# REMARK ON THE BLOW UP CONDITION TO THE INCOMPRESSIBLE VISCOELASTIC FLUID SYSTEM

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**Abstract :** In this paper, we study the blow-up criterion for smooth solutions to the incompressible viscoelastic fluid system in $\mathbb{R}^2$ by using the logarithmic Sobolev inequality. This is a refined version of the condition given by [6]. Compared with [3] and [5], the blow-up condition is expressed by a single term : the vorticity with respect to the velocity field.

## 1. INTRODUCTION

This paper is concerned with the incompressible viscoelastic fluid system in the Oldroyd model

$$(VE)\begin{cases} U_t + v\cdot\nabla U = \nabla v U, \\ v_t + v\cdot\nabla v - \triangle v = -\nabla p + \nabla\cdot(UU^T) \\ \nabla\cdot v = 0 \\ U(x,0) = U_0(x), v(x,0) = v_0 \\ (t,x)\in(0,+\infty)\times R^2 \end{cases}$$

where the matrix $U$ represents the deformation tensor, $v$ is the fluid velocity, and $p$ is the pressure. The above system is one of the basic macroscopic models for viscoelastic flows, which corresponds to the so-called Hookean linear elasticity. For the physical background to this equation and various well-posedness results, one may check [5, 6, 10, 11] and references therein for the details. In particular, in [6], they have the following necessary condition for blow-up : Let $T^\star > 0$ be a maximal time for the existence of the solution. Then, $T^\star < \infty \Rightarrow \int_0^{T^\star} ||\nabla v(t)||_{L^\infty} dt = \infty$. Recently, for the incompressible Euler equation, Planchon [12] established an improved blow-up criterion in the framework of Besov sapces : There exists a positive constant $M$ such that if

$$\lim_{\epsilon\longrightarrow 0}\sup_{j\in\mathbb{Z}}\int_{T-\epsilon}^{T} ||\triangle_j w(t)||_{L^\infty} dt \geq M$$

then $v$ cannot be continued beyond $t = T$. Motivated by this result, Cannone-Chen-Miao [3] obtained the corresponding result for the MHD equation:

$$(MHD)\begin{cases} v_t + v\cdot\nabla v = -\nabla p - \frac{1}{2}\nabla b^2 + b\cdot\nabla b \\ b_t + v\cdot\nabla b = b\cdot\nabla v \\ \nabla\cdot v = \nabla\cdot b = 0 \\ b(x,0) = b_0(x), v(x,0) = v_0 \end{cases}$$

where $v$ and $b$ describe the velocity and the magnetic field vector, respectively. Unfortunately, they cannot apply the method used in [12] directly, and they overcome this difficulty by obtaining a losing estimate for the MHD equation, which is studied in [5], and further established a blow-up criterion of smooth solution for the MHD equation : Let $(v_0, b_0) \in B^s_{p,q}$, $s > \frac{d}{p} + 1$, $1 \le p, q < \infty$. Suppose that $(v, b) \in C([0, T); B^s_{p,q}) \cap C^1([0, T); B^{s-1}_{p,q})$ is the smooth solution to (MHD). There exists an absolute constant $M > 0$ such that If

$$\lim_{\epsilon \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\epsilon}^{T} ||\triangle_j(\nabla \times v)(t)||_{L^\infty} + ||\triangle_j(\nabla \times b)(t)||_{L^\infty} dt \ge M$$

then $v$ cannot be continued beyond $t = T$. Now, we want to prove the following theorem.

**THEOREM:** Let $U_0 \in H^s$, $v_0 \in H^{s-1}$ with $s > 1$. Suppose that $v \in C([0, T); H^{s-1})$, $U \in C([0, T); H^s)$ are the smooth solutions to the incompressible viscoelastic fluid system. Then there exists a constant $M > 0$ such that
(i) If $\lim_{\sigma \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\sigma}^{T} ||\triangle_j \nabla v(t)||_{L^\infty} dt = \delta < M$, then $\delta = 0$, and the solutions can be extended beyond $T$.
(ii) If $\lim_{\sigma \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\sigma}^{T} ||\triangle_j \nabla v(t)||_{L^\infty} dt \ge M$, then the solutions blow up at $t = T$.

**Remark:** As explained in [12], we cannot say $\sup_{j \in \mathbb{Z}} \int_0^T ||\triangle_j \nabla v(t)|| dt < \infty$ as a non-blowup condition. But, if we keep $\{\sup_{j \in \mathbb{Z}}\}$ inside of the time integration in (8), then we recover the condition used in [8] in the context of the Navier-Stokes equation. Of course, our criterion is an improved version of the criterion given by [6], and our result is better than the results in [5] because we only have one term in the criterion. We started with initial data $(v_0, U_0) \in H^{s-1} \times H^s$ which is less regular than initial data used in [3,5,6,12] because we are using the Laplacian to gain some derivatives.

## 2. PROOF OF THEOREM

### (1) BIOT-SAVART LAW, LITTLEWOOD-PALEY THEORY

Since the divergence of $v$ is 0, there exists a scalar function $\psi$ such that $v = \nabla^\perp \psi$. Then the vorticity $w = \nabla \times v$ satisfies $w = -\triangle \psi$. Therefore, we can recover $v$ from $w$ by $v = \nabla^\perp \triangle^{-1} w$, which is called the Biot-Savart Law. And $\nabla v$ is the image of $w$ under the singular integral operators of the Calderon-Zygmund type. One may then freely pass from $\nabla v$ to $w$ in $\int_0^T ||\triangle_j \nabla v(t)||_{L^\infty} dt$ in the above theorem since the singular integral operators are bounded on $\dot{B}^0_{\infty.\infty}$. Now, we briefly introduce the Littlewood-Paley Theory. We first have the following Littlewood-Paley Decomposition.[5]

**Proposition 1.** Let us denote by $\mathcal{D}(\Omega)$ the space of $C^\infty$ functions whose support is compact and included in $\Omega$. Let us define $\mathcal{C}$ to be the ring of center 0 of small radius $\frac{1}{2}$ and great radius 2. There exist two nonnegative radial functions $\chi$ and $\psi$ belonging, respectively, to $\mathcal{D}(B(0, 1))$ and $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi) + \sum_{q \ge 0} \psi(2^{-q}\xi) = 1, \quad |p - q| \ge 2 \Rightarrow \psi(2^{-p}\xi) \cdot \psi(2^{-q}\xi) = 0$$

For example, one can take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,\frac{1}{2})$ and take $\psi(\xi) = \chi(2\xi) - \chi(\xi)$. Then we are able to define the Littlewood-Paley decomposition. Let $h, \tilde{h}, \triangle_q, S_q$ be defined as follow. Denoting by $\mathcal{F}$ the Fourier transform,

$$h = \mathcal{F}^{-1}\psi, \quad \tilde{h} = \mathcal{F}^{-1}\chi, \quad \triangle_q u = \mathcal{F}^{-1}(\psi(2^{-q}\xi)\hat{u}), \quad S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\hat{u}))$$

The set $\{S_q, \triangle_q\}_{q\in N\cup\{0\}}$ is the Littlewood-Paley decomposition of unity. Let $s \in R$, $p, q \in [1,\infty]$. Then the inhomogeneous and homogeneous Besov seminorms are defined, respectively, by

$$||u||_{B^s_{p,q}} = ||S_0 u||_{L^p} + (\sum_{j\geq -1} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}, \quad ||u||_{\dot{B}^s_{p,q}} = (\sum_{j\in\mathbb{Z}} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}$$

We also define time dependent seminorms.

$$||u||_{L^\rho_T B^s_{p,q}} = ||S_0 u||_{L^\rho_T L^p} + ||(\sum_{j\geq -1} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}||_{L^\rho_T}, \quad ||u||_{L^\rho_T \dot{B}^s_{p,q}} = ||(\sum_{j\in\mathbb{Z}} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}||_{L^\rho_T}$$

Let us point out that $B^s_{2,2}$ is a usual Sobolev space $H^s$ and that $B^s_{\infty,\infty}$ is the usual Holder space $C^s$. Now we have the following proposition.

**Proposition 2.** (a) Bernstein's inequality : for $1 \leq a \leq b$, $||\triangle_q f||_{L^b} \lesssim 2^{d(\frac{1}{a}-\frac{1}{b})q}||\triangle_q f||_{L^a}$

(b) Assume that $f \in L^p$, $1 \leq p \leq \infty$, and $supp\hat{f} \subset \{2^{j-2} \leq |\xi| \leq 2^j\}$. Then there exists a constant $C_k$ such that $\quad {C_k}^{-1} 2^{jk}||f||_{L^p} \leq ||D^k f||_{L^p} \leq C_k 2^{jk}||f||_{L^p}$

(c) Commutator estimate : $||[f\cdot\nabla, \triangle_j]g||_{L^p} \lesssim ||\nabla f||_{L^\infty} \cdot ||\triangle_j g||_{L^p}$

The proof is standard and can be found in [4, 5, 7].

#### (2) A NEW FORMULATION

With the introduction of the deformation tensor, the incompressibility of the fluid can be represented as $(\det U = 1)$. Moreover, if we denote $(\nabla\cdot U)_j = \partial_i U^{ij}$, we deduce from (VE) [11]

$$(\nabla\cdot U)_t + v\cdot\nabla(\nabla\cdot U) = 0$$

In two space dimension, when $\nabla\cdot U_0 = 0$, (1) ensures that $\nabla\cdot U = 0$ for all time. Therefore, we can find a vector $\phi = (\phi_1, \phi_2)$ such that [10]

$$\mathbf{U} = \begin{pmatrix} -\partial_y\phi_1 & -\partial_y\phi_2 \\ \partial_x\phi_1 & \partial_x\phi_2 \end{pmatrix}$$

Then (VE) can be reformulated as

$$(VE)\begin{cases}\phi_t + v\cdot\nabla\phi = 0\\ v_t + v\cdot\nabla v - \triangle v = -\nabla p - \sum_{i=1}^{2}\triangle\phi_i\nabla\phi_i\\ \nabla\cdot v = 0\\ \phi(x,0)=\phi_0(x), v(x,0)=v_0\end{cases}$$

**(3) A PRIORI ESTIMATE**

By taking the localized operator $\triangle_j$ to the velocity equation, multiplying by $\triangle_j v$, and integrating in the spatial variables, we have

$$\frac{1}{2}\frac{d}{dt}||\triangle_j v||_{L^2}^2 + ||\nabla\triangle_j v||_{L^2}^2 \lesssim ||[v\cdot\nabla,\triangle_j]v||_{L^2}||\triangle_j v||_{L^2} + (\triangle_j(\nabla\cdot(\nabla\phi\nabla\phi)),\triangle_j v)$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2$ space.

Using the fact $||[v\cdot\nabla,\triangle_j]v||_{L^2}\lesssim||\nabla v||_{L^\infty}\cdot||\triangle_j v||_{L^2}$, and integrating the last term by parts,

$$\frac{1}{2}\frac{d}{dt}||\triangle_j v||_{L^2}^2 + ||\nabla\triangle_j v||_{L^2}^2 \lesssim ||\nabla v||_{L^\infty}||\triangle_j v||_{L^2}^2 + ||\triangle_j(\nabla\phi\nabla\phi)||_{L^2}^2 + \frac{1}{2}||\nabla\triangle_j v||_{L^2}^2$$

Multiplying by $2^{j(s-1)}$ and adding them up, we have

$$\frac{d}{dt}||v(t)||_{H^{s-1}}^2 + ||v(t)||_{H^s}^2 \lesssim ||\nabla v(t)||_{L^\infty}||v(t)||_{H^{s-1}}^2 + ||\nabla\phi(t)||_{H^s}^4 \tag{1}$$

Similarly, $\frac{d}{dt}||\nabla\phi(t)||_{H^s}\lesssim||\nabla v(t)||_{L^\infty}||\nabla\phi(t)||_{H^s}$. Therefore,

$$\frac{d}{dt}||\nabla\phi(t)||_{H^s}^4 = 4||\nabla\phi(t)||_{H^s}^3\cdot\frac{d}{dt}||\nabla\phi(t)||_{H^s}\lesssim||\nabla v(t)||_{L^\infty}||\nabla\phi(t)||_{H^s}^4 \tag{2}$$

**(4) EQUATION OF THE VORTICITY**

By applying curl to the velocity equation, we have

$$w_t + v\cdot\nabla w - \triangle w = \nabla\times(\nabla\phi\triangle\phi)$$

Multiplying by $w$ and integrating in the spatial variables, we obtain that

$$\frac{1}{2}\frac{d}{dt}||w||_{L^2}^2 + ||\nabla w||_{L^2}^2 \lesssim ||\triangle\phi\nabla\phi||_{L^2}||\nabla w||_{L^2}\lesssim||\nabla\phi||_{L^\infty}||\triangle\phi||_{L^2}||\nabla w||_{L^2}\lesssim||\nabla\phi||_{H^s}^2||\nabla w||_{L^2}$$

Here, we use the fact that $s>1$. Therefore,

$$\frac{d}{dt}||w||_{L^2}^2\lesssim||\nabla v||_{L^\infty}||w||_{L^2}^2 + ||\nabla\phi||_{H^s}^4 \tag{3}$$

**Remark:** $||w||_{L^\infty_T L^2}$ comes from the estimate of $||\nabla v||_{L^1_T L^\infty}$ below. But, we are mentioning $||w||_{L^\infty_T L^2}$ before $||\nabla v||_{L^1_T L^\infty}$ for convenience. As we'll see later, the vorticity estimates stem from the lower frequency part of the gradient of the velocity, which is convolved with a nice function. The vorticity equation almost preserves the Navier-Stokes equation, and we are only concerned about the $L^2$ bound, which is easily obtained from (3).

**(5) CALCULATION OF** $||\nabla v||_{L^1_T L^\infty}$

By integrating (1), (2), and (3) in time, we deduce that

$$||v(t)||^2_{H^{s-1}} + \int_0^t ||v(\tau)||^2_{H^s} d\tau \lesssim ||v_0||^2_{H^{s-1}} + \int_0^t ||\nabla v(\tau)||_{L^\infty} ||v(\tau)||^2_{H^{s-1}} d\tau + \int_0^t ||\nabla \phi(\tau)||^4_{H^s} d\tau$$
$$\lesssim ||v_0||^2_{H^{s-1}} + ||\nabla v||_{L^1_T L^\infty} ||v||^2_{L^\infty_T H^{s-1}} + T \cdot ||\nabla \phi||^4_{L^\infty_T H^s} \tag{4}$$

$$||\nabla \phi(t)||^4_{H^s} \lesssim ||\nabla \phi_0||^4_{H^s} + \int_0^t ||\nabla v(\tau)||_{L^\infty} ||\nabla \phi(\tau)||^4_{H^s} d\tau \lesssim ||\nabla \phi_0||^4_{H^s} + ||\nabla v||_{L^1_T L^\infty} ||\nabla \phi||^4_{L^\infty_T H^s} \tag{5}$$

$$||w(t)||^2_{L^2} \lesssim ||w_0||^2_{L^2} + \int_0^t ||\nabla \phi(\tau)||^4_{H^s} d\tau \lesssim ||w_0||^2_{L^2} + T \cdot ||\nabla \phi||^4_{L^\infty_T H^s} \tag{6}$$

Now, we need to estimate $||\nabla v||_{L^1_T L^\infty}$. First, we decompose $\nabla v(t)$ in the following way:

$$\nabla v(t) = \nabla S_{-N} v(t) + \sum_{|j| \le N} \nabla \triangle_j v(t) + \sum_{j > N} \nabla \triangle_j v(t)$$

where $N$ will be determined later. Integrating in the spatial variables, we have that

$$||\nabla v(t)||_{L^\infty} \lesssim ||\nabla S_{-N} v(t)||_{L^\infty} + \sum_{|j| \le N} ||\nabla \triangle_j v(t)||_{L^\infty} + \sum_{j > N} ||\nabla \triangle_j v(t)||_{L^\infty}$$

On the first term, we use Bernstein's inequality so that

$$||\nabla S_{-N} v(t)||_{L^\infty} \lesssim 2^{-N} ||S_{-N} \nabla v(t)||_{L^2} \lesssim 2^{-N} ||\nabla v(t)||_{L^2} \lesssim 2^{-N} ||w(t)||_{L^2}$$

where we used the Biot-Savart Law to the last inequality. We estimate the third term by using Bernstein's inequality and Young's inequality

$$\sum_{j > N} ||\nabla \triangle_j v(t)||_{L^\infty} \lesssim \sum_{j > N} 2^{j(1 + \frac{d}{2})} ||\triangle_j v(t)||_{L^2} = \sum_{j > N} 2^{2j} ||\triangle_j v(t)||_{L^2}$$
$$= \sum_{j > N} 2^{j(-s+1)} \cdot 2^{j(s+1)} ||\triangle_j v(t)||_{L^2} \lesssim 2^{-N(s-1)} \cdot ||v(t)||_{H^{s+1}}$$

Let $\lambda = \min\{1, s-1\} > 0$. Integrating in time,

$$||\nabla v||_{L^1_T L^\infty} \lesssim 2^{-N\lambda}\{||w||_{L^1_T L^2} + ||v||_{L^1_T H^{s+1}}\} + (2N+1)\sup_{|j|\le N}\int_0^T ||\triangle_j \nabla v(t)||_{L^\infty} dt$$

$$\lesssim\ 2^{-N\lambda}\{T||w||_{L^\infty_T L^2} + ||v||_{L^1_T H^{s+1}}\} + (2N+1)\sup_{j\in\mathbb{Z}}\int_0^T ||\triangle_j \nabla v(t)||_{L^\infty} dt$$

Now, we would like to estimate $||v||_{L^1_T H^{s+1}}$. It comes from the estimate of the inhomogeneous heat equation

$$(H)\begin{cases} v_t - \triangle v = -v\cdot\nabla v - \nabla\cdot(\nabla\phi\nabla\phi) = f \\ v(x,0) = v_0 \in H^{s-1}\end{cases}$$

The solution can be expressed as an integral form :

$$v(t) = e^{t\triangle}v_0 + \int_0^t e^{(t-s)\triangle} f(s)ds$$

Since $s > 1$, $f = -v\cdot\nabla v - \triangle\phi\nabla\phi \in L^1_T H^{s-1}$. From now on, we assume that $T > 1$ and we estimate terms in the time interval $[1, T]$. Then,

$$||v||_{L^1_T H^{s+1}} \lesssim \log T\{||v(1)||_{H^{s-1}} + ||f||_{L^1_T H^{s-1}}\} \lesssim \log T\{||v||_{L^\infty_T H^{s-1}} + ||f||_{L^1_T H^{s-1}}\}$$

Therefore, by the assumption on $T > 1$,

$$||v||_{L^1_T H^{s+1}} \lesssim \log T\{||v_0||_{H^{s-1}} + ||v||^2_{L^2_T H^s} + ||\nabla\phi||^2_{L^2_T H^s}\} \lesssim T\{||v_0||_{H^{s-1}} + ||v||^2_{L^2_T H^s} + T||\nabla\phi||^2_{L^\infty_T H^s}\}$$

But, from (4), $||v||^2_{L^2_T H^s} \lesssim ||v_0||_{H^{s-1}} + ||\nabla v||_{L^1_T L^\infty}||v||^2_{L^\infty_T H^{s-1}} + T||\nabla\phi||^4_{L^\infty_T H^s}$.

$$||v||_{L^1_T H^{s+1}} \lesssim T||v_0||_{H^{s-1}} + T||v_0||^2_{H^{s-1}} + T||\nabla v||_{L^1_T L^\infty}||v||^2_{L^\infty_T H^{s-1}} + T^2||\nabla\phi||^4_{L^\infty_T H^s}$$

Therefore,

$$\begin{aligned}
||\nabla v||_{L^1_T L^\infty} &\lesssim 2^{-N\lambda}\{T||w||_{L^\infty_T L^2} + T||v_0||_{H^{s-1}} + T||v_0||^2_{H^{s-1}} + T^2||\nabla\phi||^4_{L^\infty_T H^s} + T^2||\nabla\phi||^2_{L^\infty_T H^s}\} \\
&+ 2^{-N\lambda}T||\nabla v||_{L^1_T L^\infty}||v||^2_{L^\infty_T H^{s-1}} + (2N+1)\cdot\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j\nabla v(t)||_{L^\infty}dt \\
&\lesssim 2^{-N\lambda}\{T||w_0||_{L^2} + T||v_0||_{H^{s-1}} + T||v_0||^2_{H^{s-1}} + T^2||\nabla\phi||^4_{L^\infty_T H^s} + T^2||\nabla\phi||^2_{L^\infty_T H^s}\} \\
&+ 2^{-N\lambda}T||\nabla v||_{L^1_T L^\infty}||v||^2_{L^\infty_T H^{s-1}} + (2N+1)\cdot\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j\nabla v(t)||_{L^\infty}dt \qquad (7)
\end{aligned}$$

**(6) LOGARITHMIC ESTIMATE**

We establish the logarithmic Sobolev inequality in the framework of mixed time-space Besov space.

From (7),

$$
\begin{aligned}
& \{1-2^{-N\lambda}T||v||^2_{L^\infty_T H^{s-1}}\}||\nabla v||_{L^1_T L^\infty} \\
\lesssim\ & 2^{-N\lambda}\{T||w_0||_{L^2}+T||v_0||_{H^{s-1}}+T||v_0||^2_{H^{s-1}}+T^2||\nabla\phi||^4_{L^\infty_T H^s}+T^2||\nabla\phi||^2_{L^\infty_T H^s}\} \\
+\ & (2N+1)\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j \nabla v(t)||_{L^\infty}dt \\
\lesssim\ & 2^{-N\lambda}T^2||\nabla\phi||^4_{L^\infty_T H^s}+2^{-N\lambda}\{T^2+T||w_0||_{L^2}+T||v_0||_{H^{s-1}}+T||v_0||^2_{H^{s-1}}\} \\
+\ & (2N+1)\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j \nabla v(t)||_{L^\infty}dt
\end{aligned}
$$

Let $\mu(T)=||w||^2_{L^\infty_T L^2}+||v||^2_{L^\infty_T H^{s-1}}+||\nabla\phi||^4_{L^\infty_T H^s}$.

$$
\begin{aligned}
& \{1-2^{-N\lambda}\cdot T\cdot ||v||^2_{L^\infty_T H^{s-1}}\}\cdot||\nabla v||_{L^1_T L^\infty} \\
\lesssim\ & 2^{-N\lambda}\cdot T^2\cdot\mu(T)+2^{-N\lambda}\cdot T^2\{1+||w_0||_{L^2}+||v_0||^2_{H^{s-1}}\}+(2N+1)\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j\nabla v(t)||_{L^\infty}
\end{aligned}
$$

If we choose $N\sim\frac{1}{\lambda}log(T\mu(T))$,

$$
||\nabla v||_{L^1_T L^\infty}\lesssim T+T\{1+||w_0||_{L^2}+||v_0||^2_{H^{s-1}}\}+(1+log(1+T\mu(T)))\cdot\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j\nabla v(t)||_{L^\infty}dt
$$

By (4), (5), and (6),

$$
\mu(T)\lesssim\mu(0)+T\mu(T)+T(\mu(0))^2\mu(T)+\{(1+log(1+T\mu(T)))\cdot\sup_{j\in\mathbb{Z}}\int_1^T ||\triangle_j\nabla v(t)||_{L^\infty}dt\}\mu(T)
$$

This inequality still holds if the time interval $[1,T)$ is replaced by $[T-\sigma,T)$. So, we infer that $\mu(T)$ can be dominated by $\mu(T-\sigma)$ from the following inequality :

$$
\mu(T)\lesssim\mu(T-\sigma)+g(\sigma)\cdot\mu(T)\cdot\{1+log(1+\sigma\mu(T))\} \tag{8}
$$

where $g(\sigma)=\sigma+\sigma(\mu(T-\sigma))^2+\sup_{j\in\mathbb{Z}}\int_{T-\sigma}^T ||\triangle_j\nabla v(t)||_{L^\infty}dt$ is a function such that $g(\sigma)$ tends to 0 as $\sigma$ goes to 0. Since $\sigma$ does not depend on $T$, this completes the proof of the Theorem. ■

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