Existence and uniqueness of the Hele-Shaw problem with injection

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Abstract

This paper gives a new and short proof of existence and uniqueness of the Polubarinova-Galin equation. The existence proof is an application of the main theorem in Lin [2]. Furthermore, we can conclude that every strong solution can be approximated by many strong polynomial solutions locally in time.

Keywords: Hele-Shaw flow, well-posedness.

1 Introduction

We consider a slow parallel flow of an incompressible fluid with viscosity μ between two parallel flat plates which are fixed at a small distance h. With the injection of fluid with strength Q at the origin, the flow can be described as a 2-dimension flow and the boundary of the flow is changing with time. Denote $\Omega(t)$ as the moving domain. Let $v(z,t)$ and $p(z,t)$ be the averaged velocity across the gap and averaged pressure at z for $z \in \Omega(t)$ respectively. Then by Darcy's law

$$v(z,t) = -k\nabla p, z \in \Omega(t) \tag{1.1}$$

where $k = \frac{h^2}{12\mu}$. Also because of the incompressibility of the fluid

$$\triangle p(z,t) = -\nabla \cdot v(z,t) = -Q\delta_0, z \in \Omega(t). \tag{1.2}$$

The dynamic boundary condition is

$$p(z,t) = \gamma\kappa, z \in \partial\Omega(t) \tag{1.3}$$

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for some nonnegative constant γ. We set k in (1.1) by a suitable scaling to be 1. These three equations (1.1), (1.2) and (1.3) describe Hele-Shaw flows. Furthermore, we call it the zero surface tension Hele-Shaw problem if $\gamma = 0$.

By (1.1),

$$\frac{\partial p(z,t)}{\partial n} = -\nu_n, z \in \partial\Omega(t)$$

where n is the unit outward normal vector to $\partial\Omega(t)$ and ν_n is the outward normal velocity. The motion of the zero surface tension Hele-Shaw flow can be described by

$$\begin{cases} \Delta p = -Q\delta_0 & \text{in } \Omega(t), \\ p = 0 & \text{on } \partial\Omega(t), \\ \nu_n = -\frac{\partial p}{\partial n} & \text{on } \partial\Omega(t). \end{cases}$$

The present paper is mainly devoted to the following differential equation which arises from the reformulation of the zero surface tension Hele-Shaw problem with injection 2π at the origin as in Richardson [4], that is:

$$Re[f_t(\xi,t)\overline{f'(\xi,t)\xi}] = 1, \xi \in \partial B_1(0) \tag{1.4}$$

where $f(\xi,t) : B_1(0) \to \Omega(t)$ is univalent and analytic in $\overline{B_1(0)}$, $f(0,t) = 0$, $f^{'}(0,t) > 0$ and $\Omega(t)$ is the domain of the moving fluid. This is called the Polubarinova-Galin equation since Galin and Polubarinova-Kochina first have derived (1.4) and investigated the Riemann mapping method along these lines. A solution to (1.4) is said to be a strong solution for $t \in [0,b)$ if $f(\xi,t)$ is univalent and analytic in $\overline{B_1(0)}$ and continously differentiable in $t \in [0,b)$. The existence and uniqueness of the solutions to the P-G equation (locally in time) have been proven in several different ways, in particular [5], [3] and [1]. In 1948, Vinogrado and Kufarev proved in [5] the existence and uniqueness of solutions which depend analytically on z and t under the additional assumption $f^{'}(0,t) > 0$. The proofs in [5] are fairly complicated. In [1], Gustafsson gave a less complicated and elementary proof of existence and uniqueness of strong solutions to (1.4) in the case that initial function is a polynomial or a rational function. In Reissig and von Wolfersdorf [3], the classical solvability of strong solutions may be proven using the nonlinear abstract Cauchy-Kovalevskaya Theorem and this is considered as a modern proof.

Here we offer a new proof of existence and uniqueness of the strong solution to (1.4). This proof is based on two facts proven in Gustafsson [1] in detail: one is the reformulation of the P-G equation ; that is:

$$\frac{d}{dt}f(\xi,t) = \frac{f^{'}(\xi,t)\xi}{2\pi i}\int_{\partial B_1(0)} \frac{1}{\mid f^{'}(z,t)\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0);$$

another is the fact that if $f(\xi, 0)$ is a polynomial conformal mapping, the solution $f(\xi, t)$ to (1.4) is also a polynomial conformal mapping.

In this paper, we quote the short time existence of solutions in the case that the initial function is a small perturbation of a univalent and analytic polynomial as stated in Lin [2]. We prove the existence of a solution by explaining any initial conformal mapping $f(\xi, 0)$ which is univalent and analytic in $\overline{B_1(0)}$ to be the small perturbation of some univalent and analytic polynomial in $\overline{B_1(0)}$. Furthermore, we can conclude that every strong solution can be approximated by many strong polynomial solutions locally in time. The uniqueness is also proven separately here.

2 Existence

Define

$$\Big|\sum_{i=0}^{\infty} a_i\xi^i\Big|_M = \sum_{i=0}^{\infty} \mid a_i \mid$$

$$\Big|\sum_{i=0}^{\infty} a_i\xi^i\Big|_{M(r)} = \sum_{i=0}^{\infty} \left|a_i r^i\right|$$

$H(\Omega) = \{f \mid f \text{ is analytic in } \Omega\}$

$O(\Omega) = \{f \mid f \text{ is analytic and univalent in } \Omega,\ f(0) = 0 \text{ and } f'(0) > 0\}$

$\omega(\Omega) = \{f \mid f \text{ is analytic and univalent in } \Omega,\ f(0) = 0 \text{ and } f'(0) > 0\}$

In [1], Gustafsson reformulates the Polubarinova-Galin equation, that is:

$$f_t = \frac{f'\xi}{2\pi i}\int_{\partial B_1(0)} \frac{1}{\mid f'(z)\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0). \tag{2.1}$$

As Gustafsson [1], the mathematical treatment for (2.1) only requires the local univalence of the function $f(\xi, t)$. To make a distinction, we define a solution to be a strong solution to (2.1) as follows:

Definition 2.1. A solution $f(\xi, t) \in \omega(\overline{B_1(0)})$ is a strong solution to (2.1) for $0 \le t \le b$ if $f(\xi, t)$ is continuously differentiable with respect to $t \in [0, b)$ and satisfies (2.1).

A solution $f(\xi, t) \in O(\overline{B_1(0)})$ to (2.1) must be a solution to (1.4).

Lemma 2.1. *([2]) Given* $g \in \omega(\overline{B_1(0)})$ *and* $h \in \omega(\overline{B_1(0)})$ *where*

$$\frac{d}{dt}[g] = \frac{1}{2\pi i}\xi g^{'} \int_{\partial B_1(0)} \frac{1}{|g^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}$$

$$\frac{d}{dt}[h] = \frac{1}{2\pi i}\xi h^{'} \int_{\partial B_1(0)} \frac{1}{|h^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z},$$

then

$$\Big\|\frac{d}{dt}(g-h)\Big\|_{L^2([0,2\pi])}$$

$$\leq \Big\{\max_{\partial B_1}\Big|\frac{\xi}{2\pi i}\int_{\partial B_1(0)} \frac{1}{|g^{'}|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big| + \sqrt{2}\max_{\partial B_1(0)}\big|h^{'}\big| \max_{\partial B_1(0)}\frac{|g^{'}|+|h^{'}|}{|g^{'}|^2|h^{'}|^2}\Big\}\big\|g^{'}-h^{'}\big\|_{L^2([0,2\pi])}$$

Lemma 2.2. *([2]) Given a polynomial mapping* $f(\xi,0) \in \omega(\overline{B_r(0)})$ *for some* $r>1$*, then there exists a unique strong polynomial solution* $f(\xi,t) \in \omega(\overline{B_r(0)})$ *to* (2.1) *at least for a short time. Furthermore, if the polynomial solution ceases to exist at* $t=b$*, then for any* $r>1$*,*

$$\liminf_{t\to b}(\min_{B_r(0)} \mid f^{'}(\xi,t)\mid) = 0.$$

Lemma 2.3. *([2]) Assume that* $f_{k_0}(\xi,t) \in C^1([0,t_1], H(\overline{B_r}(0))) \cap \omega(\overline{B_r(0)})$ *is a strong degree* k_0 *polynomial solution to* (2.1) *for some* $t_1>0$ *and* $r>1$ *and that* $\rho > r$ *and* $l<1$*. If* $\{b_k(0)\}_{k\geq 1}$ *satisfy the assumption*
(A)

$$\sum_{k=1}^{\infty} \mid b_k(0)\mid \rho^k k^{3/2} \leq \frac{1}{\sqrt{k_0}} l \min_{(\overline{B_r(0)},[0,t_1])}\big|f_{k_0}^{'}\big|,$$

and $b_1(0) \in R$*, then the following (a)-(d) are true:*
(a)The initial value $f_{k_0}(\xi,0) + \sum_{k=1}^{\infty} b_k(0)\xi^k$ *gives rise to a strong solution to (2.1),* $f(\xi,t)$*, at least locally in time.*
(b)Let

$$A = \Big\{h(z,t) \in \omega(\overline{B_r(0)}) \cap C^1([0,t_h], H(\overline{B_r(0)}))\ \text{a strong polynomial solution to}$$

$$(2.1), 0 < t_h \leq t_1 \Big| \max_{([0,t_h])}\big|h^{'}(z,t) - f_{k_0}^{'}(z,t)\big|_{M(r)} \leq l \min_{(\overline{B_r(0)},[0,t_1])}\big|f_{k_0}^{'}\big|\Big\}$$

and

$$M = \sup\Big\{\max_{(\partial B_1(0),[0,t_h])}\Big|\frac{1}{2\pi i}\int_{\partial B_1(0)}\frac{1}{|h^{'}(z,t)|^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}\Big|\Big|h \in A\Big\},$$

then $M < \infty$.
(c)Define

$$t_0 = \min\left\{\frac{1}{Ck_0}(\ln\rho - \ln r), t_1\right\}$$

where

$$C = \left\{M + \sqrt{2}(1+l)\frac{2}{(1-l)^3}\max_{(\partial B_1(0),[0,t_1])}\left|f'_{k_0}\right|\max_{(\partial B_1(0),[0,t_1])}\frac{1}{|f'_{k_0}|^3}\right\}.$$

Then $f(\xi,t) \in C^1([0,t_0], H(B_r(0))) \cap \omega(B_r(0))$ *and*

$$\max_{([0,t_0])}\left|f' - f'_{k_0}\right|_{M(r)} \leq l \min_{(\overline{B_r(0)},[0,t_1])}\left|f'_{k_0}\right|.$$

(d)Furthermore, if there exist $\delta > 0$ *and* j *nonnegative integer such that*

$$\sum_{k=1}^{\infty} \mid b_k(0) \mid \rho^k k^{\frac{2j+1}{2}} \leq \delta,$$

then there exists $c(j,k_0) > 0$ *such that*

$$\max_{([0,t_0])}\left|f^{(j)} - f^{(j)}_{k_0}\right|_{M(r)} \leq c(j,k_0)\delta.$$

Remark 2.2. Theorem 2.3 is also true for the suction case.

2.1 Proof of existence

Theorem 2.4. *Given* $\sum_{i=1}^{\infty} a_i(0)\xi^i \in \omega(\overline{B_r(0)}) \cap H(\overline{B_{\rho_0}(0)})$ *where* $\rho_0 > r > 1$, *then there exist* $t_0 > 0$ *and a solution* $f(\xi,t) \in C^1([0,t_0], H(\overline{B_r(0)})) \cap \omega(\overline{B_r(0)})$ *to the following equation*

$$f_t = \frac{\xi}{2\pi i}f'(\xi,t)\int_{\partial B_1(0)}\frac{1}{\mid f' \mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0) \tag{2.2}$$

with intial value $\sum_{i=1}^{\infty} a_i(0)\xi^i$ *for at least a short time.*

Proof. (a).Since $H(\overline{B_{\rho_0}(0)}) \subset M(B_{\rho_0}(0))$, there exists $M > 0$ such that

$$\mid a_i(0) \mid \leq M\rho_0^{-i}.$$

Define $f_n(\xi,0) = \sum_{i=1}^{n} a_i(0)\xi^i$. Also

$$\Big|\min_{(\overline{B_r(0)},0)}\left|f'(\xi,0)\mid - \min_{(\overline{B_r(0)},0)}\left|f'_n(\xi,0)\right|\right| \leq \sum_{i=n+1}^{\infty} i \mid a_i(0) \mid (r)^i \leq \sum_{i=n+1}^{\infty} iM(\frac{\rho}{r})^{-i}$$

where $\sum_{i=n+1}^{\infty} iM(\frac{\rho}{r})^{-i}$ approaches zero as n approaches ∞. Therefore there exists $n_0 \in N$ such that

$$\frac{1}{2}\min_{(\overline{B_r(0)},0)} \left|f^{'}(\xi,0)\right| \leq \min_{(\overline{B_r(0)},0)} \left|f_n^{'}(\xi,0)\right|$$

for $n \geq n_0$ and $f_n(\xi,0) \in \omega(\overline{B_r(0)})$. By Lemma 2.2, there exists a polynomial solution $f_n(\xi,t) \in \omega(\overline{B_r(0)})$ at least for a short time. Denote $f_n(\xi,t)$ as the polynomial solution to (2.5) which satisfies the initial condition $f_n(\xi,0) = \sum_{i=1}^{n} a_i(0)\xi^i$.

(b).Given $1 < r_0 < \frac{\rho_0}{r}$. Pick $k_0 \geq n_0$ such that for all $k \geq 0$

$$M \leq \frac{r_0^{(k+k_0+1)}}{(k+k_0+1)^{3/2}}\Big[\frac{1}{(k+1)^2}\big(\frac{1}{4}\min_{(\overline{B_r(0)},0)}\left|f^{'}\right|\big)\frac{1}{2m}\Big] \tag{2.3}$$

where $m = \sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{3}$.

(c).There exists $t_1 > 0$ such that for $0 \leq t \leq t_1$

$$\min_{(\overline{B_r(0)},[0,t])}\left|f_{k_0}^{'}(\xi,t)\right| \geq \frac{1}{4}\min_{(\overline{B_r(0)},0)}\left|f^{'}(\xi,0)\right|.$$

By above, (2.3) implies

$$M \leq \frac{r_0^{(k+k_0+1)}}{(k+k_0+1)^{3/2}}\Big[\frac{1}{(k+1)^2}\big(\min_{(\overline{B_r(0)},[0,t_1])}\left|f_{k_0}^{'}(\xi,t)\right|\big)\frac{1}{m}\Big]\frac{1}{2}, k \geq 0. \tag{2.4}$$

Then

$$\sum_{k=k_0+1}^{\infty} \mid a_k(0) \mid \Big(\frac{\rho_0}{r_0}\Big)^k k^{3/2} \leq \frac{1}{\sqrt{k_0}}\frac{1}{2}\min_{(\overline{B_r(0)},[0,t_1])}\left|f_{k_0}^{'}\right|.$$

(d).By letting $\rho = \frac{\rho_0}{r_0}$ and $l = \frac{1}{2}$, we can see that assumption (A) in Theorem 2.3 is satisfied from (c). Applying Theorem 2.3, the short time existence is proven. □

Remark 2.3. The proof also can be applied to the suction case.

The following theorem naturally holds.

Theorem 2.5. *Given* $\sum_{i=1}^{\infty} a_i(0)\xi^i \in O(\overline{B_r(0)}) \cap H(\overline{B_{\rho_0}(0)})$ *where* $\rho_0 > r > 1$*, then there exists* $b > 0$ *and a solution* $f(\xi,t) \in C^1([0,b], H(\overline{B_r(0)})) \cap O(\overline{B_r(0)})$ *to the following equation*

$$f_t = \frac{\xi}{2\pi i}f^{'}(\xi,t)\int_{\partial B_1(0)}\frac{1}{\mid f^{'}\mid^2}\frac{z+\xi}{z-\xi}\frac{dz}{z}, \xi \in B_1(0) \tag{2.5}$$

with intial value $\sum_{i=1}^{\infty} a_i(0)\xi^i$ *for at least a short time.*

3 Uniqueness

Theorem 3.1. *The strong solution to (1.4) is unique.*

Proof. (1) Assume there are two solutions f_1, f_2 with the same initial value $f(\xi, 0) \in H(\overline{B_\rho(0)}) \cap O(\overline{B_r(0)})$. There exists t_0 such that $f_1(\xi, t)$ and $f_2(\xi, t) \in O(\overline{B_r(0)})$ for $0 < t < t_0$. Denote

$$M^2 = \max_{i=1,2} \max_{t \in [0,t_0]} \int_{\partial B_r(0)} \mid f_i^{'} \mid^2 d\theta$$

, then

$$\mid \alpha_i(t) \mid \leq \frac{M}{i}(r)^{-i}, \mid \beta_i(t) \mid \leq \frac{M}{i}(r)^{-i}$$

if we denote $f_1(\xi, t) = \sum_{i=1}^{\infty} \alpha_i(t)\xi^i$ and $f_2(\xi, t) = \sum_{i=1}^{\infty} \beta_i(t)\xi^i$.
(2)By Lemma2.1, there exists $C > 0$, for $t \in [0, t_0]$

$$\begin{aligned}
&\sum_{i=1}^{\infty} [\mid (\alpha_i - \beta_i)_t \mid]^2 \\
\leq & C\Big\{ \sum_{i=1}^{\infty} [\mid (\alpha_i - \beta_i) \mid i]^2 \Big\} \\
\leq & C\Big\{ \sum_{i=1}^{k} [\mid (\alpha_i - \beta_i) \mid i]^2 + \sum_{i=k+1}^{\infty} (2M)^2 r^{-2i} \Big\} \\
\leq & C\Big\{ \sum_{i=1}^{k} [\mid (\alpha_i - \beta_i) \mid i]^2 + 4M^2 \Big(\frac{r^{-2(k+1)}}{1 - r^{-2}} \Big) \Big\}.
\end{aligned}$$

(3)Denote $D_k(t)=\sum_{i=1}^{k}[\mid(\alpha_i-\beta_i)\mid i]^2$, then

$$\begin{aligned}
D_k^{'}(t) &= \sum_{i=1}^{k} 2Re[(\alpha_i-\beta_i)\overline{(\alpha_i-\beta_i)_t}]i^2 \\
&\le 2k\sum_{i=1}^{k}\mid(\alpha_i-\beta_i)(\alpha_i-\beta_i)_t i\mid \\
&\le 2k\Big\{\sum_{i=1}^{k}[\mid(\alpha_i-\beta_i)\mid i]^2\Big\}^{1/2}\Big\{\sum_{i=1}^{k}[\mid(\alpha_i-\beta_i)_t\mid]^2\Big\}^{1/2} \\
&\le 2kCD_k^{1/2}(t)\Big\{D_k(t)+4M^2\Big(\frac{r^{-2(k+1)}}{1-r^{-2}}\Big)\Big\}^{1/2} \\
&\le 2kCD_k^{1/2}(t)\Big\{D_k^{1/2}(t)+2M\Big(\frac{r^{-(k+1)}}{(1-r^{-2})^{1/2}}\Big)\Big\} \\
&\le 2kCD_k(t)+2kMCD_k^{1/2}(t)\Big(\frac{r^{-(k+1)}}{(1-r^{-2})^{1/2}}\Big).
\end{aligned}$$

Note that $Area(t)=\pi\sum_{i=1}^{\infty}i\mid\alpha_i(t)\mid^2=\pi\sum_{i=1}^{\infty}i\mid\beta_i(t)\mid^2\le Area(0)+2t_0$. So we have $D_k(t)\le\frac{1}{\pi}4kArea(t)\le\frac{1}{\pi}4k(Area(0)+2t_0)=2kA$ for some $A>0$. Therefore

$$D_k^{'}(t)\le 2kCD_k(t)+2MC(2A)^{1/2}k^{3/2}\frac{r^{-(k+1)}}{(1-r^{-2})^{1/2}}.$$

Denote $(2A)^{1/2}(2MC)\frac{1}{(1-r^{-2})^{1/2}}=C_0$, then

$$D_k^{'}(t)\le 2kCD_k(t)+C_0r^{-(k+1)}k^{3/2}$$

$$\Big(D_k(t)e^{-2kCt}\Big)^{'}\le e^{-2kCt}C_0r^{-(k+1)}k^{3/2}$$

$$D_k(t)e^{-2Ckt}\le\frac{1-e^{-2kCt}}{2kC}C_0r^{-(k+1)}k^{3/2}$$

$$D_k(t)\le\frac{1}{2kC}\Big(e^{2kCt}\Big)C_0r^{-(k+1)}k^{3/2}=\frac{1}{2rC}\Big(e^{2Ct}r^{-1}\Big)^kk^{\frac{1}{2}}C_0.$$

If $0\le t<\frac{1}{2C}\ln r$, and let k approach ∞, we have that $D_k(t)$ approaches zero since $\frac{1}{2C}(e^{2Ct}r^{-1})^kk^{\frac{1}{2}}C_0$ approaches zero. Therefore $f_1(\xi,t)=f_2(\xi,t)$ for $t\in[0,T)$ where $T=\min\{\frac{1}{2C}\ln r,t_0\}$.
(4)Finally, the uniqueness of the short time existence is proven. □

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