Large-time rescaling behaviors for large data to the Hele-Shaw problem

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Abstract

This paper addresses rescaling behaviors of some classes of global solutions to the zero surface tension Hele-Shaw problem with injection at the origin, $\{\Omega(t)\}_{t\geq 0}$. Here $\Omega(0)$ is a small perturbation of $f(B_1(0), 0)$ if $f(\xi, t)$ is a global strong polynomial solution to the Polubarinova-Galin equation with injection at the origin and we prove the solution $\Omega(t)$ is global as well. We rescale the domain $\Omega(t)$ so that the new domain $\Omega^{'}(t)$ always has area π and we consider $\partial\Omega^{'}(t)$ as the radial perturbation of the unit circle centered at the origin for t large enough. It is shown that the radial perturbation decays algebraically as $t^{-\lambda}$. This decay also implies that the curvature of $\partial\Omega^{'}(t)$ decays to 1 algebraically as $t^{-\lambda}$. The decay is faster if the low Richardson moments vanish. We also explain this work as the generalization of Vondenhoff's work which deals with the case that $f(\xi, t) = a_1(t)\xi$. We can see that rescaling behaviors are described precisely in terms of the Richardson's complex moments.

Keywords: Hele-Shaw flows, starlike function, rescaling behavior.

1 Introduction

This paper addresses large-time behaviors for the classical zero surface tension (ZST) Hele-Shaw flows with injection at the origin. The driving mechanics, injection with a constant rate 2π at the origin, produce a family of domains $\{\Omega(t)\}_{t\geq 0}$ which is a subordination chain. In two dimensions, Galin and Polubarinova-Kochina reformulated the planar model of Hele-Shaw flows by describing the domains $\{\Omega(t)\}_{t\geq 0}$ by a family of conformal

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mappings $\{f(\xi,t)\}_{t>0}$ where $f(\xi,t): B_1(0) \to \Omega(t)$ and $f(0,t)=0, f^{'}(0,t) > 0$. This is called the Polubarinova-Galin equation and it is expressed:

$$Re\big[f_t(\xi,t)\overline{f^{'}(\xi,t)\xi}\big] = 1, \xi \in \partial B_1(0). \tag{1.1}$$

A solution to equation (1.1) is said to be a strong solution for $t \in [0,b)$ if $f(\xi,t)$ is univalent and analytic in $\overline{B_1(0)}$, $f(0,t)=0$, $f^{'}(\xi,0) > 0$ and $f(\xi,t)$ is continously differentiable in $\overline{B_1(0)} \times [0,b)$. Equivalently, we obtain a strong solution $\Omega(t) = f(B_1(0),t)$ to the ZST Hele-Shaw problem with injection, where $\Omega(t)$ has a real analytic boundary and is simply connected.

We define

$$O(E) = \{f \mid f(\xi) \text{ is analytic and univalent in } E,\ f(0)=0, f^{'}(0) > 0\}.$$

The well-posedness of this problem has been thoroughly explored. In Reissig and von Wolfersdorf [7], the authors prove the existence and uniqueness of a strong solution in $O(\overline{B_1(0)})$ if the initial function is in $O(\overline{B_1(0)})$.

In Gustafsson, Prokhorov and Vasil'ev [3] and Lin [5] , the dynamics for $b=\infty$ are discussed. In the former, it is proven that if the initial domain is a strongly starlike domain with a real analytic boundary, the global strong solution to (1.1) exists. In the latter, it is shown that the initial domain of a global strong solution can even be nonstarlike. In fact, there exists a nonstarlike polynomial function $f(\xi,0) \in O(\overline{B_1(0)})$ such that the global strong polynomial solution $f(\xi,t)$ to (1.1) is global.

In Gustafsson [2], the author proves that a strong solution to (1.1) is degree k_0 polynomial if its initial function in $O(\overline{B_1(0)})$ is also a degree k_0 polynomial. In Lin [5], we show that there is a large class of global strong polynomial solutions and also give rescaling behaviors of these solutions precisely in terms of moments.

Here we consider the initial domain $\Omega(0) = f(B_1(0))$ where $f(\xi)$ is a small perturbation of $f_{k_0}(\xi,t)\mid_{t=0}$ where $f_{k_0}(\xi,t)$ is a global degree k_0 strong polynomial solution to (1.1). Therefore, the solution $\Omega(t)$ to the problem is simply connected and has a real analytic boundary. There are two main parts of this work: **first**, we show that the solution $\Omega(t)$ is also a global strong solution to the Hele-Shaw problem with injection; **second**, we show rescaling behaviors of the solution $\Omega(t)$. In Vondenhoff [1], the author gives rescaling behaviors of global solutions in the case that the initial domain $\Omega(0)$ is a small perturbation of a disk centered at the origin for any dimension. We can consider the current work as the generalization of Vondenhoff [1] in dimension 2 by taking $f_{k_0}(\xi,t) = a_1(t)\xi$. Figure 1.1 illustrates the graph of one specific polynomial function which can give rise to a global strong polynomial solution to (1.1). This graph is a big perturbation of the unit circle centered at the origin.

In the past decades, for the weak solutions of this problem, the distance from the free boundaries to the injection source and estimates for the curvature of free boundaries in one direction are studied mainly in Sakai [9] and Gustafsson and Sakai [4] respectively. In this paper, for the subset of strong solutions stated as above, we get a more precise description of rescaling behaviors, including curvature and the boundaries to the injection source.

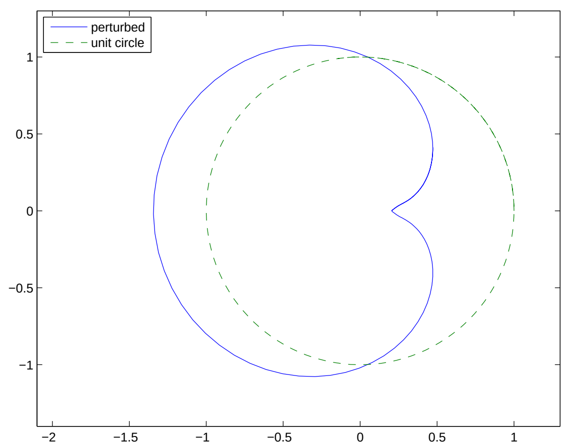

Figure 1.1: The perturbed domain is obtained by dividing $f(B_1(0))$ by the square root of $\frac{1}{\pi} \mid f(B_1(0)) \mid$ where $f(\xi) = \frac{\xi}{1.1} - \frac{15}{14}(\frac{\xi}{1.1})^2 + \frac{4}{7}(\frac{\xi}{1.1})^3 - \frac{1}{7}(\frac{\xi}{1.1})^4$ and it has area π. The function $f(\xi) \in O(B_{1.1}(0))$ is strongly starlike and can be the initial function of a strongly global solution to (1.1). Visually, the perturbed domain is quite different from the unit circle centered at the origin.

We give a short description about how we deal with the rescaling behaviors of the global strong solution $\Omega(t)$ as stated above.
(1)We rescale the domain $\Omega(t)$ by the square root of $\frac{1}{\pi} \mid \Omega(t) \mid$ so that the the new domain $\Omega^{'}(t)$ always has area equal to π.
(2)We show that there exists T_0 such that the domain $\Omega(t)$ is strong starlike for $t \geq T_0$. Then we can express the new domain $\Omega^{'}(t) = \Omega^{'}_{\overline{r}(t)} = \{x \in R^2 \backslash \{0\} : \mid x \mid < 1 + \overline{r}(t, \frac{x}{|x|})\} \cup \{0\}\}$ for some $\overline{r}(t, \cdot) : S^1 \to (-1, \infty)$.
(3)We show that $\Omega(t)$ eventually becomes the small perturbation of a disk centered at the origin with area $\mid \Omega(t) \mid$ as $t = T_0$ in the sense of Vondenhoff.

Then we apply the theorem in Vondenhoff's work by considering $\Omega(T_0)$ as the initial domain and obtain the decay rate $\|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)} = o(\frac{1}{t^\lambda})$ for any $\lambda \in (0, 1+\frac{n_0}{2})$ where $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$ as Vondenhoff [1] reports. Furthermore, by Lin [5], we show the decay rate is $\|\overline{r}(t,\cdot)\|_{C^3(S^1)} = O(\frac{1}{t^\lambda})$ for $\lambda = 1+\frac{n_0}{2}$ where $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$ if $\Omega(t)$ is the mapping of a global strong polynomial solution. The value $\lambda = 1+\frac{n_0}{2}$ is sharp.

The structure of this paper is as follows. In section 2, we show how a small perturbation of a polynomial conformal mapping domain affects the evolution of the free boundary in finite time. We assume that $\{f_{k_0}(\xi,t)\}_{t\geq 0}$ is a global strong polynomial solution to (1.1) and that $f(\xi,0)$ is the small perturbation of $f_{k_0}(\xi,0)$ in the sense stated later. Section 3 shows that starting with the initial domain $\Omega(0) = f(B_1(0),0)$, the family of domains $\{\Omega(t)\}_{t\geq 0}$ which solves the Hele-Shaw problem is global, and that there are some rescaling behaviors also. In particular, if the global strong solution is a polynomial solution, we can get a more precise description for the rescaling behaviors of the domains compared with Vondenhoff [1].

2 Main theorem

Define

$$\Big|\sum_{i=0}^{\infty} a_i\xi^i\Big|_M = \sum_{i=0}^{\infty} \mid a_i \mid$$
$$\Big|\sum_{i=0}^{\infty} a_i\xi^i\Big|_{M(r)} = \sum_{i=0}^{\infty} \left|a_i r^i\right|$$

$H(\Omega) = \{f \mid f \text{ is analytic in } \Omega\}$

$O(\Omega) = \{f \mid f \text{ is analytic and univalent in } \Omega,\ f(0)=0 \text{ and } f'(0)>0\}$

$\omega(\Omega) = \{f \mid f \text{ is analytic and univalent in } \Omega,\ f(0)=0 \text{ and } f'(0)>0\}$

In [2], Gustafsson reformulates the Polubarinova-Galin equation, that is:

$$f_t = \frac{f'\xi}{2\pi i}\int_{\partial B_1(0)} \frac{1}{\mid f'(z)\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0). \tag{2.1}$$

As Gustafsson [2], the mathematical treatment for (2.1) only requires the local univalence of the function $f(\xi,t)$. To make a distinction, we define a solution to be a strong solution to (2.1) as follows:

Definition 2.1. A solution $f(\xi,t) \in \omega(\overline{B_1(0)})$ is a strong solution to (2.1) for $0 \le t \le b$ if $f(\xi,t)$ is continuously differentiable with respect to $t \in [0,b)$ and satisfies (2.1).

A solution $f(\xi,t) \in O(\overline{B_1(0)})$ to (2.1) must be a solution to (1.1).

Lemma 2.1. *For $g \in \omega(\overline{B_r(0)})$ for some $r > 1$, Gustafsson [2] shows*

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{1}{\mid g'(z)\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z} = \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{1}{g'(z)\overline{g'}(1/z)}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0).$$

Also given $h \in \omega(\overline{B_r(0)})$, then

$$\begin{aligned}
&\max_{\partial B_1(0)}\Big|\frac{1}{2\pi i}\int_{\partial B_1(0)}\Big(\frac{1}{\mid g'(z)\mid^2}-\frac{1}{\mid h'(z)\mid^2}\Big)\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big| \\
&= \max_{\partial B_1(0)}\Big|\frac{1}{2\pi i}\int_{\partial B_r(0)}\Big(\frac{1}{g'(z)\overline{g'}(1/z)}-\frac{1}{h'(z)\overline{h'}(1/z)}\Big)\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big| \\
&\le \max_{\partial B_r(0)}\Big|\frac{1}{g'(z)\overline{g'}(1/z)}-\frac{1}{h'(z)\overline{h'}(1/z)}\Big|\frac{r+1}{r-1} \\
&= \max_{\partial B_r(0)}\Big|\frac{h'(z)\overline{h'}(1/z)-g'(z)\overline{g'}(1/z)}{h'(z)\overline{h'}(1/z)g'(z)\overline{g'}(1/z)}\Big|\frac{r+1}{r-1} \\
&= \max_{\partial B_r(0)}\Big|\frac{\overline{h'}(1/z)(h'(z)-g'(z))+g'(z)(\overline{h'}(1/z)-\overline{g'}(1/z))}{h'(z)\overline{h'}(1/z)g'(z)\overline{g'}(1/z)}\Big|\frac{r+1}{r-1} \\
&= \max_{\partial B_r(0)}\Big|\frac{(h'(z)-g'(z))}{h'(z)g'(z)\overline{g'}(1/z)}+\frac{(\overline{h'}(1/z)-\overline{g'}(1/z))}{h'(z)\overline{h'}(1/z)\overline{g'}(1/z)}\Big|\frac{r+1}{r-1}
\end{aligned}$$

Lemma 2.2. *If g is holomorphic in a neighborhood of $\partial B_1(0)$ and g is also a real function on $\partial B_1(0)$, then we have*

$$\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|_{L^2([0,2\pi])} \le \sqrt{2}\|g\|_{L^2([0,2\pi])}.$$

Proof. Let $\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i} = \sum_{i=0}^{\infty} c_i\xi^i$, then $g(\xi) = \frac{1}{2}(\sum_{i=0}^{\infty} c_i\xi^i + \sum_{i=0}^{\infty}\overline{c_i}\xi^{-i})$ on $\partial B_1(0)$. Therefore

$$\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|^2_{L^2([0,2\pi])} = 2\pi\sum_{i=0}^{\infty}\mid c_i\mid^2$$

$$\|g\|^2_{L^2[0,2\pi]} = \frac{2\pi}{4}\Big[2\Big(\sum_{i=0}^{\infty}\mid c_i\mid^2\Big)+2c_0^2\Big]$$

where

$$c_0 = \frac{1}{2\pi}\int_{\partial B_1(0)} g d\alpha.$$

$$\|g\|^2_{L^2([0,2\pi])} = \frac{2\pi}{4}\Big(\frac{2}{2\pi}\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|^2_{L^2([0,2\pi])} + 2c_0^2\Big)$$

$$\begin{aligned}
\frac{1}{2}\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|^2_{L^2([0,2\pi])} &\leq \|g\|^2_{L^2([0,2\pi])}\\
\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|^2_{L^2([0,2\pi])} &\leq 2\|g\|^2_{L^2([0,2\pi])}\\
\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|_{L^2([0,2\pi])} &\leq \sqrt{2}\|g\|_{L^2([0,2\pi])}
\end{aligned}$$

□

Remark 2.2. There exists u which is harmonic in $B_1(0)$, continuous in $\overline{B_1(0)}$, and $u = g$ on $\partial B_1(0)$. Therefore, by Theorem 17.26 in Rudin [8], it is shown that for $1 < p < \infty$, there exists $C_p > 0$ such that

$$\Big\|\int_{\partial B_1(0)} u\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|_{L^p([0,2\pi])} \leq C_p\|u\|_{L^p([0,2\pi])},$$

which means

$$\Big\|\int_{\partial B_1(0)} g\frac{z+\xi}{z-\xi}\frac{dz}{z}\frac{1}{2\pi i}\Big\|_{L^p([0,2\pi])} \leq C_p\|g\|_{L^p([0,2\pi])}.$$

Lemma 2.3. *Given that $g \in \omega(\overline{B_1(0)})$ and $h \in \omega(\overline{B_1(0)})$ satisfy*

$$\begin{aligned}
\frac{d}{dt}[g] &= \frac{1}{2\pi i}\xi g' \int_{\partial B_1(0)} \frac{1}{|g'|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\\
\frac{d}{dt}[h] &= \frac{1}{2\pi i}\xi h' \int_{\partial B_1(0)} \frac{1}{|h'|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z},
\end{aligned}$$

respectively, then we have

$$\Big\|\frac{d}{dt}(g-h)\Big\|_{L^2([0,2\pi])}$$

$$\leq \Big\{\max_{\partial B_1(0)}\Big|\frac{\xi}{2\pi i}\int_{\partial B_1(0)}\frac{1}{|g'|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big| + \sqrt{2}\max_{\partial B_1(0)}|h'|\max_{\partial B_1(0)}\frac{|g'|+|h'|}{|g'|^2|h'|^2}\Big\}\|g'-h'\|_{L^2([0,2\pi])}$$

Proof.

$$\frac{d}{dt}[g-h] = \frac{1}{2\pi i}\xi\Big\{\big[g^{'}-h^{'}\big]\int_{\partial B_1(0)}\frac{1}{|g^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}+h^{'}\Big[\int_{\partial B_1(0)}(\frac{1}{|\,g^{'}\,|^2}-\frac{1}{|h^{'}|^2})\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big]\Big\}$$

Here, by Lemma 2.2

$$\Big\|\frac{1}{2\pi i}\int_{\partial B_1(0)}(\frac{1}{|g^{'}|^2}-\frac{1}{|h^{'}|^2})\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big\|_{L^2([0,2\pi])} \leq \sqrt{2}\Big\|\frac{1}{|g^{'}|^2}-\frac{1}{|h^{'}|^2}\Big\|_{L^2([0,2\pi])}.$$

$$\Big\|\frac{d}{dt}(g-h)\Big\|_{L^2([0,2\pi])} \leq \|g^{'}-h^{'}\|_{L^2([0,2\pi])}\max_{\partial B_1(0)}\Big|\frac{\xi}{2\pi i}\int_{\partial B_1(0)}\frac{1}{|g^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big|$$

$$+\sqrt{2}\|h^{'}\|_{L^\infty([0,2\pi])}\Big\|\frac{1}{|g^{'}|^2}-\frac{1}{|h^{'}|^2}\Big\|_{L^2([0,2\pi])}$$

$$\leq \Big\{\max_{\partial B_1(0)}\Big|\frac{\xi}{2\pi i}\int_{\partial B_1(0)}\frac{1}{|g^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big|+\sqrt{2}\max_{\partial B_1(0)}\big|h^{'}\big|\max_{\partial B_1(0)}\frac{|g^{'}|+|h^{'}|}{|g^{'}|^2|h^{'}|^2}\Big\}\|g^{'}-h^{'}\|_{L^2([0,2\pi])}$$

□

The following lemma helps us to control the blow-up time of polynomial solutions to (2.1).

Lemma 2.4. *Given a polynomial mapping $f(\xi,0)\in\omega(\overline{B_r(0)})$ for some $r>1$, then there exists a unique strong polynomial solution $f(\xi,t)\in\omega(\overline{B_r(0)})$ to (2.1) at least for a short time. Furthermore, if the polynomial solution ceases to exist at $t=b$, then for any $r>1$,*

$$\liminf_{t\to b}(\min_{\overline{B_r(0)}}|\,f^{'}(\xi,t)\,|)=0.$$

Proof. (a)If not, there exists $r>1$ such that

$$\liminf_{t\to b}(\min_{\overline{B_r(0)}}|\,f^{'}(\xi,t)\,|)>0.$$

This implies that there exist $C>0$ and $1<r'\leq r$ such that

$$\min_{\overline{B_{r'}(0)}}|\,f^{'}(\xi,t)\,|>C, t\in[0,b).$$

(b)It is trivial that there exists $M>0$ such that

$$\sup_{t\in[0,b)}\max_{\xi\in\overline{B_{r'}(0)}}|\,f^{'}(\xi,t)\xi\,|\leq M,$$

since each coefficient of $f(\xi,t)$ is bounded.
(c)For $\xi \in \overline{B_1(0)}$,

$$
\begin{aligned}
&\sup_{t\in[0,b)} \Big| \frac{f^{'}(\xi,t)\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{\mid f'(\xi,t)\mid^2} \frac{z+\xi}{z-\xi}\frac{dz}{z} \Big| \\
\leq &\sup_{t\in[0,b)} \Big| \frac{f^{'}(\xi,t)\xi}{2\pi i} \int_{\partial B_{r'}(0)} \frac{1}{\mid f'(\xi,t)\mid^2} \frac{z+\xi}{z-\xi}\frac{dz}{z} \Big| \\
\leq &\sup_{t\in[0,b)} \Big(\max_{\xi\in\overline{B_1(0)}} \mid f^{'}(\xi,t)\xi \mid \cdot \max_{\xi\in\partial B_{r'}(0)} \frac{1}{\mid f'(\xi,t)\mid^2} \frac{r'+1}{r'-1} \Big) \\
\leq &M\cdot \frac{1}{C^2}\frac{r'+1}{r'-1}
\end{aligned}
$$

Therefore, for $0 \leq t_2 < t_1 < b$,

$$
\mid f(\xi,t_1) - f(\xi,t_2) \mid = \Big| \int_{t_2}^{t_1} \frac{f^{'}(\xi,t)\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{\mid f'(\xi,t)\mid^2} \frac{z+\xi}{z-\xi}\frac{dz}{z} \Big| \leq \mid t_1 - t_2 \mid \frac{M}{C^2}\frac{r'+1}{r'-1}.
$$

Therefore $\lim_{t\to b} f(\xi,t)$ exists and we define it as $f(\xi,b)$. Note that $f(\xi,b)$ satisfies $\min_{\overline{B_{r'}(0)}} \mid f^{'}(\xi,b) \mid \geq C$. Let $f(\xi,t+b)$ be the solution to (2.1) with the initial value $f(\xi,b)$ for $t \in [0,\epsilon)$. Then $f(\xi,t)$ is continuous with respect to t for $t \in [0, b+\epsilon)$ and

$$
f(\xi,t) - f(\xi,0) = \int_0^t \frac{f^{'}(\xi,t)\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{\mid f'(\xi,t)\mid^2} \frac{z+\xi}{z-\xi}\frac{dz}{z}.
$$

This implies that $f(\xi,t) \in \omega(\overline{B_1(0)})$ is continuously differentiable with respect to t for $t \in [0, b+\epsilon)$ and satisfies (2.1). Hence it is impossible that $f(\xi,t)$ blows up at $t = b$ and hence for any $r > 1$,

$$
\liminf_{t\to b} (\min_{\overline{B_r(0)}} \mid f^{'}(\xi,t) \mid) = 0.
$$

□

Theorem 2.5. *Assume that* $f_{k_0}(\xi,t) \in C^1([0,t_1], H(\overline{B_r}(0))) \cap \omega(\overline{B_r(0)})$ *is a strong degree* k_0 *polynomial solution to* (2.1) *for some* $t_1 > 0$ *and* $r > 1$ *and that* $\rho > r$ *and* $l < 1$*. If* $\{b_k(0)\}_{k\geq 1}$ *satisfy the assumption*
(A)

$$
\sum_{k=1}^{\infty} \mid b_k(0) \mid \rho^k k^{3/2} \leq \frac{1}{\sqrt{k_0}} l \min_{(\overline{B_r(0)},[0,t_1])} |f^{'}_{k_0}|,
$$

and $b_1(0) \in R$, *then the following (a)-(d) are true:*
(a)The initial value $f_{k_0}(\xi,0)+\sum_{k=1}^{\infty} b_k(0)\xi^k$ *gives rise to a strong solution to (2.1),* $f(\xi,t)$, *at least locally in time.*
(b)Let

$$A=\Big\{h(z,t)\in\omega(\overline{B_r(0)})\cap C^1([0,t_h],H(\overline{B_r(0)}))\ \textit{a strong polynomial solution to}$$

$$(2.1), 0<t_h\le t_1\Big|\max_{([0,t_h])}\big|h^{'}(z,t)-f^{'}_{k_0}(z,t)\big|_{M(r)}\le l\min_{(\overline{B_r(0)},[0,t_1])}\big|f^{'}_{k_0}\big|\Big\}$$

and

$$M=\sup\Big\{\max_{(\partial B_1(0),[0,t_h])}\Big|\frac{1}{2\pi i}\int_{\partial B_1(0)}\frac{1}{|h^{'}(z,t)|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big|\Big|h\in A\Big\},$$

then $M<\infty$.
(c)Define

$$t_0=\min\Big\{\frac{1}{Ck_0}(\ln\rho-\ln r),t_1\Big\}$$

where

$$C=\Big\{M+\sqrt{2}(1+l)\frac{2}{(1-l)^3}\max_{(\partial B_1(0),[0,t_1])}\big|f^{'}_{k_0}\big|\max_{(\partial B_1(0),[0,t_1])}\frac{1}{|f^{'}_{k_0}|^3}\Big\}.$$

Then $f(\xi,t)\in C^1([0,t_0],H(B_r(0)))\cap\omega(B_r(0))$ *and*

$$\max_{([0,t_0])}\big|f^{'}-f^{'}_{k_0}\big|_{M(r)}\le l\min_{(\overline{B_r(0)},[0,t_1])}\big|f^{'}_{k_0}\big|.$$

(d)Furthermore, if there exist $\delta>0$ *and* j *nonnegative integer such that*

$$\sum_{k=1}^{\infty}\mid b_k(0)\mid\rho^k k^{\frac{2j+1}{2}}\le\delta,$$

then there exists $c(j,k_0)>0$ *such that*

$$\max_{([0,t_0])}\big|f^{(j)}-f^{(j)}_{k_0}\big|_{M(r)}\le c(j,k_0)\delta.$$

Remark 2.3. Theorem 2.5 is also true for the suction case.

Proof. (1)We want to prove (b) by showing that $M<\infty$ as follows.

For $h \in A$, $0 \le t \le t_h$, by Lemma 2.1,

$$\max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|h'|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big|$$

$$\le \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|f'_{k_0}|^2} - \frac{1}{|h'|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big| + \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|f'_{k_0}|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big|$$

$$\le \max_{\partial B_r(0)} \Big| \frac{1}{h'(z,t)\overline{h'}(1/z,t)} - \frac{1}{f'_{k_0}(z,t)\overline{f'_{k_0}}(1/z,t)} \Big| \frac{r+1}{r-1} + \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|f'_{k_0}|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big|$$

$$\le \max_{\partial B_r(0)} \Big| \frac{h'(z,t) - f'_{k_0}(z,t)}{f'_{k_0}(z,t)\overline{f'_{k_0}}(1/z,t)h'(z,t)} + \frac{\overline{h'}(1/z,t) - \overline{f'_{k_0}}(1/z,t)}{h'(z,t)\overline{h'}(1/z,t)\overline{f'_{k_0}}(1/z,t)} \Big| \frac{r+1}{r-1}$$

$$+ \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|f'_{k_0}|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big|.$$

In order to prove that $M < \infty$, it is enough to show that there exists $B(f_{k_0}) > 0$ such that

$$\max_{(\partial B_r(0),[0,t_h])} \Big| \frac{h'(z,t) - f'_{k_0}(z,t)}{f'_{k_0}(z,t)\overline{f'_{k_0}}(1/z,t)h'(z,t)} + \frac{\overline{h'}(1/z,t) - \overline{f'_{k_0}}(1/z,t)}{h'(z,t)\overline{h'}(1/z,t)\overline{f'_{k_0}}(1/z,t)} \Big| \frac{r+1}{r-1} < B(f_{k_0}).$$

Since $h \in A$, then for $(z,t) \in (\partial B_r(0), [0, t_h])$,

$$\big| h'(z,t) - f'_{k_0}(z,t) \big| \le l \big| f'_{k_0}(z,t) \big|,$$

and

$$\big| h'(z,t) \big| \ge (1-l) \big| f'_{k_0}(z,t) \big|. \tag{2.2}$$

Also, for $(z,t) \in (\partial B_r(0), [0, t_h])$,

$$\big| \overline{h'}(1/z,t) - \overline{f'_{k_0}}(1/z,t) \big| \le l \big| \overline{f'_{k_0}}(1/z,t) \big|,$$

$$\big| \overline{h'}(\frac{1}{z},t) \big| \ge (1-l) \big| \overline{f'_{k_0}}(\frac{1}{z},t) \big|. \tag{2.3}$$

Therefore by (2.2) and (2.3),

$$\max_{(\partial B_r(0),[0,t_h])} \Big| \frac{h'(z,t) - f'_{k_0}(z,t)}{f'_{k_0}(z,t)\overline{f'_{k_0}}(1/z,t)h'(z,t)} + \frac{\overline{h'}(1/z,t) - \overline{f'_{k_0}}(1/z,t)}{h'(z,t)\overline{h'}(1/z,t)\overline{f'_{k_0}}(1/z,t)} \Big| \frac{r+1}{r-1}$$

$$\le 2l \Big[\max_{(\partial B_r(0),[0,t_1])} \Big| \frac{1}{|f'_{k_0}(z,t)||\overline{f'_{k_0}}(1/z,t)|(1-l)^2} \Big| \Big] \frac{r+1}{r-1}.$$

Therefore $M < \infty$.
(2)We want to prove (a) and (c) in the following, by showing that there exists a strong solution $f(\xi, t) \in \omega(B_r(0))$ to (2.1) for $0 \le t \le t_0$, where $f(\xi, 0) = f_{k_0}(\xi, 0) + \sum_{i=1}^{\infty} b_i(0)\xi^i$.

By assumption (A), there exist $\{d_k\}_{k \ge 0}$ nonnegative and $\sum_{k=0}^{\infty} d_k = 1$ such that $\mid b_i(0) \mid \le M_i \rho^{-i}$ for $i \ge 1$ where

$$M_{k+1} \le \frac{1}{\sqrt{k_0}} \frac{1}{(k+1)^{3/2}} d_k \min_{(B_r(0),[0,t_1])} |f_{k_0}^{'}| l, k \ge 0.$$

Denote the polynomial solution to (2.1) with the initial value $f_{k_0}(\xi, 0) + \sum_{i=1}^{k} b_i(0)\xi^i$ by $g_k(\xi, t)$. And the solution $g_k(\xi, t) \in \omega(\overline{B_r(0)})$ exists for at least a short time since $f_{k_0}(\xi, 0) + \sum_{i=1}^{k} b_i(0)\xi^i$ is in $\omega(\overline{B_r(0)})$ for $k \ge 0$ by the assumption in (A).
Step1:

Here, we want to prove that for $k \ge 0$, $g_k(\xi, t) \in C^1([0, t_0], H(\overline{B_r(0)})) \cap \omega(\overline{B_r(0)})$ and

$$\max_{([0,t_0])} |g_k^{'} - g_{k+1}^{'}|_{M(r)} \le l d_k \min_{(\overline{B_r(0)},[0,t_1])} |g_0^{'}|$$

by induction.
(i)Assume for $0 \le k \le n-1$,

$$\max_{([0,t_0])} |g_k^{'} - g_{k+1}^{'}|_{M(r)} \le l d_k \min_{(\overline{B_r(0)},[0,t_1])} |g_0^{'}|.$$

(ii)From (i), this means for (z, t) in $(\overline{B_r(0)}, [0, t_0]), 0 \le k \le n-1$,

$$|g_{k+1}^{'}| \ge |g_0^{'}| - \sum_{j=0}^{k} |g_j^{'} - g_{j+1}^{'}| \ge |g_0^{'}| - \sum_{j=0}^{k} l d_j \min_{(\overline{B_r(0)},[0,t_1])} |g_0^{'}|$$

$$\ge |g_0^{'}| - l|g_0^{'}| = (1-l)|g_0^{'}|.$$

Similarly,

$$|g_{k+1}^{'}| \le (1+l)|g_0^{'}|.$$

Finally, for $(z, t) \in (\overline{B_r(0)}, [0, t_0])$ and $0 \le k \le n-1$,

$$(1-l)|g_0^{'}| \le |g_{k+1}^{'}| \le (1+l)|g_0^{'}|. \tag{2.4}$$

In particular, if $k = n-1$ in (2.4),

$$(1-l)|g_0^{'}| \le |g_n^{'}| \le (1+l)|g_0^{'}|. \tag{2.5}$$

Also by the assumption in (i), we have

$$\max_{([0,t_0])} \left|g_n^{'} - g_0^{'}\right|_{M(r)} \leq \sum_{k=0}^{n-1} l d_k \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right| \leq l \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right| \tag{2.6}$$

which means that g_n is in the set A stated in (b).
(iii)**Claim**:
For $t \in [0, t_0]$

$$\left|g_n^{'} - g_{n+1}^{'}\right|_{M(r)} \leq l d_n \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right|. \tag{2.7}$$

Proof. (of claim) Assume that (2.7) holds for $0 \leq t \leq s_n \leq t_1$. Hence for $(z,t) \in (\overline{B_r(0)}, [0, s_n])$,

$$(1-l)\left|g_0^{'}\right| \leq \left|g_{n+1}^{'}\right| \leq (1+l)\left|g_0^{'}\right|. \tag{2.8}$$

We need to show $s_n \geq t_0$.

By Lemma 2.3, we have

$$\begin{aligned}
&\left\|\frac{d}{dt}[g_n - g_{n+1}]\right\|_{L^2([0,2\pi])} \\
\leq &\Big\{ \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|g_n^{'}|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big| + \sqrt{2} \max_{\partial B_1(0)} |g_{n+1}^{'}| \max_{B_1(0)} \frac{|g_n^{'}| + |g_{n+1}^{'}|}{|g_n^{'}|^2 |g_{n+1}^{'}|^2} \Big\} \\
&\times \left\| [g_n^{'} - g_{n+1}^{'}] \right\|_{L^2([0,2\pi])}.
\end{aligned} \tag{2.9}$$

Due to (2.5), (2.6) and (2.8), there exists $C(f_{k_0}) > 0$ as defined in (c) such that for $0 \leq t \leq \min\{s_n, t_0\}$,

$$\Big\{ \max_{\partial B_1(0)} \Big| \frac{\xi}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|g_n^{'}|^2} \frac{z+\xi}{z-\xi} \frac{dz}{z} \Big| + \sqrt{2} \max_{\partial B_1(0)} \left|g_{n+1}^{'}\right| \max_{\partial B_1(0)} \frac{|g_n^{'}| + |g_{n+1}^{'}|}{|g_n^{'}|^2 |g_{n+1}^{'}|^2} \Big\} \leq C.$$

Therefore, (2.9) implies for $0 \leq t \leq \min\{s_n, t_0\}$

$$\left\|\frac{d}{dt}[g_n - g_{n+1}]\right\|_{L^2([0,2\pi])} \leq C \left\|g_n^{'} - g_{n+1}^{'}\right\|_{L^2([0,2\pi])}.$$

Assume first that $n \geq k_0$. Denote $g_n = \sum_{i=1}^{n} \alpha_i(t)\xi^i$ and $g_{n+1} = \sum_{i=1}^{n+1} \beta_i(t)\xi^i$. Define

$$D(t) = \left\|g_{n+1}^{'} - g_n^{'}\right\|^2_{L^2([0,2\pi])} = 2\pi \Big\{ \sum_{i=1}^{n} [\mid \alpha_i(t) - \beta_i(t) \mid]^2 i^2 + \mid \beta_{n+1}(t) \mid^2 (n+1)^2 \Big\}.$$

$$
\begin{aligned}
D^{'}(t) &= 2\pi \cdot 2\Big\{ \sum_{i=1}^{n} Re\big[(\alpha_i-\beta_i)\overline{(\alpha_i-\beta_i)_t}\big] i^2 + Re\big[(\beta_{n+1})\overline{(\beta_{n+1}(t))_t}\big](n+1)^2 \Big\} \\
&\le 2\pi \cdot 2(n+1)\Big\{ \sum_{i=1}^{n} \mid (\alpha_i-\beta_i) \mid\mid (\alpha_i-\beta_i)_t \mid i + \mid (\beta_{n+1}) \mid\mid (\beta_{n+1}(t))_t \mid (n+1) \Big\} \\
&\le 2\pi \cdot 2(n+1)\Big\{ \sum_{i=1}^{n} \mid (\alpha_i-\beta_i) \mid^2 i^2 + \mid (\beta_{n+1}) \mid^2 (n+1)^2 \Big\}^{\frac{1}{2}} \\
&\quad \times \Big\{ \sum_{i=1}^{n} \mid (\alpha_i-\beta_i)_t \mid^2 + \mid (\beta_{n+1}(t))_t \mid^2 \Big\}^{\frac{1}{2}}.
\end{aligned}
$$

We conclude that for $0 \le t \le \min\{t_0, s_n\}$,

$$
\begin{aligned}
D^{'}(t) &\le 2(n+1)\Big\| \frac{d}{dt}[g_n - g_{n+1}] \Big\|_{L^2([0,2\pi])} \big\| [g_n^{'} - g_{n+1}^{'}] \big\|_{L^2([0,2\pi])} \\
&\le 2C(n+1)\big\| [g_n^{'} - g_{n+1}^{'}] \big\|^2_{L^2([0,2\pi])}
\end{aligned}
$$

$$
\begin{aligned}
D^{'}(t) &\le 2C(n+1)D(t). \\
(D(t)e^{-2C(n+1)t})^{'} &\le 0. \\
D(t)e^{-2C(n+1)t} - D(0) &\le 0. \\
D(t) &\le D(0)e^{2Ct(n+1)}.
\end{aligned}
$$

If $n < k_0$, similarly,

$$
D(t) \le D(0)e^{2Ct(k_0)}.
$$

Note that if $s_n < t_0$, then the following (R_1) and (R_2) must hold:
(R_1) At time $t = s_n$,

$$
\big|g_n^{'} - g_{n+1}^{'}\big|_{M(r)} = d_n \min_{(\overline{B_r(0)},[0,t_1])} \big|g_0^{'}\big| l.
$$

(R_2) Also for $t = s_n^{+}$,

$$
\big|g_n^{'} - g_{n+1}^{'}\big|_{M(r)} > d_n \min_{(\overline{B_r(0)},[0,t_1])} \big|g_0^{'}\big| l.
$$

If $s_n < t_0$, then for $0 \le t \le s_n$,

$$\begin{aligned}
&\left|g_n^{'} - g_{n+1}^{'}\right|_{M(r)} \\
\le&\sqrt{D(t)(n+1)k_0}r^{(n)} \\
\le&\sqrt{(n+1)k_0 D(0) e^{2Ctk_0(n+1)} r^{2(n)}} \\
\le&\sqrt{(n+1)k_0 D(0) e^{2Cs_n k_0(n+1)} r^{2(n)}} \\
<&\sqrt{(n+1)k_0 D(0) e^{2Ct_0 k_0(n+1)} r^{2(n)}}.
\end{aligned}$$

Since

$$D(0)(n+1)k_0 \le (\rho)^{-2(n+1)}(d_n)^2 \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right|^2 l^2,$$

we have

$$\begin{aligned}
&\max_{([0,s_n])} \left|g_n^{'} - g_{n+1}^{'}\right|_{M(r)} \\
\le&\sqrt{(n+1)k_0 D(0) e^{2Ct_0 k_0(n+1)} r^{2(n)}} \\
<&d_n \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right| l
\end{aligned}$$

which contradicts the remark (R_1). Therefore, $s_n \ge t_0$. □

Step2:
By Step 1, for $k \ge 1$

$$\max_{([0,t_0])} \left|g_k^{'} - g_0^{'}\right|_{M(r)} \le l \sum_{n=0}^{\infty} d_n \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right| \le l \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right|.$$

Let k go to ∞. There exists $f(\xi,t) \in C([0,t_0], \omega(B_r(0)) \cap C(\overline{B_r(0)}))$ such that

$$\max_{([0,t_0])} \left|f^{'} - g_0^{'}\right|_{M(r)} \le l \min_{(\overline{B_r(0)},[0,t_1])} \left|g_0^{'}\right|.$$

Still, we have to show $f(\xi,t)$ satisfies (2.1). Fix $1 < r^{'} < r$. For $\xi \in B_{r'}(0)$ and $0 \le t \le t_0$,

$$\frac{d}{dt} g_k(\xi,t) = \frac{g_k^{'}(\xi,t)\xi}{2\pi i} \int_{\partial B_{r'}(0)} \frac{1}{g_k^{'}(z,t)\overline{g_k^{'}}(\frac{1}{z},t)} \frac{z+\xi}{z-\xi} \frac{dz}{z}.$$

Integrating this equation with respect to t, we have that for $\xi \in B_{r'}(0)$ and $0 \le t \le t_0$,

$$g_k(\xi,t) - g_k(\xi,0) = \int_0^t \frac{g_k'(\xi,t)\xi}{2\pi i} \int_{\partial B_{r'}(0)} \frac{1}{g_k'(z,t)\overline{g_k'}(\frac{1}{z},t)} \frac{z+\xi}{z-\xi} \frac{dz}{z} dt.$$

Let $k \to \infty$. For ξ in any compact subset of $B_{r'}(0)$,

$$f(\xi,t) - f(\xi,0) = \int_0^t \frac{f'(\xi,t)\xi}{2\pi i} \int_{\partial B_{r'}(0)} \frac{1}{f'(z,t)\overline{f'}(\frac{1}{z},t)} \frac{z+\xi}{z-\xi} \frac{dz}{z} dt \qquad (2.10)$$

for some $f(\xi,t) \in C([0,t_0], \omega(B_r(0)) \cap C(\overline{B_r(0)}))$. The identity (2.10) shows that $f(\xi,t) \in C^1([0,t_0], H(B_r(0)) \cap C(\overline{B_r(0)}))$, and also we have $f(\xi,t) \in \omega(B_r(0))$ since $f(\xi,t) \in C([0,t_0], \omega(B_r(0)) \cap C(\overline{B_r(0)}))$.
(3)Now assume (d). Then

$$| b_i(0) | \le M_i \rho^{-i}, i \ge 1$$

where

$$M_{k+1} \le \frac{1}{(k+1)^{\frac{1}{2}+j}} d_k \delta, k \ge 0.$$

First we look at the case $j = 2$. Under (d),

$$\begin{aligned}
&\max_{([0,t_0])} \left| g_n'' - g_{n+1}'' \right|_{M(r)} \\
\le &\sqrt{(n+2)^3 (k_0+1)^3 \frac{1}{3} D(0) e^{2Ct_0 k_0 (n+1)}} r^{n-1} \\
= &\Big(\frac{n+2}{n+1}\Big)^{\frac{3}{2}} \frac{1}{\sqrt{3}} (k_0+1)^{\frac{3}{2}} \sqrt{D(0)(n+1)^3 e^{2ct_0 k_0 (n+1)}} r^{n-1} \\
\le &\Big(\frac{n+2}{n+1}\Big)^{\frac{3}{2}} \frac{1}{\sqrt{3}} (k_0+1)^{\frac{3}{2}} d_n \delta, n \ge 0.
\end{aligned}$$

Therefore, we have for $n \ge 1$

$$\max_{([0,t_0])} \left| g_0'' - g_n'' \right|_{M(r)} \le \frac{1}{\sqrt{3}} 2^{\frac{3}{2}} (k_0+1)^{\frac{3}{2}} \delta.$$

Assume $j \ge 2$ now. Under the assumption of (d), there exists $c(j,k_0) > 0$ such that

$$\begin{aligned}
&\max_{([0,t_0])} \left| g_n^{(j)} - g_{n+1}^{(j)} \right|_{M(r)} \\
\le &c(j,k_0) \sqrt{(n+1)^{2j-1} D(0) e^{2Ct_0 k_0 (n+1)}} \\
\le &c(j,k_0) d_n \delta.
\end{aligned}$$

Therefore, we have

$$\max_{([0,t_0])} \left| g_0^{(j)} - g_n^{(j)} \right|_{M(r)} \leq c(j,k_0)\delta.$$

Let $n \to \infty$,

$$\max_{([0,t_0])} \left| g_0^{(j)} - f^{(j)} \right|_{M(r)} \leq c(j,k_0)\delta.$$

□

We will give two applications of Theorem 2.5 in the rest of this section. Define

$$\|v\|_{\rho,n} = \sum_{j=1}^{\infty} \mid v_j \mid \rho^j j^{\frac{1}{2}+n}, v = \sum_{j=1}^{\infty} v_j \xi^j.$$

Theorem 2.6. *Given a strong global polynomial solution* $f_{k_0}(\xi,t)$ *to* (1.1)*, then there exists* $r > 1$ *such that for* $t \geq 0$*,*

$$f_{k_0}(\xi,t) \in O(\overline{B_r(0)}).$$

Also given $\epsilon > 0, T_0 > 0, k \in N$ *and* $1 < r^{'} < r$*, there exist* $\delta(f_{k_0}) > 0$ *and* $\rho(f_{k_0}) > 1$ *such that if* $\|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,k} < \delta$ *where* $f(0,0) = 0$ *and* $f^{'}(0,0) > 0$*, then the strong solution* $f(\xi,t)$ *to* (1.1) *satisfies*

$$f(\xi,t) \in O(\overline{B_{r'}(0)}) \cap C^1([0,T_0], H(B_r(0))),$$

and for $0 \leq n \leq k$*,* $0 \leq t \leq T_0$*,*

$$\left| f_{k_0}^{(n)}(\xi,t) - f^{(n)}(\xi,t) \right|_{M(r)} < \epsilon.$$

Proof. (a)There exists $r > 1$ such that $f_{k_0}(\xi,t) \in O(\overline{B_r(0)})$ for all $t > 0$.
(b)By Theorem 4.1 in the appendix, there exists $\eta(f_{k_0}, T_0, r^{'}) > 0$ such that if $f(\xi,t)$ satisfies

$$f(\xi,t) \in C^1([0,T_0], H(B_r(0))) \quad \text{and} \quad \max_{([0,T_0])} \mid f_{k_0}'(\xi,t) - f'(\xi,t) \mid_{M(r)} \leq \eta,$$

then $f(\xi,t) \in O(\overline{B_{r'}(0)})$ for $t \in [0,T_0]$.
(c)We apply Theorem 2.5 by letting $t_1 = T_0$, $l = \frac{1}{2}$, δ small enough such that

$$\delta < \min_{1 \leq j \leq k} \left\{ \frac{\epsilon}{c(j,k_0)} \right\}, \delta < \frac{l}{\sqrt{k_0}} \min_{(B_r(0),[0,T_0])} \mid f_{k_0}^{'}(\xi,t) \mid, \delta < \min_{1 \leq j \leq k} \left\{ \frac{\eta}{c(j,k_0)} \right\}$$

and $\rho > 1$ large enough such that $\frac{1}{Ck_0}(\ln\rho - \ln r) \geq T_0$. We get that for $0 \leq n \leq k$, $0 \leq t \leq T_0$, the strong solution $f(\xi, t)$ to (2.1) satisfies

$$\left| f_{k_0}^{(n)}(\xi,t) - f^{(n)}(\xi,t) \right|_{M(r)} < \min\{\epsilon, \eta\}.$$

This shows that $f(\xi,t) \in O(\overline{B_{r'}(0)})$ and hence $f(\xi,t)$ solves (1.1). □

Corollary 2.7. *Let $f_{k_0}(\xi,t)$ be a strong global polynomial solution to* (1.1). *Then there exist $\rho > 1$ and $\delta > 0$ such that if $\|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,1} < \delta$ where $f(0,0) = 0$ and $f^{'}(0,0) > 0$, then $f(\xi,t)$ is a strong global solution to* (1.1) *as well.*

Proof. (1)Denote $f_{k_0}(\xi,t) = \sum_{i=1}^{k_0} a_i(t)\xi^i$. There exists $T_0 > \frac{1}{2}$ such that

$$\sum_{i=2}^{k_0} i \mid a_i(t) \mid \leq \frac{1}{2}, t \geq T_0.$$

(2)Take $\epsilon = \frac{1}{2}, k = 1$.
(3)For T_0, ϵ and k as above, by Theorem 2.6, there exist $\rho > 1$ and $\delta > 0$ such that if $\|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,1} < \delta$, then $f(\xi,t) \in O(B_r(0)) \cap C^1([0,T_0], H(B_r(0)))$ for some $r > 1$ and

$$\left| f_{k_0}^{'}(\xi,t) - f^{'}(\xi,t) \right|_M < \epsilon = \frac{1}{2}, 0 \leq t \leq T_0.$$

In particular, the above inequality implies that as $t = T_0$,

$$\sum_{i=2}^{\infty} i \mid b_i(T_0) \mid < \frac{1}{2} + \sum_{i=2}^{k_0} i \mid a_i(T_0) \mid \leq 1$$

where $f(\xi,t) = \sum_{i=1}^{\infty} b_i(t)\xi^i \in O(B_r(0))$ for $0 \leq t \leq T_0$.
(4)Finally,

$$\sum_{i=2}^{\infty} i \mid b_i(T_0) \mid < 1 < 2T_0 < b_1(T_0).$$

This is a sufficient condition for a strongly starlike function as shown in Pommerenke [6] and hence $f(\xi,T_0)$ is a strongly starlike function of order < 1. Therefore $f(\xi,t)$ must be a global strong solution to (1.1).

Remark 2.4. As shown in Theorem 2.5 in Lin [5], $f_{k_0}(\xi,0)$ can be a non-starlike function.

□

3 Rescaling behaviors and the geometric meaning

Here, we deal with strong global solutions to the Hele-Shaw problem, which are simply connected and have real analytic boundaries. Equivalently, for each moving domain $\Omega(t)$ here, there exists a global strong solution $f(\xi, t)$ to (1.1) such that $f(B_1(0), t)$ represents $\Omega(t)$. Here we denote the Richardson complex moments by $\{M_k(t)\}_{k\geq 0}$.

Recalling the rescaling in Vondenhoff [1] for two dimensions, we represent it in terms of $f(\xi, t)$. The author defines the quantity $r(t, \theta) = \frac{|f(\xi,t)|}{\sqrt{2t+R^2}} - 1$ where $\theta = \arg \frac{f(\xi,t)}{|f(\xi,t)|}$ in the case that the initial domain is a small perturbation of a open disk $B_R(0)$. In this case, the asymptotic behavior of $r(t, \theta)$ is discussed in Vondenhoff [1].

Inspired by the ideas in Vondenhoff [1], we define a rescaling $\overline{r}(t, \theta) = \frac{|f(\xi,t)|}{\sqrt{2t+M_0(0)}} - 1$ where $\theta = \arg \frac{f(\xi,t)}{|f(\xi,t)|}$ for ξ on S^1 and $M_0(0)$ is the zero moment at $t = 0$. The value $\overline{r}(t, \theta)$ is well-defined if the function $f(\xi, t)$ is strongly starlike. We show in this paper that the solution becomes strongly starlike eventually though it initially might not be. The boundary $\partial\Omega(t)$ has the polar coordinate equation $(S_2(1 + \overline{r}(t, \theta)), \theta)$ where $S_2 = \sqrt{2t + M_0(0)}$ if $\Omega(t)$ becomes starlike. After the rescaling, domains with the boundary polar coordinate equation $\{(1 + \overline{r}(t, \theta), \theta)\}_{t\geq 0}$ have area π as stated in the introduction. We can see that $\overline{r}(\theta, t) = \overline{r}(\theta + 2\pi, t)$. Here we write $\xi = e^{i\alpha}$.

Lemma 3.1 and Lemma 3.2 address the relation of $\overline{r}(t, \theta)$ and $f(\xi, t)$, and of $r(t, \theta)$ and $f(\xi, t)$. In this section, we first calculate the decay rate of $\|\overline{r}(t, \cdot)\|_{C^3(S^1)}$ in the case that the family of moving domains is a family of mappings of a global strong polynomial solution. Second, starting with the initial domain $\Omega(0)$, which is the mapping of a function of $O(\overline{B_1(0)})$ constructed by perturbing the initial function of a global strong polynomial solution to (1.1), we show that the solution $\{\Omega(t)\}_{t\geq 0}$ is global as well. The domain $\Omega(t)$ becomes a small perturbation of some open disk centered at the origin at some later time $T_0 > 0$, and we can apply the results in Vondenhoff [1].

Recall the definition used in Vondenhoff [1].

$$h^{2,\alpha}(\overline{\Omega}) := \big\{ r \in C^{2,\alpha}(\overline{\Omega}) \mid \forall \beta, \mid \beta \mid = 2, \partial^\beta r \in h^{0,\alpha}(\overline{\Omega}) \big\},$$

where

$$h^{0,\alpha}(\overline{\Omega}) := \Big\{ r \in C^{0,\alpha}(\overline{\Omega}) \mid \lim_{\epsilon\to 0} \sup_{x,y\in\overline{\Omega}, |x-y|<\epsilon} \frac{\mid r(x) - r(y) \mid}{\mid x - y \mid^\alpha} = 0 \Big\}.$$

Lemma 3.1. *If $f(\xi) : B_1(0) \to \Omega$ is a strongly starlike function of order < 1 and $f(\xi) \in O(\overline{B_1(0)})$, then*

$$r(\theta), \overline{r}(\theta) \in C^\infty(S^1) = C^\infty(S^1).$$

Furthermore, $r(\theta)$ and $\overline{r}(\theta)$ are not well-defined if the domain is a nonstarlike domain.

Proof. As defined, $\theta = \arg \frac{f(\xi,t)}{|f(\xi,t)|}$. Since f is a strongly starlike function of order < 1,

$$\partial_\alpha \theta = Im \partial_\alpha \Big(\ln \frac{f(\xi, t)}{| f(\xi, t) |} \Big) = Im \Big(\frac{i f'(\xi)\xi}{f(\xi)} \Big) = Re \Big(\frac{f'(\xi)\xi}{f(\xi)} \Big) > 0.$$

That means there exists $F : S^1 \to S^1$,

$$\theta = F(\alpha) \in C^\infty(S^1) \quad \text{and} \quad \alpha = F^{-1}(\theta) \in C^\infty(S^1).$$

Similarly, this argument also holds for $\overline{r}(t, \theta)$. □

Lemma 3.2.

$$C^\infty \subset h^{2,\alpha}.$$

3.1 Rescaling behaviors for small data

First, we rewrite the results in Vondenhoff [1] as Theorems 3.3 and 3.4, in the case that the injection strength 2π, $N = 2$ and the initial domain is a small perturbation of the open disk $B_R(0)$. The function $r(t, \theta) = \frac{|f(\xi,t)|}{\sqrt{2t+R^2}} - 1$ where $\theta = \arg f(\xi, t)$.

Theorem 3.3. *([1]) Fix $\lambda \in (0, 1)$. There exist $\delta(\lambda) > 0$ and $M(\lambda) > 0$ such that the problem with $r(0) = r_0 \in h^{2,\alpha}(S^1)$ and $\|r_0\|_{C^{2,\alpha}(S^1)} \leq \delta$ has a solution $r \in C([0, \infty), h^{2,\alpha}(S^1)) \cap C^1([0, \infty), h^{1,\alpha}(S^1))$ satisfying*

$$\|r(t, \cdot)\|_{C^{2,\alpha}(S^1)} \leq M(2t + R^2)^{-\lambda} \|r_0\|_{C^{2,\alpha}(S^1)}.$$

In the case that the domain is a small perturbation of $B_R(0)$ **but has area** $R^2\pi$, if the lower Richardson moments vanish, there is a better estimate for the decay rate. Also, $\overline{r} = r$ in this case. Let $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$.

Theorem 3.4. *([1]) Fix $\lambda \in (0, 1 + \frac{n_0}{2})$. Assume all the initial domains, which are small perturbations of $B_R(0)$ here, have area $R^2\pi$. There exist $\delta_{n_0}(\lambda) > 0$ and $M_{n_0}(\lambda) > 0$ such that the problem with $\overline{r}(0) = \overline{r}_0 \in h^{2,\alpha}(S^1)$ and $\|\overline{r}_0\|_{C^{2,\alpha}(S^1)} \leq \delta_{n_0}(\lambda)$ has a solution $\overline{r} \in C([0,\infty), h^{2,\alpha}(S^1)) \cap C^1([0,\infty), h^{1,\alpha}(S^1))$ satisfying*

$$\|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)} \leq M_{n_0}(\lambda)(2t + R^2)^{-\lambda}\|\overline{r}_0\|_{C^{2,\alpha}(S^1)}.$$

Corollary 3.5. *Assume all the initial domains, which are the small perturbations of $B_R(0)$ here, have area $R^2\pi$, then there exists $\delta > 0$, such that if $\|r_0\|_{C^{2,\alpha}(S^1)} \leq \delta$ and $r_0 \in h^{2,\alpha}(S^1)$,*

$$\limsup_{t\to\infty} \|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)}(2t)^{\lambda} = 0, \forall \lambda \in (0, 1 + \frac{n_0}{2})$$

where $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$.

Proof. There exist $\delta(\frac{1}{4})$ and $M(\frac{1}{4}) > 0$ as stated in Theorem 3.3 such that if $\|\overline{r}_0\|_{C^{2,\alpha}(S^1)} \leq \delta(\frac{1}{4})$, then there exists a global solution such that

$$\|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)} \leq M(\frac{1}{4})(2t + R^2)^{\frac{-1}{4}}\|\overline{r}_0\|_{C^{2,\alpha}(S^1)}.$$

For $\lambda \in (0, 1 + \frac{n_0}{2})$, there exist $M_{n_0}(\lambda)$ and $\delta_{n_0}(\lambda)$ as stated in Theorem 3.4. Pick T_λ such that

$$M(\frac{1}{4})(2T_\lambda + R^2)^{\frac{-1}{4}}\|\overline{r}_0\|_{C^{2,\alpha}(S^1)} \leq \delta_{n_0}(\lambda).$$

Applying Theorem 3.4 again with the initial value $\overline{r}(T_\lambda)$, then we have that for $t \geq T_\lambda$,

$$\|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)} \leq M_{n_0}(\lambda)(2T + R^2)^{-\lambda}M(\frac{1}{4})(2T_\lambda + R^2)^{\frac{-1}{4}}\|\overline{r}_0\|_{C^{2,\alpha}(S^1)}.$$

We conclude that Theorem 3.4 implies that there exists $\delta > 0$ such that if $\|\overline{r_0}\|_{C^{2,\alpha}(S^1)} \leq \delta$,

$$\limsup_{t\to\infty} \|\overline{r}(t,\cdot)\|_{C^{2,\alpha}(S^1)}(2t)^{\lambda} = 0, \forall \lambda \in (0, 1 + \frac{n_0}{2}).$$

□

3.2 Rescaling behaviors for large data

It is proven that given a global strong polynomial solution to equation (1.1), called $\{f_{k_0}(\xi,t)\}_{t\geq 0}$, then for any ξ

$$\lim_{t\to\infty}\big[f_{k_0}(\xi,t)-\sqrt{2t+M_0(0)}\xi\big](2t)^{\frac{n_0+1}{2}}=\overline{M_{n_0}}\xi^{n_0+1},$$

where $n_0=\min\{k\geq 1\mid M_k\neq 0\}$. Furthermore,

$$\lim_{t\to\infty}\max_{\xi\in\overline{B_1(0)}}\big[f_{k_0}(\xi,t)-\sqrt{2t+M_0(0)}\xi\big](2t)^{\frac{n_0+1}{2}}=\mid M_{n_0}\mid. \tag{3.1}$$

In this case, we have a result corresponding to Corollary 3.5.

Theorem 3.6. *Given a global strong dergree k_0 polynomial solution to equation (1.1), called $f_{k_0}(\xi,t)$, then there exists $T_0(f_{k_0})>0$ such that $f_{k_0}(\xi,t)\in O(\overline{B_1(0)})$ is a strongly starlike function of order <1 for $t\geq T_0$ and*

$$\limsup_{T_0\leq t\to\infty}\|\overline{r}(t,\cdot)\|_{C^3(S^1)}(2t)^{\frac{3}{2}}<\infty.$$

Furthermore,

$$\limsup_{T_0\leq t\to\infty}\|\overline{r}(t,\cdot)\|_{C^3(S^1)}(2t)^{1+\frac{n_0}{2}}=n_0^3\mid M_{n_0}\mid$$

where $n_0=\min\{k\geq 1\mid M_k\neq 0\}$.

Proof. Denote $f_{k_0}(\xi,t)=\sum_{i=1}^{k_0}a_i\xi^i\in O(\overline{B_1(0)})$. There exists $T_0>0$ such that $\{f_{k_0}(\xi,t)\}_{t\geq T_0}$ is a strongly starlike function of order <1 since $\{a_k(t)\}_{2\leq k\leq k_0}$ decay to zero. Denote $S_2=\sqrt{2t+M_0(0)}$. By (3.1),

$$\lim_{T_0\leq t\to\infty}\|\overline{r}(t,\cdot)\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}=\mid M_{n_0}\mid. \tag{3.2}$$

For the first derivative,

$$\partial_\theta(\overline{r}(t,\theta))=(\partial_\theta\alpha)(\partial_\alpha\overline{r}(t,\theta))=(\partial_\theta\alpha)\partial_\alpha\Big(\frac{\mid f_{k_0}(\xi,t)\mid}{S_2}\Big)$$

$$=\frac{1}{Re(\frac{f_{k_0}'\xi}{f_{k_0}})}\frac{1}{S_2\mid f_{k_0}\mid}Re\big[(f_{k_0})_\alpha\overline{f_{k_0}}\big]=\frac{-1}{Re(\frac{f_{k_0}'\xi}{f_{k_0}})}\frac{1}{S_2\mid f_{k_0}\mid}Im\big(\xi f_{k_0}'\overline{f_{k_0}}\big).$$

Therefore, the first derivative of $\overline{r}(t,\theta)$ is

$$\partial_\theta(\overline{r}(t,\theta))=\frac{-1}{Re(\frac{f_{k_0}'\xi}{f_{k_0}})}\frac{1}{S_2\mid f_{k_0}\mid}Im\big(\xi f_{k_0}'\overline{f_{k_0}}\big). \tag{3.3}$$

Write

$$f_{k_0}(\xi,t)=\sqrt{2t+M_0(0)}\xi+\overline{M_{n_0}}\xi^{n_0+1}(\frac{1}{2t})^{\frac{n_0+1}{2}}+o\Big(\frac{1}{2t}\Big)^{\frac{n_0+1}{2}}.$$

For ξ on $\partial B_1(0)$,

$$Im\big[\xi f^{'}_{k_0}\overline{f_{k_0}}\big]=Im\Big[\sqrt{2t+M_0(0)}\overline{M_{n_0}}n_0\xi^{n_0}\Big(\frac{1}{2t}\Big)^{\frac{n_0+1}{2}}+o\Big(\frac{1}{2t}\Big)^{\frac{n_0}{2}}\Big]. \quad (3.4)$$

Therefore, by (3.4), (3.3) implies

$$\begin{aligned}&\lim_{T_0\le t\to\infty}\|\partial_\theta(\overline{r}(t,\cdot))\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\ =&\lim_{T_0\le t\to\infty}\Big\|\frac{-1}{(Re\frac{f^{'}_{k_0}\xi}{f_{k_0}})}\frac{1}{S_2\mid f_{k_0}\mid}Im\big(f^{'}_{k_0}\xi\overline{f_{k_0}}\big)\Big\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\ =&\mid M_{n_0}\mid n_0. \end{aligned} \quad (3.5)$$

For the second derivative,

$$\partial_\theta[\partial_\theta(\overline{r}(t,\theta))] \quad (3.6)$$

$$=\frac{-1}{Re\frac{f^{'}_{k_0}\xi}{f_{k_0}}}\partial_\alpha\Big[\Big(\frac{1}{Re\frac{f^{'}_{k_0}\xi}{f_{k_0}}}\frac{1}{S_2\mid f_{k_0}\mid}\Big)Im\big(\xi f^{'}_{k_0}\overline{f_{k_0}}\big)\Big]. \quad (3.7)$$

$$=\frac{-1}{Re\frac{f^{'}_{k_0}\xi}{f_{k_0}}}\Big[\frac{-\partial_\alpha(Re\frac{f^{'}_{k_0}\xi}{f_{k_0}})}{(Re\frac{f^{'}_{k_0}\xi}{f_{k_0}})^2}\frac{1}{S_2\mid f_{k_0}\mid}Im\big(f^{'}_{k_0}\xi\overline{f_{k_0}}\big)+\frac{-1}{Re\frac{f^{'}_{k_0}\xi}{f_{k_0}}}\frac{Im(f^{'}_{k_0}\xi\overline{f_{k_0}})}{S_2\mid f_{k_0}\mid^3}Im\big(f^{'}_{k_0}\xi\overline{f_{k_0}}\big) \quad (3.8)$$

$$+\frac{1}{Re\frac{f^{'}_{k_0}\xi}{f_{k_0}}}\frac{1}{S_2\mid f_{k_0}\mid}\partial_\alpha Im(f^{'}_{k_0}\xi\overline{f_{k_0}})\Big]. \quad (3.9)$$

$$\begin{aligned}&\lim_{T_0\le t\to\infty}\|\partial^2_\theta(\overline{r}(t,\cdot))\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\ =&\lim_{T_0\le t\to\infty}\Big\|\frac{-1}{(Re\frac{f^{'}_{k_0}\xi}{f_{k_0}})^2}\frac{1}{S_2\mid f_{k_0}\mid}\partial_\alpha Im(f^{'}_{k_0}\xi\overline{f_{k_0}})\Big\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\ =&\mid M_{n_0}\mid n_0^2.\end{aligned} \quad (3.10)$$

Similarly,

$$\begin{aligned}
&\lim_{T_0\le t\to\infty}\|\partial_\theta^3(\overline{r}(t,\cdot))\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\
=&\lim_{T_0\le t\to\infty}\left\|\frac{-1}{(Re\frac{f'_{k_0}\xi}{f_{k_0}})^3}\frac{1}{S_2\mid f_{k_0}\mid}\partial_\alpha^2 Im(f'_{k_0}\xi\overline{f_{k_0}})\right\|_{C^0(S^1)}(2t)^{\frac{n_0}{2}+1}\\
=&\mid M_{n_0}\mid n_0^3. \qquad (3.11)
\end{aligned}$$

By (3.5), (3.2) and (3.11),

$$\lim_{T_0\le t\to\infty}\|\overline{r}(t,\cdot)\|_{C^3(S^1)}(2t)^{\frac{n_0}{2}+1}=n_0^3\mid M_{n_0}\mid .$$

□

Lemma 3.7. *Define $M_0\pi$ as the area of $f(B_1(0))$ for some $f(\xi)=\sum_{i=1}^\infty a_i\xi^i$ in $O(\overline{B_1(0)})$. Given $\delta>0$, there exists $\epsilon_0>0$ such that if $\mid\frac{f^{(j)}(\xi)}{a_1}\mid_M<\epsilon_0$ for $2\le j\le 3$, then $f(\xi)$ is strongly starlike of order <1 and $\|\overline{r}_0\|_{C^{2,\alpha}(S^1)}\le\delta$ where $\overline{r}_0(\theta)=\frac{|f(\xi)|}{\sqrt{M_0}}-1$ and $\theta=\arg f(\xi)$. So we can consider the domain $f_{k_0}(B_1(0))$ as a small perturbation of $B_{\sqrt{M_0}}(0)$.*

Define $\aleph_{n_0}=\{f(\xi)\in O(\overline{B_1(0)})\mid\{M_k(f)\}_{k=1}^{n_0-1}=0,M_{n_0}(f)\ne 0\}$.

Theorem 3.8. *Given a global strong degree k_0 polynomial solution to* (1.1) *$\{f_{k_0}(\xi,t)\}_{t\ge0}$.*
(a)There exist $\rho(f_{k_0})>1,\epsilon(f_{k_0})>0,T_0(f_{k_0})>0$ such that if $\|f(\xi,0)-f_{k_0}(\xi,0)\|_{\rho,3}<\epsilon$, then the solution to (1.1) *$f(\xi,t)$ is global and is a strongly starlike function of order <1 for $t\ge T_0$. And*
(b)if $f(\xi,0)\in\aleph_{n_0}$, then

$$\lim_{T_0\le t\to\infty}\|\overline{r}(t,\cdot)\|_{h^{2,\alpha}(S^1)}(t)^\lambda=0,\forall\lambda\in(0,1+\frac{n_0}{2}),$$

where $\overline{r}(t,\theta)=\frac{|f(\xi,t)|}{\sqrt{2t+M_0(0)}}-1$ and $\theta=\arg f(\xi,t)$, which are well-defined for $t\ge T_0$.

Proof. Denote

$$f(\xi,t)=\sum_{i=1}^\infty b_i(t)\xi^i;f_{k_0}(\xi,t)=\sum_{i=1}^{k_0}a_i(t)\xi^i.$$

Note that $b_1^2(t)\ge b_1^2(0)+2t$ and $a_1^2(t)\ge a_1^2(0)+2t$.
(1)There exists $\delta>0$ as stated in Corollary 3.5.

(2)For such $\delta > 0$, we can find $\epsilon_0 > 0$ as stated in Lemma 3.7.
(3)Given $\epsilon_0 > 0$, there exists $T_0 > \frac{1}{2}$ such that for $t \geq T_0$,

$$\left|f_{k_0}^{(2)}(\xi,t)\right|_M < \frac{1}{4}\epsilon_0 \quad \text{and} \quad \left|f_{k_0}^{(3)}(\xi,t)\right|_M < \frac{1}{4}\epsilon_0 \tag{3.12}$$

since the coefficients $\{a_i(t)\}_{i\geq 2}$ decay to zero algebraically as shown in section 3.
(4)By Theorem 2.6, for such T_0 and ϵ_0, there exist $\rho > 1$ and $\epsilon > 0$ such that for $0 \leq t \leq T_0$, $2 \leq j \leq 3$,

$$\left|f_{k_0}^{(j)}(\xi,t) - f^{(j)}(\xi,t)\right|_M < \frac{1}{4}\epsilon_0 \quad \text{if} \quad \|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,3} < \epsilon$$

where $\{f(\xi,t)\}_{0\leq t\leq T_0} \subset O(\overline{B_1(0)})$ is a strong solution to (1.1). That means, since $b_1(T_0) \geq 1$ and by (3.12),

$$\left|\frac{f^{(j)}(\xi,T_0)}{b_1(T_0)}\right|_M \leq \frac{1}{2}\epsilon_0, 2 \leq j \leq 3 \quad \text{if} \quad \|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,3} < \epsilon.$$

Due to the fact in (2), $f(\xi,T_0)$ is starlike of order < 1 and

$$\|\overline{r}(T_0,\cdot)\|_{C^{2,\alpha}(S^1)} < \delta \quad \text{if} \quad \|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,3} < \epsilon$$

where $\overline{r}(t,\theta) = \frac{|f(\xi,t)|}{\sqrt{M_0(t)}} - 1$ and $\theta = \arg f(\xi,t)$.
(5)By (1)(2)(3)(4), we conclude that there exist $T_0 > 0$, $\rho > 1$, $\epsilon > 0$ such that $f(\xi,T_0) \in O(\overline{B_1(0)})$ is a strongly starlike function of order < 1 and

$$\|\overline{r}(T_0,\cdot)\|_{C^{2,\alpha}(S^1)} < \delta \quad \text{if} \quad \|f(\xi,0) - f_{k_0}(\xi,0)\|_{\rho,3} < \epsilon.$$

By Theorem 2.1 in Gustafsson, Prokhorov and Vasil'ev [3], the solution $f(\xi,t)$ must be global and $\{f(\xi,t)\}_{t\geq T_0}$ has strictly decreasing strongly starlike order $\alpha(t)$ for $t \geq T_0$ since $f(\xi,T_0) \in O(\overline{B_1(0)})$ and is a strongly starlike function. This also implies that $\overline{r}(t,\cdot)$ is well-defined for $t \geq T_0$.
(6)Combining these arguments with Corollary 3.5,

$$\limsup_{T_0\leq t\to\infty} \|\overline{r}(t,\cdot)\|_{h^{2,\alpha}(S^1)}(2t)^\lambda = 0, \forall \lambda \in (0, 1 + \frac{n_0}{2}).$$

□

3.3 Geometric meaning of the rescaling

If a curve has the polar coordinate equation $R(\theta)$, then the curvature

$$\kappa(\theta) = \frac{R^2 + 2(R^{'})^2 - RR^{''}}{(R^2 + (R^{'})^2)^{3/2}}.$$

We replace R by $\overline{r}(t,\theta) + 1$ which is defined in Theorem 3.8 and get

$$\begin{aligned}
&\mid \kappa(t,\theta) - 1 \mid \\
=&\Big|\frac{(1+\overline{r})^2 + 2(\overline{r}^{'})^2 - \overline{r}^{''}}{[(1+\overline{r})^2 + (\overline{r}^{'})^2]^{\frac{3}{2}}} - 1\Big| \\
\le&\frac{1}{[(1+\overline{r})^2 + (\overline{r}^{'})^2]^{\frac{3}{2}}}\big|(1+\overline{r})^2 + 2(\overline{r}^{'})^2 - \overline{r}^{''}(1+\overline{r}) - [(1+\overline{r})^2 + (\overline{r}^{'})^2]^{\frac{3}{2}}\big| \\
\le&\frac{1}{[(1+\overline{r})^2 + (\overline{r}^{'})^2]^{\frac{3}{2}}}\big[2\big|\overline{r}^{'}\big|^2 + \big|\overline{r}^{''}(1+\overline{r})\big| + \big|(1+\overline{r})^2 - 1\big| + \big|((1+\overline{r})^2 + (\overline{r}^{'})^2)^{\frac{3}{2}} - 1\big|\big] \\
\le&\frac{1}{[(1+\overline{r})^2 + (\overline{r}^{'})^2]^{\frac{3}{2}}}\big[2\big|\overline{r}^{'}\big|^2 + \big|\overline{r}^{''}(1+\overline{r})\big| + \big|(2+\overline{r})\overline{r}\big| + \big|((1+\overline{r})^2 + (\overline{r}^{'})^2)^{\frac{3}{2}} - 1\big|\big].
\end{aligned}$$

One of the term

$$\begin{aligned}
&\big|((1+\overline{r})^2 + (\overline{r}^{'})^2)^{\frac{3}{2}} - 1\big| \\
\le&\frac{3}{2}\big|(1+\overline{r})^2 + (\overline{r}^{'})^2 + 1\big|^{\frac{1}{2}}\big|(1+\overline{r})^2 + (\overline{r}^{'})^2 - 1\big| \\
\le&\frac{3}{2}\big|(1+\overline{r})^2 + (\overline{r}^{'})^2 + 1\big|^{\frac{1}{2}}\big|(2+\overline{r})\overline{r} + (\overline{r}^{'})^2\big| \\
\le&\frac{3}{2}\big|(1+\overline{r})^2 + (\overline{r}^{'})^2 + 1\big|^{\frac{1}{2}}\big(\big|(2+\overline{r})\overline{r}\big| + \big|\overline{r}^{'}\big|^2\big).
\end{aligned}$$

Under the assumptions and results of Theorem 3.8, the rescaled domain is $\Omega^{'}(t) = \{x \in R^N\backslash\{0\} :\mid x \mid < 1 + \overline{r}(t, \frac{x}{|x|})\} \cup \{0\}$ which has area π always. It has curvature $\kappa(t,z)$ which satisfies

$$\max_{z\in\partial\Omega^{'}(t)} \mid \mid z \mid -1| = o\Big(\frac{1}{t}\Big)^{\lambda}, \forall\lambda \in (0, 1 + \frac{n_0}{2})$$

$$\max_{z\in\Omega^{'}(t)} |\kappa(t,z) - 1| = o\Big(\frac{1}{t}\Big)^{\lambda}, \forall\lambda \in (0, 1 + \frac{n_0}{2}).$$

Similarly, under the assumptions of Theorem 3.6, $\partial\Omega^{'}(t)$ has the curvature $\kappa(t,z)$ for $z \in \partial\Omega^{'}(t)$ which satisfies

$$\max_{z\in\partial\Omega^{'}(t)} \mid \mid z \mid -1| = O\Big(\frac{1}{t}\Big)^{1+\frac{n_0}{2}},$$

and

$$\max_{z\in\Omega'(t)} |\kappa(t,z)-1| = O\Big(\frac{1}{t}\Big)^{1+\frac{n_0}{2}}.$$

Furthermore, since we can carry out these calculations by the expression that $| z |=| f(\xi) |$ and $\kappa(z) = \frac{1}{|f'|} Re\Big(1+\frac{f''\xi}{f'}\Big)$ if $z = f(\xi)$, then

$$\limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} || z | -1|(2t)^{1+\frac{n_0}{2}} =| M_{n_0} |,$$

and

$$\limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\kappa(t,z)-1|(2t)^{1+\frac{n_0}{2}} = (n_0+1)(n_0-1) \mid M_{n_0} \mid .$$

Note that if $n_0 = 1$, then

$$\limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\kappa(t,z)-1|t^{1+\frac{n_0}{2}} = 0.$$

(a)In the case that $n_0 = 1$ and $M_2 \neq 0$, then

$$0 < \limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\kappa(t,z)-1|t^{2} < \infty.$$

(b)If $n_0 = 1$ and $M_2 = 0$ and $M_3 \neq 0$, then

$$0 < \limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\kappa(t,z)-1|t^{1+\frac{3}{2}} < \infty.$$

(c)If $n_0 = 1$ and $M_2 = 0, M_3 = 0$, then

$$0 < \limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\kappa(t,z)-1|t^{1+\frac{4}{2}} < \infty,$$

no matter how $\{M_k\}_{k\geq 4}$ behave. We put the calculations for the sharp decay rate of $\kappa - 1$ in the appendix.

Therefore, the rate $1 + \frac{n_0}{2}$ is optimal. We conclude that the boundary of the domain is getting better and better.

4 Appendix

$$\begin{aligned}
f_{k_0}(\xi,t) &= a_1(t)\xi + a_2(t)\xi^2 + a_3(t)\xi^3 + a_4(t)\xi^4 + \dots \\
&= [\sqrt{2t+M_0(0)} + A(t)]\xi + a_2(t)\xi^2 + a_3(t)\xi^3 + a_4(t)\xi^4 + \cdots \\
&= [\sqrt{2t+M_0(0)}\xi + a_2(t)\xi^2 + a_3(t)\xi^3 + a_4(t)\xi^4 + \cdots] + A(t)\xi
\end{aligned}$$

Let

$$g(\xi,t) = \frac{f_{k_0}(\xi,t)}{\sqrt{2t+M_0(0)}}$$
$$= \Big[\xi + \frac{a_2}{\sqrt{2t+M_0(0)}}\xi^2 + \frac{a_3}{\sqrt{2t+M_0(0)}}\xi^3 + \cdots\Big] + \Big(\frac{A(t)}{\sqrt{2t+M_0(0)}}\xi\Big)$$

$$g^{'}(\xi,t) = \Big[1 + \frac{2a_2}{\sqrt{2t+M_0(0)}}\xi + \frac{3a_3}{\sqrt{2t+M_0(0)}}\xi^2 + \cdots\Big] + \Big(\frac{A(t)}{\sqrt{2t+M_0(0)}}\Big)$$
$$g^{''}(\xi,t)\xi = \Big[\frac{2a_2}{\sqrt{2t+M_0(0)}}\xi + \frac{6a_3}{\sqrt{2t+M_0(0)}}\xi^2 + \frac{12a_4}{\sqrt{2t+M_0(0)}}\xi^3 + \cdots\Big]$$

Denote

$$P = \Big[\frac{2a_2}{\sqrt{2t+M_0(0)}}\xi + \frac{3a_3}{\sqrt{2t+M_0(0)}}\xi^2 + \frac{4a_4}{\sqrt{2t+M_0(0)}}\xi^3 + \cdots\Big] + \Big(\frac{A(t)}{\sqrt{2t+M_0(0)}}\Big)$$
$$Q = \Big[\frac{3a_3}{\sqrt{2t+M_0(0)}}\xi^2 + \frac{8a_4}{\sqrt{2t+M_0(0)}}\xi^3 + \cdots\Big] - \Big(\frac{A(t)}{\sqrt{2t+M_0(0)}}\Big)$$

Then $g^{'}(\xi,t) = 1 + P$ and $g^{''}(\xi,t)\xi = P + Q$.

$$\mid \kappa - 1 \mid = \frac{1}{\mid g^{'} \mid} Re\Big(1 + \frac{g^{''}\xi}{g^{'}}\Big) - 1$$
$$= \Big(\frac{1}{\mid g^{'} \mid} - 1\Big) + \frac{1}{\mid g^{'} \mid} Re\Big(\frac{g^{''}\xi}{g^{'}}\Big)$$
$$= \frac{(1 - \mid g^{'} \mid)(1 + \mid g^{'} \mid)}{\mid g^{'} \mid (1 + \mid g^{'} \mid)} + \frac{1}{\mid g^{'} \mid} Re\Big(\frac{P+Q}{1+P}\Big)$$
$$= \frac{-2ReP - \mid P \mid^2}{\mid g^{'} \mid (1 + \mid g^{'} \mid)} + \frac{1}{\mid g^{'} \mid} Re(P - P^2 + P^3 - \cdots) + \frac{1}{\mid g^{'} \mid} Re(Q - QP + QP^2 - \cdots)$$
$$= \Big[\frac{-2ReP}{\mid g^{'} \mid (1 + \mid g^{'} \mid)} + \frac{ReP}{\mid g^{'} \mid}\Big] + \frac{ReQ}{\mid g^{'} \mid} - \Big[\frac{\mid P \mid^2}{\mid g^{'} \mid (1 + \mid g^{'} \mid)} + \frac{1}{\mid g^{'} \mid} Re(P^2)\Big]$$
$$+ \frac{1}{\mid g^{'} \mid} Re(P^3 - P^4 + \cdots) + \frac{1}{\mid g^{'} \mid} Re(-QP + QP^2 + \cdots)$$
$$= \frac{ReP}{\mid g^{'} \mid}\Big(\frac{2ReP + \mid P \mid^2}{(1 + \mid g^{'} \mid)^2}\Big) + \frac{ReQ}{\mid g^{'} \mid} - \Big[\frac{\mid P \mid^2}{\mid g^{'} \mid (1 + \mid g^{'} \mid)} + \frac{1}{\mid g^{'} \mid} Re(P^2)\Big]$$
$$+ \frac{1}{\mid g^{'} \mid} Re(P^3 - P^4 + \cdots) + \frac{1}{\mid g^{'} \mid} Re(-QP + QP^2 + \cdots)$$

Note that

$$a_2 = \frac{\overline{M_1}}{a_1^2} + O(\frac{1}{a_1^6})$$
$$a_3 = \frac{\overline{M_2}}{a_1^3} + O(\frac{1}{a_1^7})$$
$$a_4 = \frac{\overline{M_3}}{a_1^4} + O(\frac{1}{a_1^8})$$
$$a_5 = \frac{\overline{M_4}}{a_1^5} + O(\frac{1}{a_1^9})$$
$$a_6 = \frac{\overline{M_4}}{a_1^6} + O(\frac{1}{a_1^{10}})$$

(i)Assume that $M_1 \neq 0$.
(a)If $M_2 \neq 0$, then the sharp decay rates of Q and P are t^{-2} and $t^{-3/2}$ respectively. Also $A(t)/\sqrt{2t + M_0(0)}$ decays like t^{-3}. Therefore the sharp decay is

$$\mid \kappa - 1 \mid = O(\frac{1}{t^2}).$$

(b)If $M_2 = 0$ and $M_3 \neq 0$, then $Q = O(\frac{1}{t^{7/2}})$ and the sharp decay rate of Q and P are $t^{-5/2}$ and $t^{-3/2}$ respectively. Also $A(t)/\sqrt{2t + M_0(0)}$ decays like t^{-3}. Therefore the sharp decay is

$$\mid \kappa - 1 \mid = O(\frac{1}{t^{\frac{5}{2}}}).$$

(c)If $M_2 = 0$, $M_3 = 0$ and $M_4 \neq 0$, the sharp decay rate is

$$\mid \kappa - 1 \mid = O(\frac{1}{t^3}).$$

(d)Others. In this case, $M_2 = M_3 = M_4 = 0$, then $Q = O(t^{-7/2})$ and the sharp decay rate of P is $t^{-3/2}$. In this case, the sharp decay is

$$\mid \kappa - 1 \mid = O(\frac{1}{t^3}).$$

(ii)If $M_1 = 0$ and $n_0 = \min\{k \geq 1 \mid M_k \neq 0\}$, then the sharp decay rate of Q and P are both $t^{-(1+n_0/2)}$. In this case, the sharp decay is

$$\mid \kappa - 1 \mid = O(\frac{1}{t^{1+\frac{n_0}{2}}}).$$

In fact, we can calculate and get

$$\lim_{t\to\infty}\max_{z\in\partial\Omega'(t)} \mid \kappa(t,z)-1 \mid = (n_0-1)(n_0+1) \mid M_{n_0} \mid .$$

Theorem 4.1. *Given $f_{k_0}(\xi,t)\in C^1([0,T_0],H(\overline{B_r(0)}))\cap O(\overline{B_r})$ and $1<r^{'}<r$, there exists $\eta(f_{k_0},T_0,r^{'})>0$ such that if*

$$\max_{([0,T_0])} \mid f^{'}(\xi,t)-f^{'}_{k_0}(\xi,t) \mid_{M(r)} \leq \eta$$

where $f(\xi,t)\in C^1([0,t_1],H(B_r(0)\cap C(\overline{B_r(0)})))$ and $f^{'}(\xi,t)\neq 0$ for $(\xi,t)\in(B_r(0),[0,T_0])$, then for $0\leq t\leq T_0$,

$$f(\xi,t)\in O(\overline{B_{r'}(0)}).$$

Proof. First assume that

$$\max_{([0,T_0])} \mid f^{'}(\xi,t)-f^{'}_{k_0}(\xi,t) \mid_{M(r)} \leq \frac{1}{2}\min_{(\overline{B_r(0)},[0,T_0])} \mid f^{'}_{k_0}(z,t) \mid .$$

We want to show that there exists $r_0>0$ such that

$$f(\cdot,t):\overline{B_{r_0}(z_0)}\to f(\overline{B_{r_0}(z_0)})$$

is univalent for all $z_0\in\overline{B_{r'}(0)}$. It is sufficient to prove that

$$Re\frac{f^{'}(z,t)(z-z_0)}{f(z,t)-f(z_0,t)}\geq\frac{1}{2}, z\in B_{r_0}(z_0)$$

which means the function is starlike with respect to the point z_0 for $z\in\overline{B_{r_0}(z_0)}$, since a starlike function in $\overline{B_{r_0}(z_0)}$ is univalent in $\overline{B_{r_0}(z_0)}$.

Since $f(\xi,t)$ is analytic in $\overline{B_r(0)}$,

$$f(z,t)=f(z_0,t)+\sum_{n=1}^{\infty}\frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^n.$$

Let

$$l=\min\{r^{'},r-r^{'}\}, M=\max_{(z,t)\in(\overline{B_r(0)},[0,T_0])} \mid f(z,t) \mid, m=\min_{(z,t)\in(\overline{B_r(0)},[0,T_0])} \mid f^{'}(z,t) \mid .$$

Since

$$\max_{([0,T_0])} \mid f^{'}(\xi,t)-f^{'}_{k_0}(\xi,t) \mid_{M(r)} \leq \frac{1}{2}\min_{(\overline{B_r(0)},[0,T_0])} \mid f^{'}_{k_0}(z,t) \mid,$$

then we can get that $M \leq C(f_{k_0}, T_0)$ and $m \geq D(f_{k_0}, T_0) > 0$. For $z_0 \in \overline{B_{r'}(0)}$,

$$| \frac{f^{(n)}(z_0)}{n!} | \leq M l^{-(n)}, n \geq 1.$$

$$\begin{aligned}
&\Big| \frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)} - 1 \Big| \\
=&\Big| \frac{\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^{n-1} n}{\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^{n-1}} - 1 \Big| \\
=&\Big| \frac{\sum_{n=2}^{\infty} \frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^{n-1}(n-1)}{f'(z_0,t) + \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0,t)}{n!}(z-z_0)^{n-1}} \Big| \\
\leq& \frac{\sum_{n=2}^{\infty} M l^{-n} \mid z-z_0 \mid^{n-1} (n-1)}{m - \sum_{n=2}^{\infty} M l^{-n} \mid z-z_0 \mid^{n-1}}
\end{aligned}$$

if $m > \sum_{n=2}^{\infty} M l^{-n} \mid z - z_0 \mid^{n-1}$. Pick $0 < r_0 < l$ such that

$$\sum_{n=2}^{\infty} M l^{-n} r_0^{n-1}(n-1) \leq \frac{m}{4}.$$

This implies

$$\Big| \frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)} - 1 \Big| \leq \frac{1}{2}, z \in B_{r_0}(z_0),$$

and it follows that

$$Re \frac{f'(z,t)(z-z_0)}{f(z,t)-f(z_0,t)} \geq \frac{1}{2}, z \in B_{r_0}(z_0).$$

Assume that there exist η_k goes to zero as k goes to ∞, and $f^k(\xi_k^1, t_k) = f^k(\xi_k^2, t_k)$, $\xi_k^1 \neq \xi_k^2$, $\xi_k^1, \xi_k^2 \in \overline{B_{r'}(0)}$ such that

$$| f^k(\xi_k^1, t_k) - f_{k_0}(\xi_k^1, t_k) | \leq \eta_k, | f^k(\xi_k^2, t_k) - f_{k_0}(\xi_k^2, t_k) | \leq \eta_k.$$

Without loss of generality, assume t_k converges to t_0, ξ_k^1 converges to ξ^1 and ξ_k^2 converges to ξ^2. Note that $| \xi^1 - \xi^2 | \geq r_0$. This implies

$$f_{k_0}(\xi^1, t_0) = f_{k_0}(\xi^2, t_0).$$

This contradicts with the result that $f_{k_0}(\xi, t_0)$ is univalent in $\overline{B_r(0)}$. Therefore,

$$f(\xi, t) \in O(\overline{B_{r'}(0)}).$$

□

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