{t\to\infty}\Big(a_1(t)\sqrt{2t}-\big(\sqrt{2t}\big)^2-\frac{M_0(0)}{2}\Big)(2t)=0. \tag{4.13}$$

Proof. (of claim) The proof for (4.13) is easy. We will now focus on the proof of (4.12). Equation (4.6) states

$$\overline{a_k}=\frac{1}{a_1^k}\Big[M_{k-1}-\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j\neq 1}i_1a_{i_1}a_{i_2}\cdots a_{i_k}\overline{a_{i_1+\cdots+i_k}}\Big].$$

We multiply the above identity by $(\sqrt{2t})^k$ and obtain

$$\overline{a_k}\big(\sqrt{2t}\big)^k=\Big(\frac{\sqrt{2t}}{a_1}\Big)^k\Big[M_{k-1}-\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j\neq 1}i_1a_{i_1}a_{i_2}\cdots a_{i_k}\overline{a_{i_1+\cdots+i_k}}\Big].$$

By subtracting M_{k-1} from the above identity, we have

$$\begin{aligned}&\big[\overline{a_k}\big(\sqrt{2t}\big)^k-M_{k-1}\big]\\&=M_{k-1}\Big[\Big(\frac{\sqrt{2t}}{a_1}\Big)^k-1\Big]-\Big[\Big(\frac{\sqrt{2t}}{a_1}\Big)^k-1\Big]\Big[\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j\neq 1}i_1a_{i_1}a_{i_2}\cdots a_{i_k}\overline{a_{i_1+\cdots+i_k}}\Big]\\&-\Big[\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j\neq 1}i_1a_{i_1}a_{i_2}\cdots a_{i_k}\overline{a_{i_1+\cdots+i_k}}\Big].\end{aligned} \tag{4.14}$$

In order to estimate the right-hand side of (4.14), we estimate

$$\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1 \quad \text{and} \qquad \sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}}$$

separately.

(a)

By Mean Value Theorem, there exists $\theta_k(t) \in [0,1]$ such that

$$\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1 = k\Big(1 + \theta_k(t)\Big(\frac{\sqrt{2t}}{a_1} - 1\Big)\Big)^{k-1}\Big(\frac{\sqrt{2t}}{a_1} - 1\Big).$$

For this term $\frac{\sqrt{2t}}{a_1} - 1$, we have

$$\frac{\sqrt{2t}}{a_1} - 1 = \frac{\sqrt{2t} - a_1}{a_1} = \frac{1}{a_1}\frac{2t - a_1^2}{\sqrt{2t} + a_1} = \frac{1}{a_1}\frac{g(t) - M_0(0)}{\sqrt{2t} + a_1}.$$

Therefore,

$$\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1 = k\Big(1 + \theta_k(t)\Big(\frac{\sqrt{2t}}{a_1} - 1\Big)\Big)^{k-1}\frac{1}{a_1\big(\sqrt{2t} + a_1\big)}(-M_0(0) + g(t)).$$

This means

$$\lim_{t\to\infty}\Big[\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1\Big]\big(\sqrt{2t}\big)^2 = \frac{-M_0(0)}{2}(k). \tag{4.15}$$

(b)

$$\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}} = O\Big(\frac{1}{a_1^4}\Big).$$

Therefore

$$\lim_{t\to\infty}\Big[\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}}\Big]\big(\sqrt{2t}\big)^3 = 0 \tag{4.16}$$

(c)

By $(4.15), (4.16)$

$$\Big[\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1\Big]\Big[\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j \neq 1} i_1 a_{i_1} a_{i_2} \cdots a_{i_k} \overline{a_{i_1+\cdots+i_k}}\Big] = o\Big(\frac{1}{2t}\Big),$$

and

$$\Big[\sum_{i_1,\cdots,i_k;\sqcap_{j=1}^k i_j\neq 1} i_1 a_{i_1} a_{i_2}\cdots a_{i_k}\overline{a_{i_1+\cdots+i_k}}\Big] = o\Big(\frac{1}{2t}\Big),$$

and by (4.15)

$$M_{k-1}\Big[\Big(\frac{\sqrt{2t}}{a_1}\Big)^k - 1\Big](2t) = -\frac{M_0(0)}{2}(k)M_{k-1}.$$

Therefore in (4.14), for $k \geq 2$,

$$\lim_{t\to\infty}\big(\overline{a_k}\big(\sqrt{2t}\big)^k - M_{k-1}\big)\big(\sqrt{2t}\big)^2 = -\frac{M_0(0)}{2}(k)M_{k-1}.$$

Taking complex conjugate of the above identity, we have for $k \geq 2$

$$\lim_{t\to\infty}\big(a_k\big(\sqrt{2t}\big)^k - \overline{M_{k-1}}\big)\big(\sqrt{2t}\big)^2 = -\frac{M_0(0)}{2}(k)\overline{M_{k-1}}.$$

□

5 Rescaling behaviors for some nonpolynomial solutions

In Gustafsson [2], the P-G equation (1.1) is reformulated by

$$\frac{d}{dt}f(\xi,t) = \frac{f'(\xi,t)\xi}{2\pi i}\int_{\partial B_1(0)}\frac{1}{\mid f'(z,t)\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0). \tag{5.1}$$

A strong solution to (1.1) must be a strong solution to (5.1).

Theorem 5.1. *Assume $f(\xi,t) = a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots$ is a global strong solution to (1.1) and $\sum_{i=2}^\infty \mid a_i(t)\mid i \leq M$ for some $M > 0$, then for $i \geq 2$,*

$$a_i(t) = O\Big(t^{-\frac{1}{2}}\Big).$$

Furthermore, if $\sum_{i=2}^\infty \mid a_i(t)\mid i \leq M\frac{1}{\sqrt{t}}$, then for $i \geq 2$,

$$a_i(t) = O\Big(t^{-1}\Big).$$

Proof.
Step1:
Denote $A = \frac{1}{a_1}(2a_2\xi + 3a_3\xi^2 + \cdots)$. It is obvious $\mid A \mid \approx O(\frac{1}{a_1})$ and $\mid A \mid << 1$ as t goes large since $\sqrt{a_1^2(0) + 2t + g(0)} \geq a_1(t) \geq \sqrt{a_1^2(0) + 2t}$.
Step2:
There exists $M > 0$ such that $\sum_{i=1}^{\infty} i \mid a_i(t) \mid \leq M$. Since we are looking at the large time behavior, we can just assume that $|\frac{a_1^2(t)}{(a_1(t)-M)^2}| \leq 4$. Without loss of generality, we assume $\mid A \mid < 1$ since we are looking at the behavior of $f(\xi, t)$ at large time t. Define $(f')^*(\xi, t) = \overline{f'}(\frac{1}{\xi}, t)$. For ξ on $\partial B_1(0)$,

$$
\begin{aligned}
&\frac{1}{f'}\frac{1}{(f')^*} \\
=&\frac{1}{a_1^2}\big(1 - A - A^2 - \cdots\big)\big(1 - A^* - (A^*)^2 - \cdots\big) \\
=&\frac{1}{a_1^2}\big(1 - A - A^*\big) + \frac{1}{a_1^2}\big(A + A^2 + \cdots\big)\big(A^* + (A^*)^2 + \cdots\big) \\
&- \frac{1}{a_1^2}\big(A^2 + A^3 + \cdots\big) - \frac{1}{a_1^2}\big((A^*)^2 + (A^*)^3 + \cdots\big).
\end{aligned}
$$

Denote the regular part of $(A + A^2 + \cdots)(A^* + (A^*)^2 + \cdots)$ and $(A + A^2 + \cdots)$ by $\sum_{i=1}^{\infty} b_i\xi^i$ and $\sum_{i=1}^{\infty} d_i\xi^i$ respectively. Then

$$
\sum_{i=1}^{\infty} \mid b_i \mid \leq \Big(\frac{\frac{M}{a_1}}{1 - \frac{M}{a_1}}\Big)^2 \leq \frac{4M^2}{a_1^2}.
$$

Similarly,

$$
\sum_{i=1}^{\infty} \mid d_i \mid \leq \Big(\frac{\left(\frac{M}{a_1}\right)^2}{1 - \frac{M}{a_1}}\Big) \leq \frac{2M^2}{a_1^2}.
$$

Therefore

$$
\begin{aligned}
&\int_{\partial B_1(0)} \frac{1}{\mid f' \mid^2}\frac{z + \xi}{z - \xi}\frac{dz}{z}\frac{1}{2\pi i} \\
=&\frac{1}{a_1^2}\Big[1 - \frac{2 \cdot 2a_2}{a_1}\xi - \frac{2 \cdot 3a_3}{a_1}\xi^2 - \frac{2 \cdot 4a_4}{a_1}\xi^3 - \cdots\Big] + \sum_{k=0}^{\infty} c_k(t)\Big(\frac{1}{a_1^4}\Big)\xi^k,
\end{aligned}
$$

where $\sum_{k=0}^{\infty} \mid c_k(t) \mid \leq 12M^2$.
Step3:

If $k=0$,

$$\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0} = \frac{1}{a_1^2}+c_0(t)\Big(\frac{1}{a_1^4}\Big)$$

and if $k\geq 1$,

$$\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0} = \frac{1}{a_1^2}(-1)\frac{2(k+1)!}{a_1}a_{k+1}+c_k(t)\Big(\frac{1}{a_1^4}\Big)k!.$$

$$\xi f^{'} = a_1\xi + 2a_2\xi^2+\cdots$$
$$\big(\xi f^{'}\big)^m\big|_{\xi=0} = m(m!)a_m.$$

If $k\neq 0$, $k\neq s-1$,

$$\begin{aligned}&\Big|(\xi f^{'})^{(s-k)}\mid_{\xi=0}\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0}\Big|\\ =&\Big|(s-k)(s-k)!a_{s-k}\Big[\frac{1}{a_1^2}(-1)\frac{2(k+1)!}{a_1}a_{k+1}+k!c_k(t)\frac{1}{a_1^4}\Big]\Big|\\ =&(s-k)!k!\Big|(s-k)a_{s-k}\Big[\frac{-2(k+1)a_{k+1}+\frac{c_k(t)}{a_1}}{a_1^3}\Big]\Big|.\end{aligned}$$

Define

$$D_s(f) = \sum_{k=1,k\neq s-1}^{s} C_k^s\big(\xi f^{'}\big)^{(s-k)}\big|_{\xi=0}\Big(\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0}\Big).$$

Then

$$\begin{aligned}\mid D_s(f)\mid \leq &\sum_{k=1,k\neq s-1}^{s} C_k^s\Big|\big(\xi f^{'}\big)^{(s-k)}\big|_{\xi=0}\Big(\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0}\Big)\Big|\\ = &\sum_{k=1,k\neq s-1}^{s} C_k^s(s-k)!k!\Big|(s-k)a_{s-k}\Big[\frac{-2(k+1)a_{k+1}+\frac{c_k(t)}{a_1}}{a_1^3}\Big]\Big|\\ \leq & \, s!M\frac{2M+\frac{12M^2}{a_1}}{a_1^3}.\end{aligned}$$

Therefore for $s \geq 2$,

$$\Big(\xi f^{'} \int_{\partial B_1(0)} \frac{1}{\mid f^{'} \mid^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \frac{1}{2\pi i}\Big)^{(s)}\Big|_{\xi=0}$$

$$=\sum_{k=0}^{s} C_k^s (\xi f^{'})^{(s-k)}\big|_{\xi=0}\Big(\Big(\int_{\partial B_1(0)} \frac{1}{\mid f^{'} \mid^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \frac{1}{2\pi i}\Big)^{(k)}\Big|_{\xi=0}\Big)$$

$$=C_0^s\Big(\frac{1}{a_1^2}+c_0(t)\frac{1}{a_1^4}\Big)a_s s(s)! + C_{s-1}^s\Big[\frac{-1}{a_1^2}\frac{2(s)!}{a_1}a_s + c_{s-1}(t)\Big(\frac{1}{a_1^4}\Big)(s-1)!\Big]a_1 + D_s(f)$$

$$=\frac{1}{a_1^2}C_0^s(s!sa_s) - 2C_{s-1}^s a_1 s!\frac{a_s}{a_1^3} + C_0^s\Big(c_0(t)\frac{1}{a_1^4}\Big)a_s s(s)! + C_{s-1}^s\Big[c_{s-1}(t)\Big(\frac{1}{a_1^4}\Big)(s-1)!\Big]a_1 + D_s(f)$$

$$=\frac{1}{a_1^2}C_0^s(s!sa_s) - 2C_{s-1}^s a_1 s!\frac{a_s}{a_1^3} + (s)!\Big[\frac{sa_s c_0(t)}{a_1^4} + \frac{c_{s-1}(t)}{a_1^3} + \frac{D_s(f)}{s!}\Big]$$

$$=\frac{1}{a_1^2}C_0^s(s!sa_s) - 2C_{s-1}^s a_1 s!\frac{a_s}{a_1^3} + s!G_s(t)\Big(\frac{1}{a_1^3}\Big)$$

where

$$\mid G_s(t) \mid = \Big|\Big[\frac{sa_s c_0(t)}{a_1} + c_{s-1}(t) + \frac{a_1^3 D_s(f)}{s!}\Big]\Big| \leq \Big|\Big[\frac{sa_s c_0(t)}{a_1^3} + c_{s-1}(t) + M\Big(2M + \frac{12M^2}{a_1}\Big)\Big]\Big| \leq C^{'}$$

for some $C^{'}(M, a_1(0))$ which only depends on M and $a_1(0)$.

$$s!a_s^{'} = \frac{1}{a_1^2}(s!sa_s) - 2sa_1 s!\frac{a_s}{a_1^3} + G_s(t)\Big(\frac{1}{a_1^3}\Big)$$
$$\Rightarrow a_s^{'} = -s\frac{a_s}{a_1^2} + G_s(t)\Big(\frac{1}{a_1^3}\Big).$$

By solving this ODE $a_s^{'} = -s\frac{a_s}{a_1^2} + G_s(t)(\frac{1}{a_1^3})$, we get

$$\Big|a_s(t)e^{\int_0^t \frac{s}{a_1^2}dw} - a_s(0)\Big| \leq \int_0^t e^{\int_0^z \frac{s}{a_1^2}dw} C^{'}(M)\Big(\frac{1}{a_1^3}\Big)dz. \tag{5.2}$$

Now we have to estimate $\int_0^t \frac{1}{a_1^2}dw$ and $\int_0^t e^{\int_0^z \frac{s}{a_1^2}dw}(\frac{1}{a_1^3})dz$ in order to estimate $a_s(t)$ in (5.2). The estimate for the former is

$$\int_0^t \Big(\frac{1}{\sqrt{a_1^2(0)+g(0)+2w}}\Big)^2 dw \leq \int_0^t \frac{1}{a_1^2(w)}dw \leq \int_0^t \Big(\frac{1}{\sqrt{a_1^2(0)+2w}}\Big)^2 dw.$$

This implies

$$\frac{1}{2}\Big(\ln\frac{a_1^2(0)+2t+g(0)}{a_1^2(0)+g(0)}\Big) \leq \int_0^t \frac{1}{a_1^2(w)}dw \leq \frac{1}{2}\Big(\ln\frac{a_1^2(0)+2t}{a_1^2(0)}\Big).$$

The estimate for the latter is

$$\begin{aligned}
&\int_0^t e^{\int_0^z \frac{s}{a_1^2(w)}dw}\frac{1}{a_1^3(z)}dz\\
\leq&\int_0^t \Big(\frac{a_1^2(0)+2z}{a_1^2(0)}\Big)^{s/2}\Big(\frac{1}{\sqrt{a_1^2(0)+2z}}\Big)^3 dz\\
=&\frac{1}{s/2-1/2}\frac{1}{a_1^s(0)}\Big[\big(a_1^2(0)+2t\big)^{s/2-1/2}-\big(a_1^2(0)\big)^{s/2-1/2}\Big].
\end{aligned}$$

Therefore by these estimates

$$\begin{aligned}
&e^{-\int_0^t \frac{s}{a_1^2(w)}dw}\int_0^t e^{\int_0^z \frac{s}{a_1^2(w)}dw}\frac{1}{a_1^3(z)}dz\\
\leq&\Big(\frac{a_1^2(0)+g(0)}{a_1^2(0)+g(0)+2t}\Big)^{\frac{s}{2}}\frac{1}{s/2-1/2}\frac{1}{a_1^s(0)}\Big[\big(a_1^2(0)+2t\big)^{s/2-1/2}-\big(a_1^2(0)\big)^{s/2-1/2}\Big]\\
=&O\Big(\frac{1}{\sqrt{t}}\Big).
\end{aligned}$$

Hence the term $a_s(t)$ in (5.2) satisfies

$$\mid a_s(t)\mid\leq\mid a_s(0)\mid\Big(\frac{a_1^2(0)+g(0)}{a_1^2(0)+2t+g(0)}\Big)^{\frac{s}{2}}+O\Big(\frac{1}{\sqrt{t}}\Big)$$

and $a_s=O\Big(\frac{1}{\sqrt{t}}\Big)$.

Step4:

Furthermore, if $\sum_{i=2}^{\infty} i\mid a_i(t)\mid\leq\frac{M}{\sqrt{t}}$, by repeating the same process but with that $\mid A\mid\approx O(\frac{1}{a_1^2(t)})$, we can get for $s\geq 2$

$$a_s^{'}=-s\frac{a_s}{a_1^2}+O\Big(\frac{1}{a_1^5}\Big).$$

By solving this ODE again, we have for $s=2$,

$$\mid a_s(t)\mid=O\Big(\frac{1}{t^{2/2}}\Big)+\mid a_s(0)\mid O\Big(\frac{1}{t^{s/2}}\Big).$$

For $s=3$

$$\mid a_s(t)\mid=O\Big(\frac{1}{t^{s/2}}\ln t\Big)+\mid a_s(0)\mid O\Big(\frac{1}{t^{s/2}}\Big).$$

For $s\geq 4$

$$\mid a_s(t)\mid=O\Big(\frac{1}{t^{3/2}}\Big)+\mid a_s(0)\mid O\Big(\frac{1}{t^{s/2}}\Big).$$

Therefore, for $s \geq 2$,

$$a_s(t) = O\Big(\frac{1}{t}\Big).$$

□

Now, we want to look for global strong solutions to (1.1) which satisfy the conditions of this theorem. It is trivial that the global strong polynomial solutions satisfy the conditions of the theorem. Moreover, the following two types of functions can have some rescaling behaviors, by Theorem 5.1 and the properties of functions themselves.

5.1 The rational function with single pole rescaling behavior

In this section, we focus on rescaling behaviors of the solutions of the following type

$$\sum_{i=0}^{n_0} a_i(t)\xi^i + \sum_{j=1}^{k_0} \frac{a_{-j}(t)}{(\xi - \xi_1(t))^j}.$$

In Gustafsson [2], it is proven that given a rational type initial value, the solution to (1.1) has the same pole structure as that of its initial function. The poles cannot collide or disappear. Therefore, the form above will be kept. We can rewrite the above form to be $\sum_{k=1}^{\infty} b_k \xi^k$ where

$$b_k = \sum_{j=1}^{k_0} \frac{a_{-j}(-1)^j}{\xi_1^{k+j}} \Big(\frac{(k+j-1)!}{k!(j-1)!}\Big), k \geq n_0 + 1.$$

Since $\sum_{i=2}^{\infty} \mid a_i(t) \mid^2 i$ is decreasing, there exists $M > 0$ such that for $n_0 + 1 \leq k \leq n_0 + k_0$,

$$\Big| \sum_{j=1}^{k_0} \frac{a_{-j}(-1)^j}{\xi_1^{k+j}} \frac{(k+j-1)!}{k!(j-1)!} \Big| \leq M. \tag{5.3}$$

Denote

$$A = [(a_{i,j})]_{k_0 \times k_0}, a_{i,j} = \frac{(n_0 + i + j - 1)!}{(n_0 + i)!(j-1)!}.$$

(5.3) means that

$$A\begin{bmatrix} \frac{-a_{-1}}{\xi_1^{n_0+1+1}} \\ \frac{a_{-2}}{\xi_1^{n_0+1+2}} \\ \cdot \\ \cdot \\ \frac{(-1)^n a_{-n}}{\xi_1^{n_0+1+n}} \\ \cdot \\ \cdot \\ \frac{(-1)^{k_0} a_{-k_0}}{\xi_1^{n_0+1+k_0}} \end{bmatrix} = \begin{bmatrix} O(1) \\ O(\xi_1) \\ \cdot \\ \cdot \\ O(\xi_1^{n-1}) \\ \cdot \\ \cdot \\ O(\xi_1^{k_0-1}) \end{bmatrix}$$

We will show that $\det A = 1$ first by performing row reductions. If $k_0 = 1$, it is trivial. Assume $k_0 \geq 2$ now. Note that $a_{i,1} = 1$. For $j \geq 2$

$$\begin{aligned} & a_{i+1,j} - a_{i,j} \\ =& \frac{(n_0+i+j)!}{(n_0+i+1)!(j-1)!} - \frac{(n_0+i+j-1)}{(n_0+i)!(j-1)!} \\ =& \frac{(n_0+i+j-1)!}{(n_0+i+1)!(j-2)!} \\ =& \frac{(n_0+I+J-1)!}{(n_0+I)!(J-1)!}, \text{where } I = i+1, J = j-1. \end{aligned}$$

For $j = 1$,

$$a_{i+1,1} - a_{i,1} = 0.$$

Hence by row reductions,

$$\det A = \det[b_{I,J}]_{(k_0-1)\times(k_0-1)}, b_{I,J} = \frac{(n_0+I+J-1)!}{(n_0+I)!(J-1)!}.$$

We can perform the row reductions again until the new matrix becomes a 1×1 matrix and hence $\det A = 1$.

Since $\det A = 1$ which is nonzero, $\{\frac{(-1)^n a_{-n}}{\xi_1^{n_0+1+n}}\}_{1 \leq n \leq k_0}$ can be solved and

$$\frac{a_{-n}}{\xi_1^{n+n_0+1}} = O\big(\xi_1^{k_0-1}\big), 1 \leq n \leq k_0.$$

This implies that

$$\frac{a_{-n}}{\xi_1^{n+n_0+1+k_0-1}} = \frac{a_{-n}}{\xi_1^{n+n_0+k_0}} = O(1)$$

and there exists $M_0 > 0$ such that

$$\max_{1 \le n \le k_0} \left| \frac{a_{-n}}{\xi_1^{n_0+2k_0}} \right| \le M_0.$$

Claim1:
There exists $m_0 > 0$ such that for $t \ge 0$, we have

$$\sum_{k=2}^{\infty} k \mid b_k(t) \mid \le m_0.$$

Proof. (of claim1) There exists $r_0 > 1$ such that $\mid \xi_1 \mid > r_0$ since the radius of analyticity is increasing as stated in Gustafsson, Prokhorov and Vasil'ev [3]. Denote $2(n_0 + 2k_0) - 1$ by s.

$$\begin{aligned}
\sum_{k=s}^{\infty} k \mid b_k \mid &= \sum_{k=s}^{\infty} k \Big| \sum_{j=1}^{k_0} \frac{a_{-j}(-1)^j}{\xi_1^{k+j}} \frac{(k+j-1)!}{k!(j-1)!} \Big| \\
&\le \sum_{j=1}^{k_0} \sum_{k=s}^{\infty} k \Big| \frac{a_{-j}}{\xi_1^{k+j}} (k+1)(k+2)\cdots(k+k_0) \Big| \\
&\le \sum_{j=1}^{k_0} \sum_{k=s}^{\infty} \Big| \frac{a_{-j}}{\xi_1^{k+j}} (k+k_0)^{k_0+1} \Big| \\
&\le \max_{k \ge s} \left(r_0^{-\frac{k+1}{2}} (k+k_0)^{k_0+1} \right) \sum_{j=1}^{k_0} \sum_{k=s}^{\infty} \Big| \frac{a_{-j}}{\xi_1^{\frac{k+j}{2}}} \Big| \\
&\le \max_{k \ge s} \left(r_0^{-\frac{k+1}{2}} (k+k_0)^{k_0+1} \right) \sum_{j=1}^{k_0} \frac{\mid a_{-j} \mid \left(\frac{1}{\sqrt{|\xi_1|}} \right)^{s+1}}{1 - \sqrt{\frac{1}{|\xi_1|}}} \\
&\le \max_{k \ge s} \left(r_0^{-\frac{k+1}{2}} (k+k_0)^{k_0+1} \right) \sum_{j=1}^{k_0} \frac{\mid a_{-j} \mid \left(\frac{1}{\sqrt{|\xi_1|}} \right)^{s+1}}{1 - \sqrt{\frac{1}{r_0}}} \\
&\le \max_{k \ge s} \left(r_0^{-\frac{k+1}{2}} (k+k_0)^{k_0+1} \right) \frac{k_0 M_0}{1 - \sqrt{\frac{1}{r_0}}} < \infty
\end{aligned}$$

Also $\sum_{k=1}^{s-1} k \mid b_k \mid$ is uniformly bounded since $g(t) \le g(0)$. Therefore the claim is proven. □

Claim2:
There exists $m_1 > 0$ such that

$$\max_{1\le n\le k_0}\left|\frac{a_{-n}}{\xi_1^{n+n_0+k_0}}\right| \le \Big(\frac{m_1}{\sqrt{t}}\Big).$$

Proof. (of claim2) By Theorem 5.1, $| b_k | \le C_0(\frac{1}{\sqrt{t}})$ for some $C_0 > 0$ and $n_0 + 1 \le k \le n_0 + k_0$. Then claim2 is proven by the similar argument as before. □

Claim3:
There exists $m_2 > 0$ such that for $t > 0$, we have

$$\sum_{k=2}^{\infty} k \mid b_k(t) \mid \le \frac{m_2}{\sqrt{t}}.$$

Proof. (of claim3) The proof is similar to that of claim1. □

Claim4:

$$\max_{1\le n\le k_0}\left|\frac{a_{-n}}{\xi_1^{n+n_0+k_0}}\right| = O\Big(\frac{1}{t}\Big).$$

Proof. (of claim4) By Theorem 5.1 and the similar argument as claim2's. □

Claim5:
There exists $m_3 > 0$ such that for $t > 0$, we have

$$\sum_{k=2}^{\infty} k \mid b_k(t) \mid \le \frac{m_3}{t}.$$

Proof. (of claim5) The proof is similar to claim1's. □

Then we can show that:

Theorem 5.2. *For $\xi \in \overline{B_1(0)}$, the strong global solution of the form*

$$\sum_{i=0}^{n_0} a_i \xi^i + \sum_{j=1}^{k_0} \frac{a_{-j}}{(\xi-\xi_1)^j} \in O(\overline{B_1(0)}),$$

has the rescaling behavior:

$$f(\xi,t) - a_1\xi = O\Big(\frac{1}{t}\Big), f(\xi,t) - \sqrt{M_0(0)+2t}\xi = O\Big(\frac{1}{t}\Big).$$

Furthermore,

$$\limsup_{t\to\infty}\max_{\xi\in\overline{B_1(0)}}\Big|\Big[f(\xi,t) - \sqrt{2t}\xi + \frac{M_0(0)}{2\sqrt{2t}}\xi\Big]\Big|(t) < \infty.$$

5.2 The rational function with several simple pole rescaling behavior

Another example is

$$\sum_{i=0}^{n} a_i(t)\xi^i + \sum_{i=1}^{k_0} \frac{a_{-i}(t)}{\xi - \xi_i(t)} \in O(\overline{B_1(0)})$$

where $\{\xi_i(0)\}_{1\leq i\leq k_0}, \{a_{-i}(0)\}_{1\leq i\leq k_0}$ are of the same sign within each set. For example, $\{\xi_i(0)\}_{1\leq i\leq k_0} \subset R^+$ and $\{a_{-i}(0)\}_{1\leq i\leq k_0} \subset R^-$.

Lemma 5.3. *Assume $f(\xi,0) = \sum_{i=1}^{\infty} b_i(0)\xi^i \in O(\overline{B_1(0)})$ has real coefficients $b_i(0)$, then the strong solution $f(\xi,t) = \sum_{i=1}^{\infty} b_i(t)\xi^i \in O(\overline{B_1(0)})$ to* (1.1) *has real coefficients $b_i(t)$, too.*

Proof. Assume that $f(\xi,t)$ is a strong polynomial solution of degree n. In this case, by assuming that $f(\xi,t)$ has real coefficients, we get a solution by writing the P-G equation as n ordinary differential equations. More explicitly, if $f(\xi,t) = \sum_{i=1}^{n} b_i\xi^i$ is the solution where $b_i(t) \in R$, then there exists a $n \times n$ matrix $A_n(t) = [p_{i,j}]_{n\times n}$ where $\{p_{i,j}\}$ are linear polynomials of $b_1, \cdots, b_n$ such that

$$A_n \begin{bmatrix} b_1' \\ \cdot \\ \cdot \\ b_i' \\ \cdot \\ \cdot \\ b_n' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

The value $\det A_n(0) \neq 0$ since $(b_1'(0), \cdots, b_n'(0))$ is uniquely determined by $(b_1(0), \cdots, b_n(0))$ due to the reformulation (2.1) of the P-G equation. Conclusively, by Cramer's rule,

$$\frac{d}{dt} b_k = \frac{Q_k}{R}$$

where $\{Q_k(t)\}_{1\leq k\leq n}$ and $R(t) = \det A_n(t)$ are both polynomials of $b_1(t), \cdots, b_n(t)$ and $R(0) \neq 0$. Since the solution is unique as shown in Gustafsson [2], the real coefficient solution is indeed the unique solution.

Now assume that $f(\xi,0) = \sum_{i=1}^{\infty} b_i\xi^i \in O(\overline{B_1(0)})$ has real coefficients. The strong solution $f(\xi,t)$ can be approximated locally in time by polynomial solutions $g_n(\xi,t)$ with initial function $\sum_{i=1}^{n} b_i(0)\xi^i$ for large enough n, according to Lin [7]. The solution $g_n(\xi,t)$ is with real value since $b_k(0)$ is real-valued. Therefore $f(\xi,t)$ has real coefficients as well. $\square$

Remark 5.1. In the case that $n = 3$, it is shown in Huntingford [5] that

$$A_3 = \begin{bmatrix} b_1 & 2b_2 & 3b_3 \\ 2b_2 & b_1 + 3b_3 & 2b_2 \\ 3b_3 & 0 & b_1 \end{bmatrix}.$$

Remark 5.2. For the general n,

$$Re\big[f_t(\xi,t)\overline{\xi f'(\xi,t)}\big] = 1$$

implies

$$\Big(\sum_{i=1}^{n}\Big(\sum_{k,j,|k-j|=i-1} b_j^{'} k b_k\Big)\cos((i-1)\theta)\Big) = 1$$

where $\xi = e^{i\theta}$ and $f(\xi,t) = \sum_{k=1}^{n} b_i \xi^i$.

$$\begin{cases} \Big(\sum_{j=1}^{n}\sum_{k,|k-j|=0} kb_k\Big)b_j^{'} = 1 \\ \Big(\sum_{j=1}^{n}\sum_{k,|k-j|=i} kb_k\Big)b_j^{'} = 0 \quad ; 1 < i \le n \end{cases}$$

The A_n in Lemma 5.3 is

$$A_n = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix},$$

where

$$p_{ij} = \sum_{k \in N_{ij}} kb_k$$

and $N_{ij} = \{i+j-1, j-i+1 \mid 1 \le i+j-1 \le n, 1 \le j-i+1 \le n\}$.

Remark 5.3. A geometric interpretation is that, given an initial domain $\Omega(0)$ which is analytic on the boundary and is symmetric about the $x-$axis , then the strong solution $\Omega(t)$ must be symmetric about the x-axis as well.

Lemma 5.4. *The property of the solution is that* $\{\xi_i(t)\}_{1\le i\le k_0}, \{a_{-i}(t)\}_{1\le i\le k_0}$ *are of the same sign as that of* $\{\xi_i(0)\}_{1\le i\le k_0}$ *and that of* $\{a_{-i}(0)\}_{1\le i\le k_0}$ *respectively.*

Proof. If $\{\xi_i(t)\}_{1\le i\le k_0}, \{a_{-i}(t)\}_{1\le i\le k_0}$ are both real-valued, then it is trival that $\{\xi_i(t)\}_{1\le i\le k_0}$ and $\{a_{-i}(t)\}_{1\le i\le k_0}$ have the same sign as that of $\{\xi_i(0)\}_{1\le i\le k_0}$ and that of $\{a_{-i}(0)\}_{1\le i\le k_0}$ respectively since the rational function form has to be kept and poles never collide or vanish as shown in Gustafsson [2].

Therefore, we only need to prove that $\{\xi_i(t)\}_{1\le i\le k_0}$ and $\{a_{-i}(t)\}_{1\le i\le k_0}$ are both real-valued.

Denote $f(\xi,t)$ by $\sum_{k=1}^{\infty} b_k(t)\xi^k$. In this case,

$$b_k(t) = -\sum_{i=1}^{k_0} \frac{a_{-i}(t_1)}{\xi_i^{k+1}(t_1)}, k \ge n_0 + 1.$$

By Lemma 5.3, $b_i(t) \in R$ since $b_i(0) \in R$. Let

$$t_0 = \inf\{t \ge 0 \mid \; Im\xi_i(t) \ne 0 \text{ for some } 1 \le i \le k_0\}.$$

Claim: $t_0 = \infty$.

Proof. (of claim) We will prove by contradiction. Assume $t_0 < \infty$ first. $\mid \xi_i(t_0) \mid \ne \mid \xi_j(t_0) \mid$ if $i \ne j$. Without loss of generality,

$$\mid \xi_1(t_0) \mid < \mid \xi_2(t_0) \mid < \cdots < \mid \xi_{k_0}(t_0) \mid .$$

(casei)
There exists $\epsilon > 0$ such that for $t \in [t_0, t_0 + \epsilon]$,

$$\mid \xi_1(t) \mid < \mid \xi_2(t) \mid < \cdots < \mid \xi_{k_0}(t) \mid$$

and there exists $t_1 \in [t_0, t_0+\epsilon]$ such that $\arg \xi_1(t_1) = 2\pi\theta$ for some irrational value $\theta \in [0, 1)$.

In this case, there exists $\{k_j\}_{1\le j<\infty}$ such that

$$[\theta k_j] \to [\frac{\arg a_{-1}(t_1)}{2\pi} + \frac{1}{4}]$$

where $[\cdot]$ is the Gauss symbol. Since for $k \ge n_0 + 1$,

$$\sum_{i=1}^{k_0} \frac{a_{-i}(t_1)}{\xi_i^{k+1}(t_1)} \in R.$$

Therefore

$$\lim_{j\to\infty} \sum_{i=1}^{k_0} -\frac{a_{-i}(t_1)}{\left(\frac{\xi_i(t_1)}{|\xi_1(t_1)|}\right)^{k_j}} = i \mid a_{-1}(t_1) \mid \in R.$$

Since $a_{-1}(t_1)$ can not be zero, there is a contradiction. So this case never happens.

(caseii)
There exists $\epsilon > 0$ such that for $t \in [t_0, t_0 + \epsilon]$,

$$| \xi_1(t) |<| \xi_2(t) |< \cdots <| \xi_{k_0}(t) |$$

and $\xi_1(t) \in R$.

In this case, we can prove that $a_{-1}(t)$ is also real since

$$\lim_{k\to\infty} b_k(t)\big(\xi_1^{k+1}(t)\big) = -a_{-1}(t) \in R.$$

Therefore, $\xi_2(t)$ is in R due to the same argument as that of (casei). By induction, we can prove that $\{\xi_i(t)\}_{2\le i\le k_0}$ and $\{a_{-i}(t)\}_{2\le i\le k_0}$ are all real. Therefore, for $t \in [t_0, t_0+\epsilon]$, $\{\xi_i(t)\}_{2\le i\le k_0}$ are all real. This contradicts to the definition of t_0. Therefore, $t_0 = \infty$. □

□

Theorem 5.5. *Given a global solution*

$$f(\xi,t) = \sum_{k=0}^{n_0} a_k(t)\xi^k + \sum_{i=1}^{k_0} \frac{a_{-i}(t)}{\xi - \xi_i(t)}$$

where $\{a_{-i}(0)\}_{1\le i\le k_0}$ and $\{\xi_i(0)\}_{1\le i\le k_0}$ are of the same sign within each set. The rescaling behavior for the global solution $f(\xi,t)$ is

$$f(\xi,t) = \sqrt{2t}\xi + \frac{M_0(0)}{2\sqrt{2t}}\xi + O(\frac{1}{t})$$

where $M_0(0)$ is the zero moment at $t = 0$. Furthermore,

$$\limsup_{t\to\infty} \sup_{\xi\in\overline{B_1(0)}} \left|\left[f(\xi,t) - \sqrt{2t}\xi + \frac{M_0(0)}{2\sqrt{2t}}\xi\right]\right|(t) < \infty$$

Proof. Rewrite

$$f(\xi,t) = \sum_{k=1}^{\infty} b_k(t)\xi^k.$$

Since $g(t) = \sum_{k=2}^{\infty} k \mid b_k \mid^2 \le M^2$ for some $M > 0$, $\mid b_{n_0+1} \mid\le M$. The term b_{n_0+1} can be represented as

$$b_{n_0+1} = \sum_{j=1}^{k_0} \frac{-a_{-j}}{\xi_j^{n_0+1+1}}.$$

For $1 \leq j \leq k_0$,

$$\left|\frac{a_{-j}}{\xi_j^{n_0+1+1}}\right| \leq M.$$

Claim1: There exists $m_1 > 0$ such that

$$\sum_{k=2}^{\infty} k \mid b_k(t) \mid \leq m_1.$$

Proof. (of claim1) Denote $2(n_0+2)-1$ by s. Since the radius of analyticity of the domain is increasing with time, there exists $r_0 > 1$ such that $\mid \xi_j \mid \geq r_0$ for $1 \leq j \leq k_0$.

$$\begin{aligned}
\sum_{k=s}^{\infty} k \mid b_k \mid &= \sum_{k=s}^{\infty} k \left| \sum_{j=1}^{k_0} \frac{a_{-j}}{\xi_j^{k+1}} \right| \leq \sum_{j=1}^{k_0} \sum_{k=s}^{\infty} k \left| \frac{a_{-j}}{\xi_j^{k+1}} \right| \\
&\leq \sum_{j=1}^{k_0} \mid a_{-j} \mid \sum_{k=s}^{\infty} \left| \frac{k}{\xi_j^{k+1}} \right| \leq \max_{k \geq s} \left| \frac{k}{r_0^{\frac{k}{2}}} \right| \sum_{j=1}^{k_0} \sum_{k=s}^{\infty} \left| \frac{a_{-j}}{\xi_j^{\frac{k+1}{2}}} \right| \\
&\leq \max_{k \geq s} \left| \frac{k}{r_0^{\frac{k}{2}}} \right| \sum_{j=1}^{k_0} \frac{\mid a_{-j} \mid \left(\frac{1}{\sqrt{|\xi_j|}} \right)^{s+1}}{1 - \sqrt{\frac{1}{|\xi_j|}}} \\
&\leq \max_{k \geq s} \left| \frac{k}{r_0^{\frac{k}{2}}} \right| \sum_{j=1}^{k_0} \frac{1}{1 - \sqrt{\frac{1}{r_0}}} k_0 M
\end{aligned} \tag{5.4}$$

Also $\sum_{k=1}^{2(1+n_0)-1} k \mid b_k \mid$ is uniformly bounded since $g(t)$ is nonincreasing. Therefore claim1 is proven. □

Claim2: There exists $m_2 > 0$ such that

$$\mid b_{n_0+1} \mid \leq \frac{m_2}{t^{1/2}}.$$

Proof. (of claim2) By Theorem 5.1. □

Claim3: For $1 \leq j \leq k_0$,

$$\left|\frac{a_{-j}}{\xi_j^{n_0+1+1}}\right| \leq \frac{m_2}{t^{1/2}}.$$

Proof. (of claim3) This is obvious since $\mid b_{n_0+1} \mid = \sum_{j=1}^{k_0} \left| \frac{a_{-j}}{\xi_j^{n_0+1+1}} \right|$. ☐

Claim4: There exists $m_3 > 0$ such that

$$\sum_{i=2}^{\infty} i \mid b_i(t) \mid \leq \frac{m_3}{t^{1/2}}.$$

Proof. (of claim4) Apply the same arguement as claim1's. ☐

Claim5: There exists $m_4 > 0$ such that

$$\mid b_{n_0+1} \mid \leq \frac{m_4}{t}.$$

Proof. (of claim5) By Theorem 5.1. ☐

Claim6: For $1 \leq j \leq k_0$

$$\left| \frac{a_{-j}}{\xi_j^{n_0+1+1}} \right| \leq \frac{m_4}{t}.$$

Proof. (of claim6) This is obvious since $\mid b_{n_0+1} \mid = \sum_{j=1}^{k_0} \left| \frac{a_{-j}}{\xi_j^{n_0+1+1}} \right|$. ☐

Claim7: There exists $m_5 > 0$ such that

$$\sum_{i=2}^{\infty} i \mid b_i(t) \mid \leq \frac{m_5}{t}.$$

Proof. (of claim7) By the similar argument as claim1's. ☐

By above,

$$f(\xi,t) - a_1(t)\xi = O\Big(\frac{1}{t}\Big); f(\xi,t) = \sqrt{2t}\xi + \frac{M_0(0)}{2\sqrt{2t}}\xi + O\Big(\frac{1}{t}\Big).$$

$$\limsup_{t\to\infty} \sup_{\xi\in\overline{B_1(0)}} \left| \left[f(\xi,t) - \sqrt{2t}\xi + \frac{M_0(0)}{2\sqrt{2t}}\xi \right] \right| (t) < \infty.$$

☐

6 Future work

In this paper, the rescaling behaviors for all the global strong polynomial solutions are given. Also two types of rational solutions are discussed as well. We are interested in knowing if we can generalize these ideas to all the global rational function solutions, since rational function solutions have a much simpler structure.

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