THE CAUCHY-DIRICHLET PROBLEM FOR THE FENE DUMBBELL MODEL OF POLYMERIC FLUIDS

HAILIANG LIU AND JAEMIN SHIN

Abstract. The FENE dumbbell model consists of the incompressible Navier-Stokes equation for the solvent and the Fokker-Planck equation for the polymer distribution. In such a model, the polymer elongation cannot exceed a limit $\sqrt{b}$ which yields all interesting features of solutions near this limit. This work is concerned with the sharpness of boundary conditions in terms of the elongation parameter b. Through a careful analysis of the Fokker-Planck operator coupled with the Navier-Stokes equation, we establish a local well-posedness for the full coupled FENE dumbbell model under a class of Dirichlet-type boundary conditions dictated by the parameter b. For each $b > 0$ we identify a sharp boundary requirement for the underlying density distribution, while the sharpness follows from the existence result for each specification of the boundary behavior. It is shown that the probability density governed by the Fokker-Planck equation approaches zero near boundary, necessarily faster than the distance function d for $b > 2$, faster than $d|lnd|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Moreover, the sharp boundary requirement for $b \geq 2$ is also sufficient for the distribution to be a probability density.

1. Introduction

Let $N \geq 2$ be an integer. We consider a dimer – an idealized polymer chain – as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by an elongation vector $m \in \mathbb{R}^N$ (see e.g [6]), with the elastic spring potential given by

$$\Psi(m) = -\frac{Hb}{2} \log\left(1 - \frac{|m|^2}{b}\right), \quad m \in B. \tag{1.1}$$

Here $B := B(0, \sqrt{b})$ is a ball in $\mathbb{R}^N$ with radius $\sqrt{b}$ denoting the maximum dumbbell extension. In the limiting case as $b \to \infty$, this reduces to the well-known Hookean model with $\Psi(m) = H|m|^2/2$. A general bead-spring chain model may contain more than two beads coupled with elastic springs to represent a polymer chain.

Polymers as such when immersed in an incompressible, viscous, isothermal Newtonian solvent are modeled by a system coupling the incompressible Navier-Stokes equation for the macroscopic velocity field $v(t, x)$ with the Fokker-Planck equation

Date: September 29, 2010; revision June 03, 2011.

1991 *Mathematics Subject Classification.* 35Q30, 82C31, 76A05.

Key words and phrases. Fokker Planck equation, Navier-Stokes equation, FENE model, boundary conditions, well-posedness.

for the probability distribution function $f(t,x,m)$:

$$\partial_t v + (v\cdot\nabla)v + \nabla p = \nabla\cdot\tau + \nu_k \Delta v, \tag{1.2}$$

$$\nabla\cdot v = 0, \tag{1.3}$$

$$\partial_t f + (v\cdot\nabla)f + \nabla_m\cdot(\nabla v m f) = \frac{2}{\zeta}\nabla_m\cdot(\nabla_m\Psi(m)f) + \frac{2k_BT_a}{\zeta}\Delta_m f, \tag{1.4}$$

where $x\in\mathbb{R}^N$ is the macroscopic Eulerian coordinate and $m\in B\subset\mathbb{R}^N$ is the microscopic molecular configuration variable. The model describes diluted solutions of polymeric liquids with noninteracting polymer chains (dimers). Note that the Fokker-Planck equation can be conveniently augmented to incorporate other effects such as inertial forces (see [14]).

In Navier-Stokes equation (1.2), p is the hydrostatic pressure, ν_k is the kinematic viscosity coefficient, and τ is a tensor representing the polymer contribution to stress,

$$\tau = \lambda_p \int m\otimes\nabla_m\Psi(m) f dm,$$

where λ_p is the polymer density constant. In the Fokker-Planck equation (1.4), ζ is the friction coefficient of the dumbbell beads, T_a is the absolute temperature, and k_B is the Boltzmann constant. We refer to [6, 13, 43] for a comprehensive survey of the physical background, and [42] for the computational aspect.

Notice that the term $\nabla_m\Psi$ in (1.4) becomes singular (unbounded) on $\partial B = \{m, \quad |m| = \sqrt{b}\}$. The question arises as to what kind of boundary condition needs to be imposed on ∂B. When we consider the evolution of

$$\int_{\mathbb{R}^N}\int_B f dm dx,$$

we find that the evolution rate $\frac{d}{dt}\int_{\mathbb{R}^N}\int_B f dm dx$ depends both values of fv at far field in x through $-\lim_{r\to\infty}\int_{|x|=r}\int_B fv\cdot\frac{x}{r} dm dS_x$ and the total flux on ∂B. For mass conservation, we therefore might expect that fv should decay for large $|x|$ and should have zero flux on ∂B, i.e.,

$$\left[\frac{2}{\zeta}\left(k_BT_a\nabla_m f + \nabla_m\Psi f\right) - \nabla v m f\right]\cdot\frac{m}{|m|}\Big|_{\partial B} = 0.$$

This is indeed the condition which has been frequently adopted in priori works. Notice that here m is an end-to-end vector, this zero flux condition does not seem to have a definite physical interpretation. Instead, since the Fokker-Planck equation (1.4) is singular at ∂B, it is natural to ask what kind of boundary behavior one should impose or one would expect?

The singularity in the potential requires at least zero Dirichlet boundary condition

$$f|_{\partial B} = 0.$$

This is consistent with the result in [22], which states that the stochastic solution trajectory does not reach the boundary almost surely. However, the above condition is insufficient for well-posedness. In [35], C. Liu and H. Liu examined the ratio of the distribution f and the equilibrium f_{eq}, i.e.,

$$w = f/f_{eq}$$

for the microscopic FENE model, by the method of the Fichera function they were able to show that $b = 2$ is a threshold in the sense that for $b \geq 2$ any preassigned boundary value of w will become redundant, and for $b < 2$ that value has to be a priori given. As a side note we point out that there is a misprint in the statement of this result, Theorem 1.1 in [35], where the correct assertion should be about the boundary condition for w rather than for f – the proof is otherwise correct.

Our main quest in this article is what the least boundary requirement for f is so that both existence and uniqueness of the solution to the FENE model can be established, also the solution be a probability density. Upon pursuing this, we shall achieve two main objectives:

(1) to identify sharp boundary conditions for all $b > 0$.
(2) to prove well-posedness for the coupled Navier-Stokes-Fokker-Planck system under the identified boundary condition.

In order to describe the behavior of f near ∂B, our idea is to identify a rate function ν, which approaches zero near boundary at a different rate for different b. We also use another function q to quantify the relative ratio of f/ν near boundary. More precisely, we impose the following boundary condition

$$f(t,x,m)\nu^{-1}|_{\partial B} = q(t,x,m)|_{\partial B}. \tag{1.5}$$

This boundary condition is a boundary behavior requirement, instead of the Dirichlet data in the usual sense. Nevertheless, the pair (ν, q), once known, determines the behavior of f near ∂B.

We shall investigate solvability of the coupled system (1.2)-(1.4) subject to (1.5) and the initial data

$$v(0,x) = v_0(x), \tag{1.6}$$

$$f(0,x,m) = f_0(x,m). \tag{1.7}$$

In fact, for each b, we are able to identify the form of ν

$$\nu = \begin{cases} \rho^{b/2}, & 0 < b < 2, \\ \rho \ln \frac{e}{\rho}, & b = 2, \\ \rho, & b > 2, \end{cases} \tag{1.8}$$

where $\rho = b - |m|^2$ plays a role of the distance function $d = \sqrt{b} - |m|$. We should point out that $\nu = \rho$ when $b > 2$ was identified in [36] where the Fokker-Planck equation (1.4) alone was studied. In this work, with some regularity requirement on q as well as on initial data, we prove local well-posedness for the above Cauchy-Direchlet problem in a weighted Sobolev space for each given q. Our results indicate that simply requiring $f = 0$ on boundary does not guarantee uniqueness of solutions.

As is known, the singularity of the Fokker-Planck equation near ∂B makes the boundary issue rather subtle, and presents various challenges. To address the difficulties caused, several transformations relating to the equilibrium solution have been introduced in literature (see, e.g. [16, 24, 35, 36]). For the Dirichlet-type boundary condition (1.5), our strategy is to study a transformed problem via

$$w = \frac{f}{\nu} - q$$

with ν defined in (1.8) so that $w|_{\partial B} = 0$. Once w is known, one can extract f from $\nu(w+q)$. Inspired from [38], for the coupled FENE system we use weak norm in m and strong norm in x, this enables us to prove well-posedness for all cases of $b > 0$ and any given smooth q.

We identify a sharp boundary requirement for each $b > 0$ for the underlying density distribution, while the sharpness is a consequence of the existence result for each q. For $b \geq 2$, we show that f is a density distribution if and only if $q|_{\partial B} = 0$. In particular, our result asserts that near boundary the probability density governed by the Fokker-Planck equation approaches zero, necessarily faster than the distance function d for $b > 2$, faster than $d|lnd|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Unfortunately, within our current framework we have not been able to identify a non-trivial q for $0 < b < 2$ such that the corresponding solution is a density distribution.

We remark that the sharp boundary condition presented in this work provides a threshold on the boundary requirement: subject to the sharp requirement or a stronger condition incorporated through some weighted function spaces [47, 38], the Fokker-Planck dynamics will select the physically relevant solution, which is a probability density, any weaker boundary requirement can lead to many solutions, each depending on the ratio of f/ν near boundary.

This article is organized as follows. In Section 2, we state our main results and main ideas of the proofs. In Section 3, we study the Fokker-Planck operator and well-posedness of the initial boundary value problem for the Fokker-Planck equation alone. This part alone improves upon our previous work [36]. The main result is summarized in Theorem 13. The Fokker-Planck problem involving spatial variable x is investigated in Section 4. Well-posedness of the coupled system is proved in Section 5. In Section 6, we sketch the proof of well-posedness for the coupled system with $b \geq 6$ in a different function space than what was used in Section 5. Some concluding remarks are drawn in Section 7.

We conclude this section by some bibliographical remarks.

Existence results for the FENE model are usually limited to local in time existence and uniqueness of strong solutions. We refer to [44] for the local existence on some related coupled systems, [22] for the FENE model (in the setting where the Fokker-Planck equation is formulated by a stochastic differential equation) with $b > 6$, [17] for a polynomial force. More related to this paper are the work by Zhang and Zhang [47] for the FENE model when $b > 76$, and Masmoudi [38] for $b > 0$. Global existence results are usually limited to solutions near equilibrium, see [28, 33], or to some 2D simplified models [10, 12, 27, 41]. For results concerning the existence of weak solutions to the coupled FENE system we refer to [2, 3, 4, 5, 34, 39, 45, 48].

Boundary behavior of the polymer distribution governed by the FENE model is also essential in several other aspects, including the study of large time behavior, see [1, 20, 23, 45]; and development of numerical methods, see, e.g., [8, 9, 16, 24, 37, 46]. We also refer to [21] for references on numerical aspects of polymeric fluid models.

There are also some interesting works on non-Newtonian fluid models derived through a closure of the linear Fokker-Planck equation (see, e.g., [15, 16]). We can refer to the pioneering work [18, 19], and more recently to [11, 29, 30, 31, 32].

However, none of these works is concerned with the sharpness of boundary conditions in terms of the elongation parameter.

2. Main results

After a suitable scaling and choice of parameters we arrive at the following Cauchy-Dirichlet problem for the coupled system

$$\partial_t v + (v\cdot\nabla)v + \nabla p = \nabla\cdot\tau + \Delta v, \quad x\in\mathbb{R}^N, \quad t>0, \tag{2.1a}$$

$$\nabla\cdot v = 0, \tag{2.1b}$$

$$\partial_t f + (v\cdot\nabla)f + \nabla_m\cdot(\nabla v m f) = \frac{1}{2}\nabla_m\cdot\left(\frac{bm}{\rho}f\right) + \frac{1}{2}\Delta_m f, \quad m\in B, \tag{2.1c}$$

$$\tau = \int m\otimes\frac{bm}{\rho} f\,dm, \tag{2.1d}$$

$$v(0,x) = v_0(x), \tag{2.1e}$$

$$f(0,x,m) = f_0(x,m), \tag{2.1f}$$

$$f(t,x,m)\nu^{-1}|_{\partial B} = q(t,x,m)|_{\partial B}. \tag{2.1g}$$

To present our main results we first fix notations to be used throughout this article. We fix an exponent s, which is an integer in the range $s > N/2+1$. We use C to denote various constants depending on s, b and some other quantities which we will indicate in the sequel. A b-dependent weight function is defined as

$$\mu = \begin{cases} \rho^{b/2}, & 0<b<2, \\ \rho\ln^2\frac{e}{\rho}, & b=2, \\ \rho^{2-b/2}, & b>2. \end{cases} \tag{2.2}$$

For $b\ge 6$, we also use

$$\mu_0 = \rho^\theta, \ -1<\theta<1, \ b\ge 6. \tag{2.3}$$

Other notations are listed as below as well.

- $L^2_\mu = \left\{\phi : \int_B \phi^2\mu dm < \infty\right\}$
- $H^1_\mu = \{\phi : \phi, \partial_{m_j}\phi \in L^2_\mu, j = 1\cdots N\}$
- $\overset{\circ}{H}{}^1_\mu$ denotes the completion of C_c^∞ with H^1_μ norm.
- H^* is a dual space of H
- H^s_x is the usual Sobolev space with respect to x
- $$|v|_s^2 = \sum_{|\alpha|\le s}\int_{\mathbb{R}^N}|\partial^\alpha v|^2 dx,$$
$$|w|_{0,s}^2 = \sum_{|\alpha|\le s}\int_{\mathbb{R}^N}\int_B |\partial^\alpha w|^2\mu dm dx,$$
$$|w|_{1,s}^2 = |w|_{0,s}^2 + |\nabla_m w|_{0,s}^2,$$
$$\|w\|_{1,1,s}^2 = \sup_t(|w|_{1,s}^2 + |\partial_t w|_{1,s}^2),$$
$$\|q\| = \|q\|_{H^1_\mu} + \|\partial_t q\|_{H^1_\mu}.$$

- $H^s_x L^2_\mu = \{\phi : |\phi|_{0,s} < \infty\}, \quad H^s_x H^1_\mu = \{\phi : |\phi|_{1,s} < \infty\}$.
- $L^2_t H = L^2((0,T); H), \quad C_t H = C([0,T]; H)$ for $0 < t < T$.
- $\mathcal{H} = \{\phi : \|\phi\|_{L^2_t H^1_\mu} + \|\phi_t\|_{L^2_t (H^1_\mu)^*} < \infty\}$,
  $\overset{\circ}{\mathcal{H}} = \{\phi(t,\cdot) \in \overset{\circ}{H}{}^1_\mu : \|\phi\|_{L^2_t H^1_\mu} + \|\phi_t\|_{L^2_t (\overset{\circ}{H}{}^1_\mu)^*} < \infty\}$.
-
$$\mathbf{X}_\mu = [C_t H^s_x \cap L^2_t H^{s+1}_x] \times [C_t H^s_x L^2_\mu \cap L^2_t H^s_x H^1_\mu].$$
- For a generic constant C, independent of T and $a \in L^2_t$, we define

$$F(a) = C\left(T + \int_0^T |a(t)|^2 dt\right). \tag{2.4}$$

  Due to such a constant, any two instances of F should be presumed to be with different constants.

We now state our main theorem as follows:

Theorem 1. *Let $b > 0$ and s be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H^s_x$, $f_0 \nu^{-1} \in H^s_x L^2_\mu$, and $q \in C^1_t H^{s+1}_x H^1_\mu$. Then, for some $T > 0$ there exists a unique solution (v, f) to the coupled problem* (2.1) *such that*

$$(v, f\nu^{-1}) \in \mathbf{X}_\mu.$$

It is known from [25] that $\overset{\circ}{H}{}^1_\mu = H^1_\mu$ for $b \geq 6$ with μ defined in (2.2). Thus, boundary condition (2.1g) is nothing but the zero Dirichlet boundary condition under the assumption on q in Theorem 1. For non-trivial q when $b \geq 6$, we show the well-posedness in a different weighted Sobolev space. The result is summarized as below.

Theorem 2. *Let $b \geq 6$ and s be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H^s_x$, $f_0 \nu^{-1} \in H^s_x L^2_{\mu_0}$, and $q \in C^1_t H^{s+1}_x H^1_{\mu_0}$ with μ_0 defined in (2.3). Then, for some $T > 0$ there exists a unique solution (v, f) to the coupled problem* (2.1) *such that*

$$(v, f\nu^{-1}) \in \mathbf{X}_{\mu_0}.$$

Theorem 1 and 2 tell us that for each given q, which denotes the rate of f approaching to zero relative to ν near ∂B, there exists a unique solution (v, f). Also, they indicate that any weaker boundary requirement may lead to more than one solutions to (2.1). For instance, the boundary condition

$$f\nu^{-1}\rho^\varepsilon|_{\partial B} = 0, \quad \varepsilon > 0$$

gives infinitely many solutions to (2.1). Precisely we state the following non-uniqueness result.

Theorem 3. *Let $\tilde{\nu}$ be a smooth function of ρ such that*

$$\lim_{\rho \to 0} \frac{\nu}{\tilde{\nu}} = 0. \tag{2.5}$$

Then, the coupled problem (2.1) *with* (2.1g) *replaced by*

$$f(t,x,m)\tilde{\nu}^{-1}|_{\partial B} = 0 \tag{2.6}$$

has infinitely many solutions in $\mathbf{X}_\mu$ and $\mathbf{X}_{\mu_0}$ for $0 < b < 6$ and $b \geq 6$ respectively.

A natural question is for what q the obtained distribution f is a probability distribution. The answer when $b \geq 2$ is given in the following theorem.

Theorem 4. *Suppose that $b \geq 2$ and $q|_{\partial B} \geq 0$. Under the assumption of Theorem 1 or 2, the unique solution f to the Cauchy-Dirichlet problem* (2.1) *is a probability distribution if and only if $q|_{\partial B} = 0$. That is, $f \geq 0$ if $f_0 \geq 0$, and for any $t > 0$, $x \in \mathbb{R}^N$,*

$$\int_B f(t,x,m)dm = \int_B f_0(x,m)dm. \tag{2.7}$$

Theorem 1 is proven by a fixed point argument, which is now outlined. Given (u, g), we first solve the Navier-Stokes equation (NSE):

$$\begin{aligned}
\partial_t v + (u\cdot\nabla)v + \nabla p &= \nabla\cdot\tau + \Delta v, && \text{(2.8a)}\\
\nabla\cdot v &= 0, && \text{(2.8b)}\\
v(0,x) &= v_0(x), && \text{(2.8c)}\\
\tau &= \int m\otimes\frac{bm}{\rho} g dm. && \text{(2.8d)}
\end{aligned}$$

With the obtained v we solve the Fokker-Planck equation (FPE):

$$\begin{aligned}
\partial_t f + (v\cdot\nabla)f + \nabla_m\cdot(\nabla v m f) &= \frac{1}{2}\nabla_m\cdot\left(\frac{bm}{\rho}f\right) + \frac{1}{2}\Delta_m f, && \text{(2.9a)}\\
f(0,x,m) &= f_0(x,m), && \text{(2.9b)}\\
f(t,x,m)\nu^{-1}|_{\partial B} &= q(t,x,m)|_{\partial B}. && \text{(2.9c)}
\end{aligned}$$

The above two systems define a mapping $(u, g) \to (v, f)$, the existence of problem (2.1) is equivalent to the existence of a fixed point of this mapping.

The main difficulty lies in monitoring the boundary behavior of f. Our strategy is to apply the transformation

$$f = \nu(w+q), \tag{2.10}$$

to (2.9) to obtain a w-problem

$$\begin{aligned}
\mu(\partial_t + v\cdot\nabla)w + L[w] &= \mu h, && \text{(2.11a)}\\
w(0,x,m) &= w_0(x,m), && \text{(2.11b)}\\
w(t,x,m)|_{\partial B} &= 0. && \text{(2.11c)}
\end{aligned}$$

Here the operator L is induced from the Fokker-Planck operator, ν and μ are weights depending on the distance functions defined in (1.8) and (2.2), respectively. The source term is obtained from q

$$h = -\partial_t q - (v\cdot\nabla)q - \mu^{-1}L[q], \tag{2.12}$$

and the initial data is given by

$$w_0(x,m) := f_0(x,m)\nu^{-1} - q(0,x,m). \tag{2.13}$$

For given (u, ϖ) with $g = \nu(\varpi + q)$, we arrive at a map $\mathcal{F}$.

$$\begin{aligned}
\mathcal{F}: \quad \mathbf{M} &\to \mathbf{M}\\
(u,\varpi) &\mapsto (v,w)
\end{aligned}$$

Here $\mathbf{M}$ is a subset of

$$C_tH_x^s \times [C_tH_x^sL_\mu^2 \cap L_t^2H_x^s\overset{\circ}{H}{}_\mu^1]$$

such that

$$\mathbf{M} = \left\{(v,w) : \sup_{0\le t\le T}|v|_s^2 \le A_1,\ \sup_{0\le t\le T}|w|_{0,s}^2 + \frac{1}{2}\int_0^T |\nabla_m w|_{0,s}^2 dt \le A_2\right\}.$$

The strategy for the fixed point proof, which we implement in sections to follow, is to first prove that $\mathcal{F}$ is well defined for some T, A_1 and A_2, then to show that $\mathcal{F}$ is actually a contraction map in a weak norm. Moreover, we will show that

$$\mathcal{F}(\mathbf{M}) \subset \mathbf{X}_\mu. \tag{2.14}$$

These prove Theorem 1 for

$$q \in C_t^1H_x^{s+1}H_\mu^1 \subset [C_tH_x^sL_\mu^2 \cap L_t^2H_x^sH_\mu^1].$$

Theorem 2 is proved in the same manner. A sketch of the proof is presented in Section 6.

In order to prove Theorem 3, we pick $q(t,x,\cdot) \in C^\infty(B)\cap C(\overline{B})$ and $q|_{\partial B} \neq 0$ such that

$$q \in \begin{cases} C_t^1H_x^{s+1}H_\mu^1, & 0 < b < 6, \\ C_t^1H_x^{s+1}H_{\mu_0}^1, & b \ge 6. \end{cases}$$

Note that the existence of such a q follows from the density of the weighted Sobolev space (see [25] for details). Then for each q we have a unique solution (v,f) to the coupled problem (2.1) from Theorem 1 and Theorem 2. Finally, we turn to examine the boundary condition (2.6),

$$f\tilde{\nu}^{-1}|_{\partial B} = f\nu^{-1}\frac{\nu}{\tilde{\nu}}|_{\partial B} = q\frac{\nu}{\tilde{\nu}}|_{\partial B},$$

which vanishes since $q|_{\partial B}$ is bounded and condition (2.5) holds. This proves Theorem 3.

Theorem 4 follows from Proposition 16 and 17 via a flow map to be described in Section 4. The case for $b \ge 6$ can be proved by a simple modification, which is also sketched in Section 6.

3. The Fokker-Planck operator

We start with (2.9) when x is not involved. In such a case it reduces to the following problem:

$$\partial_t f + \mathcal{L}[f] = 0, \quad m \in B, t > 0, \tag{3.1a}$$

$$f(0,m) = f_0(m), \tag{3.1b}$$

$$f(t,m)\nu^{-1}|_{\partial B} = q(t,m)|_{\partial B}. \tag{3.1c}$$

Here

$$\mathcal{L}[f] := \nabla\cdot(\kappa m f) - \frac{1}{2}\nabla\cdot\left(\frac{bm}{\rho}f\right) - \frac{1}{2}\Delta f, \tag{3.2}$$

$\kappa = \kappa(t)$ is a square integrable matrix function such that $\mathrm{Tr}(\kappa) = 0$. We omit m from ∇_m in (3.2) for notational convenience.

The goal of this section is two folds:

(1) to provide tools for subsequent sections.
(2) to elaborate on this model alone as an extension of our previous work [36].

3.1. **Transformed operator.** The transformation (2.10) leads to

$$\partial_t w\mu + L[w] = \mu h, \quad m \in B,\ t > 0, \tag{3.3a}$$

$$w(0,m) = w_0, \tag{3.3b}$$

$$w(t,m)|_{\partial B} = 0, \tag{3.3c}$$

with the transformed operator L determined by

$$L[w] = \mu\nu^{-1}\mathcal{L}[\nu w]. \tag{3.4}$$

The source term is given as $h = -\partial_t q - \mu^{-1}L[q]$ and initial data for w is $w_0 = f_0\nu^{-1} - q(0,m)$.

From a direct calculation with the choice of μ in (2.2), and ν in (1.8), (3.4) can be expressed as

$$L[w] = -\frac{1}{2}\nabla\cdot(\nabla w\mu) + \nabla\cdot(\kappa m w\mu) - Kw, \tag{3.5}$$

where

$$K = \begin{cases} 0, & 0 < b < 2, \\ (N + 2\kappa m\cdot m)\ln\dfrac{e}{\rho}, & b = 2, \\ (N + 2\kappa m\cdot m)(b/2 - 1)\rho^{1-b/2}, & b > 2. \end{cases} \tag{3.6}$$

Associated with the operator L, we define its time-dependent bilinear form

$$\mathcal{B}[w,\phi;t] := \int \left(\frac{1}{2}\nabla w\cdot\nabla\phi\mu - w\mu\kappa m\cdot\nabla\phi - Kw\phi\right) dm \tag{3.7}$$

for $\phi, w \in \overset{\circ}{H}{}^1_\mu$ and fixed $t > 0$.

We now describe the weak solution which we are looking for.

Definition 5. *A function $w \in \overset{\circ}{\mathcal{H}}$ is a weak solution of w-problem (3.3), provided*

(1) *for each $\phi \in \overset{\circ}{H}{}^1_\mu$ and almost every $0 \le t \le T$,*

$$(\partial_t w, \phi)_{\overset{\circ}{H}{}^1_\mu} + \mathcal{B}[w,\phi;t] = (h,\phi)_{\overset{\circ}{H}{}^1_\mu}. \tag{3.8}$$

(2) *$w(0,m) = w_0(m)$ in L^2_μ sense, i.e.,*

$$\int_B |w(0,m) - w_0(m)|^2\mu dm = 0.$$

Remark 6. *In (3.8), $(\psi,\phi)_{\overset{\circ}{H}{}^1_\mu}$ is a dual pair for $\psi \in (\overset{\circ}{H}{}^1_\mu)^*$ and $\phi \in \overset{\circ}{H}{}^1_\mu$, and can be regarded as L^2_μ inner product. Indeed, from the Riesz representaiton theorem, for each $\psi \in (\overset{\circ}{H}{}^1_\mu)^*$ there exists a unique $u \in \overset{\circ}{H}{}^1_\mu$ such that*

$$(\psi,\phi)_{\overset{\circ}{H}{}^1_\mu} = \int_B (\nabla u\cdot\nabla\phi + u\phi)\mu dm.$$

Formally, the right hand side will be

$$\int_B (\nabla \cdot (\nabla u\mu)\mu^{-1} + u)\phi\mu dm.$$

We identify ψ as $\nabla \cdot (\nabla u\mu)\mu^{-1} + u$ and the dual pair will be the L^2_μ inner product.

Remark 7. *With the weight function μ so chosen as (2.2), we observe that if $\phi \in H^1_\mu$, then $\phi \in W^{1,1}$ since*

$$\int_B (|\phi| + |\nabla\phi|)dm \le C\left(\int_B (|\phi|^2 + |\nabla\phi|^2)\mu dm\right)^{1/2}\left(\int_B \mu^{-1}dm\right)^{1/2} < \infty.$$

From the standard trace theorem, the map

$$\begin{aligned} \mathcal{T}: \quad H^1_\mu(B) &\to L^1(\partial B) \\ \phi \quad &\mapsto \phi|_{\partial\Omega} \end{aligned}$$

is well defined. Thus, the element in $\overset{\circ}{H}{}^1_\mu$ is characterized by the zero trace, and the Dirichlet data (3.3c) makes sense.

The well-posedness of problem (3.3) is stated in the following.

Theorem 8. *Suppose that $w_0 \in L^2_\mu$, $h \in L^2_t(\overset{\circ}{H}{}^1_\mu)^*$ and $\kappa \in L^2_t$ with $Tr(\kappa) = 0$. Then the w-problem (3.3) has a unique weak solution in $\overset{\circ}{\mathcal{H}}$ such that*

$$\|w\|^2_{\mathcal{H}} \le e^{F(|\kappa|)}\left(\|w_0\|^2_{L^2_\mu} + \|h\|^2_{L^2_t(\overset{\circ}{H}{}^1_\mu)^*}\right) \tag{3.9}$$

with F defined in (2.4).

This result when $b > 2$ and $q = 0$ was proved in [36]. For general case we proceed in several steps.

An embedding theorem. We define

$$\mu^* = \begin{cases} \rho^{b/2-2}, & 0 < b < 2, \\ \rho^{-1}, & b = 2, \\ \rho^{-b/2}, & b > 2. \end{cases} \tag{3.10}$$

We call μ^* as the conjugate of μ due to the Sobolev inequalities in the following lemma.

Lemma 9. *If $\phi \in \overset{\circ}{H}{}^1_\mu$, then*

$$\int |\phi|^2\mu^* dm \le C\int (|\phi|^2 + |\nabla\phi|^2)\mu dm. \tag{3.11}$$

Also, if $\phi \in H^1_{\rho^\theta}$ for $\theta \le 1$, then for any $\delta > 0$

$$\int |\phi|^2\rho^{-1+\delta}dm \le C\int (|\phi|^2 + |\nabla\phi|^2)\rho^\theta dm. \tag{3.12}$$

Proof. We refer to [25] for a proof of (3.11) when $b \neq 2$, as well as (3.12). Here, we prove only the case $b = 2$.

First for $C = \max_{1\le\rho\le 2}[\rho\mu]^{-1}$ we have

$$\int_B |\phi|^2/\rho dm \le C\int_{1\le\rho\le 2} |\phi|^2\mu dm + \int_{0\le\rho\le 1}|\phi|^2/\rho dm$$
$$\le C\int_B |\phi|^2\mu dm + \int_0^1 \frac{G^2}{\rho}d\rho,$$

where we have used the spherical coordinate representation with $\rho = 2 - r^2$ and

$$G^2(\rho) = -\int_{|\xi|=1} |\phi(r\xi)|^2 r^{N-1}dS_\xi \cdot \left(\frac{d\rho}{dr}\right)^{-1} = \frac{1}{2}\int_{|\xi|=1}|\phi(r\xi)|^2 r^{N-2}dS_\xi. \tag{3.13}$$

Note that from $\phi \in \overset{\circ}{H}{}^1_\mu$ one can verify that $G(0) = 0$. It is known (see [26])) that

$$\int_0^1\left(\int_0^x g(t)dt\right)^2 \frac{1}{x}dx \le C\int_0^1 g^2(x)x|\ln x|^2dx.$$

Thus,

$$\int_0^1 \frac{G^2}{\rho}d\rho \le C\int_0^1 (G_\rho)^2\rho|\ln\rho|^2 d\rho \le C\int_0^1\frac{G_r^2}{r^2}\mu d\rho \le C\int_0^1 (G_r)^2\mu d\rho, \tag{3.14}$$

where we have used the fact that $\rho|\ln\rho|^2 \le \mu = \rho\ln^2(e/\rho)$. Differentiation of (3.13) in term of r leads to

$$2GG_r = \int_{|\xi|=1}\phi\nabla\phi\cdot\xi r^{N-2}dS_\xi + \frac{N-2}{2}\int_{|\xi|=1}|\phi(r\xi)|^2 r^{N-3}dS_\xi.$$

Squaring both sides and using the Cauchy-Schwartz inequality we obtain

$$4G^2(G_r)^2 \le 2\int_{|\xi|=1}\phi^2 r^{N-2}dS_\xi\int_{|\xi|=1}|\nabla\phi|^2 r^{N-2}dS_\xi + \frac{(N-2)^2}{2}\left(\int_{|\xi|=1}\phi^2 r^{N-2}dS_\xi\right)^2,$$

where we have used the fact $r \ge 1$. Hence

$$(G_r)^2 \le \int_{|\xi|=1}|\nabla\phi(r\xi)|^2 r^{N-2}dS_\xi + \frac{(N-2)^2}{2}G^2,$$

which inserted into (3.14) ensures that the term $\int_0^1 \frac{G^2}{\rho}d\rho$ is also bounded by $C\|\phi\|^2_{H^1_\mu}$. The proof is now complete. □

Energy estimates. We return now to the bilinear operator $\mathcal{B}$.

Lemma 10 (Energy estimates)**.** *For any $t > 0$, there exists a constant C which is dependent on N, b such that*

(1) *for $w(t,\cdot) \in \overset{\circ}{H}{}^1_\mu$*

$$\frac{1}{4}\int |\nabla w|^2\mu dm \le \mathcal{B}[w,w;t] + C(1+|\kappa|^2)\int w^2\mu dm; \tag{3.15}$$

(2) *for $\psi(t,\cdot) \in H^1_\mu$ and $\phi \in \overset{\circ}{H}{}^1_\mu$,*

$$|\mathcal{B}[\psi,\phi;t]| \le C(1+|\kappa|)\|\psi\|_{H^1_\mu}\|\phi\|_{H^1_\mu}. \tag{3.16}$$

Proof. From (3.7) it follows

$$\frac{1}{2}\int \nabla w\cdot\nabla\phi\mu dm \;=\; \mathcal{B}[w,\phi;t]+\int \kappa m\cdot\nabla\phi w\mu dm+\int Kw\phi dm, \tag{3.17}$$

where K is given in (3.6).

Case 1. If $0<b<2$, then $K=0$; hence

$$\begin{aligned}\frac{1}{2}\int|\nabla w|^2\mu dm &= \mathcal{B}[w,w;t]+\int\kappa m\cdot\nabla ww\mu dm\\ &\le \mathcal{B}[w,w;t]+\frac{1}{4}\int|\nabla w|^2\mu dm+b|\kappa|^2\int w^2\mu dm\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}|\mathcal{B}[\psi,\phi;t]| &\le \frac{1}{2}\int|\nabla\psi||\nabla\phi|\mu dm+\sqrt{b}|\kappa|\int|\psi||\nabla\phi|\mu dm\\ &\le C(1+|\kappa|)\|\psi\|_{H^1_\mu}\|\nabla\phi\|_{L^2_\mu}.\end{aligned}$$

Case 2. For $b\ge 2$, it suffices to estimate the K-related term. If $b=2$, we have

$$K=(N+2\kappa m\cdot m)\ln\frac{e}{\rho}\le(N+2b|\kappa|)\sqrt{\mu\mu^*}.$$

If $b>2$, we have

$$\begin{aligned}K&=\left(\frac{b}{2}-1\right)\rho^{1-b/2}(N+2\kappa m\cdot m)\\ &\le\left(\frac{b}{2}-1\right)(N+2b|\kappa|)\sqrt{\mu\mu^*}.\end{aligned}$$

Hence for $b\ge 2$ we have

$$\begin{aligned}\int Kw^2dm&\le C(1+|\kappa|)\int w^2\sqrt{\mu\mu^*}dm\\ &\le\varepsilon\int w^2\mu^*dm+C_\varepsilon(1+|\kappa|^2)\int w^2\mu dm.\end{aligned}$$

This when added upon right side of (3.18) using (3.11) with some small ε leads to (3.15). Using (3.11) again we have

$$\left|\int K\psi\phi dm\right|\le C(1+|\kappa|)\int|\psi||\phi|\sqrt{\mu\mu^*}dm\le C(1+|\kappa|)\|\psi\|_{H^1_\mu}\|\phi\|_{H^1_\mu},$$

which when combined with the above estimate for $b<2$ gives (3.16). □

A priori estimate.

Lemma 11 (A priori estimates). *Let w be a weak solution to (3.3). Then*

$$\sup_t\|w(t,\cdot)\|^2_{L^2_\mu}+\frac{1}{2}\|w\|^2_{L^2_tH^1_\mu}\le e^{F(|\kappa|)}\left(\|w_0\|^2_{L^2_\mu}+\|h\|^2_{L^2_t(\mathring{H}^1_\mu)^*}\right)\tag{3.19}$$

with F defined in (2.4), and furthermore

$$\|w\|^2_{\mathcal{H}}\le e^{F(|\kappa|)}\left(\|w_0\|^2_{L^2_\mu}+\|h\|^2_{L^2_t((\mathring{H}^1_\mu)^*)}\right).\tag{3.20}$$

Proof. From the weak solution definition in (3.8) we have for any $\phi \in \overset{\circ}{H}{}^1_\mu$

$$(\partial_t w, \phi)_{\overset{\circ}{H}{}^1_\mu} + \mathcal{B}[w, \phi; t] = (h, \phi)_{\overset{\circ}{H}{}^1_\mu}. \tag{3.21}$$

By (3.16), $(\partial_t w, \phi)_{\overset{\circ}{H}{}^1_\mu}$ is bounded by

$$\|h\|_{(\overset{\circ}{H}{}^1_\mu)^*}\|\phi\|_{H^1_\mu} + C(1+|\kappa|)\|w\|_{H^1_\mu}\|\phi\|_{H^1_\mu}.$$

Hence

$$\|\partial_t w\|_{(\overset{\circ}{H}{}^1_\mu)^*} \leq \|h\|_{(\overset{\circ}{H}{}^1_\mu)^*} + C(1+|\kappa|)\|w\|_{H^1_\mu}. \tag{3.22}$$

Next we set $\phi = w$ in (3.21) and use (3.15) to have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2_\mu} + \frac{1}{4}\int |\nabla w|^2 \mu dm &\leq \|h\|_{(\overset{\circ}{H}{}^1_\mu)^*}\|w\|_{H^1_\mu} + C(1+|\kappa|^2)\|w\|^2_{L^2_\mu} \\ &\leq 2\|h\|^2_{(\overset{\circ}{H}{}^1_\mu)^*} + \frac{1}{8}\|w\|^2_{H^1_\mu} + C(1+|\kappa|^2)\|w\|^2_{L^2_\mu}. \end{aligned}$$

Hence

$$\frac{d}{dt}\|w\|^2_{L^2_\mu} + \frac{1}{4}\|w\|^2_{H^1_\mu} \leq C(1+|\kappa|^2)\|w\|^2_{L^2_\mu} + 4\|h\|^2_{(\overset{\circ}{H}{}^1_\mu)^*}, \tag{3.23}$$

and therefore by Gronwall's inequality,

$$\sup_t \|w(t,\cdot)\|^2_{L^2_\mu} + \frac{1}{2}\|w\|^2_{L^2_t H^1_\mu} \leq e^{C(T+\int_0^T |\kappa|^2 dt)}\left(\|w_0\|^2_{L^2_\mu} + \|h\|^2_{L^2_t(\overset{\circ}{H}{}^1_\mu)^*}\right),$$

which together with (3.22) yields (3.20). □

Proof of Theorem 8. We construct a weak solution to (3.3) using the Galerkin approximation. Let $\{\phi_i\}$ be a basis of $\overset{\circ}{H}{}^1_\mu$ and L^2_μ. Then an approximate solution w_l in a finite dimensional space is defined as $w_l = \sum_{i=1}^{l} d^l_i(t)\phi_i$. Here $d^l_i(t)$ is a unique solution to a system of linear differential equations,

$$\begin{aligned} (\partial_t w_l, \phi_j)_{\overset{\circ}{H}{}^1_\mu} + \mathcal{B}[w_l, \phi_j; t] &= (h, \phi_j)_{\overset{\circ}{H}{}^1_\mu}, \\ d^l_i(0) &= ((\phi_i, \phi_j)^{-1}_{L^2_\mu}(w_0, \vec{\phi})_{L^2_\mu})_i, \end{aligned}$$

where $\vec{\phi} = (\phi_1, \cdots, \phi_l)^\top$. Using the same argument as that in the proof of Lemma 11, we obtain

$$\|w_l\|^2_{L^2_t H^1_\mu} + \|\partial_t w_l\|^2_{L^2_t(\overset{\circ}{H}{}^1_\mu)^*} \leq e^{F(|\kappa|)}\left(\|w_0\|^2_{L^2_\mu} + \|h\|^2_{L^2_t(\overset{\circ}{H}{}^1_\mu)^*}\right).$$

Extracting a subsequence and passing to the limit give a weak solution w in $\overset{\circ}{\mathcal{H}}$. The uniqueness follows from the a priori estimate (3.20). □

To return to the Fokker-Planck problem (3.1) we will also need the following Lemma.

Lemma 12. *Let $h = -\partial_t q - \mu^{-1}L[q]$. If $q \in C_t^1 H_\mu^1$ and $\kappa \in L_t^2$ with $Tr(\kappa) = 0$, then*

$$\|h\|^2_{L_t^2(\mathring{H}_\mu^1)^*} \leq C \int_0^T (1 + |\kappa|^2)\|q(t)\|^2 d\tau. \tag{3.24}$$

Proof. For $q \in C_t^1 H_\mu^1$, it is obvious that $\partial_t q \in L_t^2(\mathring{H}_\mu^1)^*$ since $H_\mu^1 \subset (H_\mu^1)^* \subset (\mathring{H}_\mu^1)^*$. In order to show $\mu^{-1}L[q] \in L_t^2(\mathring{H}_\mu^1)^*$, we use integration by parts and (3.16) to get

$$\left| \int \mu^{-1} L[q]\phi\mu dm \right| = |\mathcal{B}[q, \phi; t]| \leq C(1 + |\kappa|)\|q(t, \cdot)\|_{H_\mu^1}\|\phi\|_{H_\mu^1}, \ \forall \phi \in C_c^\infty.$$

Since C_c^∞ is a dense subset of $\mathring{H}_\mu^1$, for any $\phi \in \mathring{H}_\mu^1$ with $\|\phi\|_{H_\mu^1} = 1$, we have

$$|(\mu^{-1}L[q], \phi)_{\mathring{H}_\mu^1}| \leq C(1 + |\kappa|)\|q(t, \cdot)\|_{H_\mu^1}. \tag{3.25}$$

Taking the L^2 norm in t leads to the desired estimate. □

Theorem 8 and Lemma 12 lead to the following result for problem (3.1) with a general Dirichlet boundary condition.

Theorem 13. *Suppose that $f_0\nu^{-1} \in L_\mu^2$, $q \in C_t^1 H_\mu^1$ and $\kappa \in L_t^2$ with $Tr(\kappa) = 0$. Then for any $T > 0$ the Fokker-Planck problem (3.1) has a unique solution f such that*

$$f = \nu(w + q) \quad \text{with } w \in \mathcal{H} \text{ for } 0 < t \leq T. \tag{3.26}$$

Moreover, for F defined in (2.4),

$$\sup_t \|w(t, \cdot)\|^2_{L_\mu^2} + \frac{1}{2}\|w\|^2_{L_t^2 H_\mu^1} \leq e^{F(|\kappa|)} \left(\|w_0\|^2_{L_\mu^2} + \int_0^T (1 + |\kappa(t)|^2)\|q(t)\|^2 dt \right). \tag{3.27}$$

Proof. The estimate (3.27) follows from (3.19) and the estimate in Lemma 12, with Fe^F replaced by e^F.

We now prove the uniqueness of f in terms of $q|_{\partial B}$. Let $f_i (i = 1, 2)$ be two solutions with q_i such that $q_1|_{\partial B} = q_2|_{\partial B}$ and initial data f_0. Set $w = (f_2 - f_1)\nu^{-1}$, then w solves w-problem (3.3) with $w_0 \equiv h \equiv 0$. Hence $w \equiv 0$, leading to $f_1 = f_2$. □

Remark 14. *As mentioned in Section 2 that $\mathring{H}_\mu^1 = H_\mu^1$ if $b \geq 6$,i.e., the trace of $q \in H_\mu^1$ vanishes if $b \geq 6$. Thus, boundary condition* (3.1c) *is nothing but a zero Dirichlet boundary condition. In Section 6, we show the well-posedness with a nonzero Dirichlet boundary condition for $b \geq 6$ using yet a different transformation.*

Remark 15. *The condition $Tr(\kappa) = 0$ comes from the divergence free velocity field of the Navier-Stokes equations. Many of the above arguments, however, do not use this condition explicitly.*

3.2. **Probability density function.** So far we have discussed well-posedness of the initial-boundary value problem (3.1) for $b > 0$ and any given q. We now turn to the question of which q corresponds to the probability density, i.e., non-negative solution with constant mass for all time.

Proposition 16. *Let $f(t, m)$ be the solution to problem* (3.1) *obtained in Theorem 13. If $f_0 \geq 0$ and $q(t, m)|_{\partial B} \geq 0$ almost everywhere, then f remains nonnegative for $t > 0$.*

Proof. We adapt an idea from [7]. Let $f^{\pm}$ be the positive and negative parts of the solution f such that $f = f^+ - f^-$. Obviously, $w^{\pm} := f^{\pm}\nu^{-1} \in H^1_\mu$ and $q|_{\partial B} \geq 0$. This implies that the trace of w^- at the boundary vanishes, so

$$w^- \in \overset{\circ}{H}{}^1_\mu.$$

From the equation

$$\partial_t w\mu + L[w] = 0,$$

which is transformed from (3.1a) it follows that

$$(\partial_t w, w^-)_{\overset{\circ}{H}{}^1_\mu} + B[w, w^-; t] = 0.$$

Since $(\partial_t w^+, w^-)_{\overset{\circ}{H}{}^1_\mu}$ and $\int L[w^+]w^- dm$ vanish, hence

$$\frac{1}{2}\frac{d}{dt}\left(\int |w^-|^2 \mu dm\right) + \mathcal{B}[w^-, w^-; t] = 0.$$

The coercivity of $\mathcal{B}$, (3.15), gives

$$\frac{1}{2}\frac{d}{dt}\left(\int |w^-|^2 \mu dm\right) + \frac{1}{4}\int |\nabla w^-|^2 \mu dm \leq C(1+|\kappa|^2)\int |w^-|^2 \mu dm.$$

Hence

$$\sup_t \|w^-(t,\cdot)\|^2_{L^2_\mu} \leq \|w_0^-\|^2_{L^2_\mu} e^{F(|\kappa|)}$$

for $T > 0$. Since $w_0^- = 0$, $\|w^-(t,\cdot)\|^2_{L^2_\mu} = 0$ for all $0 \leq t \leq T$. □

Proposition 17. *Let f be a solution to the Fokker-Planck problem* (3.1) *obtained in Theorem 13. Suppose $b \geq 2$ and $q(t,m)|_{\partial B} \geq 0$. If $q|_{\partial B} = 0$ for all $t \in [0,T]$, then*

$$\int f(t,\cdot)dm = \int f_0 dm, \quad t \in [0,T],$$

and vice versa.

Proof. It suffices to prove the claim for smooth enough f since the general case can be treated by an approximation as in [36]. We rewrite (3.1a) as

$$\partial_t f = -\nabla\cdot(\kappa m f) + \nabla\cdot\left(\rho^{b/2}\nabla\frac{f}{\rho^{b/2}}\right).$$

First, we take a test function $\phi_\varepsilon(m) = \phi_\varepsilon(|m|) \in C_c^\infty(\mathbb{R}^N)$ converging to χ_B as $\varepsilon \to 0$ such that

$$\phi_\varepsilon(|m|) = \begin{cases} 1, & |m| \leq \sqrt{b}-\varepsilon \\ 0, & |m| \geq \sqrt{b}-\varepsilon/2 \end{cases}, \quad |\nabla\phi_\varepsilon| \leq C\frac{1}{\varepsilon}$$

and for any smooth g

$$\int_{\sqrt{b}-\epsilon}^{\sqrt{b}-\epsilon/2} g(r)\phi'_\varepsilon(r)dr \to -g(\sqrt{b}) \quad \text{as } \varepsilon \to 0, \tag{3.28}$$

where $\phi'_\varepsilon(r) = \nabla\phi_\varepsilon \cdot \dfrac{m}{|m|}$.

One can construct such a ϕ_ε by mollifiers, for example

$$\phi_\varepsilon(m) = \int_{B_{\sqrt{b}-3\varepsilon/4}} \eta_{\varepsilon/4}(m-m')dm'$$

where

$$\eta_\varepsilon(m) = \frac{1}{\varepsilon^N}\eta(m/\varepsilon), \quad \eta(m) = \begin{cases} Ce^{-\frac{1}{1-|m|^2}}, & |m|<1 \\ 0, & |m|\geq 1 \end{cases},$$

and C is a normalizing constant.

Since $\nabla\phi_\varepsilon$ is supported in $B^\varepsilon := B_{\sqrt{b}-\varepsilon/2} \setminus B_{\sqrt{b}-\varepsilon}$, hence

$$\frac{d}{dt}\int_B f\phi_\varepsilon dm = \int_{B^\varepsilon} f\kappa m\cdot\nabla\phi_\varepsilon dm - \int_{B^\varepsilon} \rho^{b/2}\nabla\left(\frac{f}{\rho^{b/2}}\right)\cdot\nabla\phi_\varepsilon dm. \tag{3.29}$$

By $w = f\nu^{-1}$, the right hand side reduces to

$$\int_{B^\varepsilon}(w\kappa m - \nabla w)\cdot\nabla\phi_\varepsilon \nu dm - \int_{B^\varepsilon} w\rho^{b/2}\nabla\phi_\varepsilon\cdot\nabla(\nu\rho^{-b/2})dm. \tag{3.30}$$

The first term converges to 0. Indeed,

$$\left|\int_{B^\varepsilon}(w\kappa m-\nabla w)\cdot\nabla\phi_\varepsilon\nu dm\right| \leq \left(\int_{B^\varepsilon}|w\kappa m-\nabla w|^2\mu dm\right)^{1/2}\left(\int_{B^\varepsilon}|\nabla\phi_\varepsilon|^2\frac{\nu^2}{\mu}dm\right)^{1/2}.$$

Since $\nu^2/\mu = \rho^{b/2}$ for $b\geq 2$, by mean value theorem there exists $r\in(\sqrt{b}-\varepsilon, \sqrt{b}-\varepsilon/2)$ such that

$$\int_{B^\varepsilon}|\nabla\phi_\varepsilon|^2\frac{\nu^2}{\mu}dm = \frac{\varepsilon}{2}\int_{\partial B_r}|\nabla\phi_\varepsilon|^2\rho^{b/2}dS \leq C\varepsilon^{b/2-1},$$

which is uniformly bounded for $b\geq 2$. Using $w\in H^1_\mu$, we obtain $\int_{B^\varepsilon}|w\kappa m - \nabla w|^2\mu dm \to 0$ as $\varepsilon\to 0$. Hence the first term in (3.30) converges to 0.

On the other hand, for $C_0 = \begin{cases} -2, & b=2 \\ 2-b, & b>2 \end{cases}$

$$\begin{aligned} -\int_{B^\varepsilon} w\rho^{b/2}\nabla\phi_\varepsilon\cdot\nabla(\nu\rho^{-b/2})dm &= C_0\int_{B^\varepsilon} w\nabla\phi_\varepsilon\cdot m dm \\ &= C_0\int_{\sqrt{b}-\epsilon}^{\sqrt{b}-\epsilon/2}\int_{\partial B_r} wr\phi_\varepsilon'(r)dSdr \\ &= C_0\int_{\sqrt{b}-\epsilon}^{\sqrt{b}-\epsilon/2}\left(r\int_{\partial B_r} wdS\right)\phi_\varepsilon'(r)dr. \end{aligned}$$

Due to (3.28) this converges to

$$-C_0\sqrt{b}\int_{\partial B} wdS = -C_0\sqrt{b}\int_{\partial B} qdS.$$

Since $C_0\neq 0$, this shows that $\frac{d}{dt}\int_B fdm = 0$ if and only if $\int_{\partial B} qdS = 0$, or $q|_{\partial B} = 0$. $\square$

Remark 18. *In Proposition 17, the assumption $b \geq 2$ is sharp. In the case $b < 2$, we need to consider nontrivial $q \neq 0$ since the equilibrium profile $f_{eq} = \rho^{b/2}$ satisfies*

$$q\big|_{\partial B} = \rho^{b/2}\nu^{-1}|_{\partial B} = 1.$$

This requirement is also consistent with [35], in which it was shown that when $b < 2$, $f\nu^{-1}|_{\partial B} = q|_{\partial B}$ is necessarily prescribed and each solution depends on the choice of q. It would be interesting to figure out a particular q for which the corresponding solution when $b < 2$ is a probability density.

4. The Fokker-Planck equation

In this section, we show the well-posedness of the FPE (2.9) including x variable. The result is stated as follows.

Theorem 19. *Suppose that for $b > 0$ and any integer $s > N/2 + 1$, $\nabla \cdot v = 0$ and*

$$v \in C_t H_x^s \cap L_t^2 H_x^{s+1}, \quad f_0\nu^{-1} \in H_x^s L_\mu^2, \quad q \in C_t^1 H_x^{s+1} H_\mu^1,\ 0 < t < T \tag{4.1}$$

for any $T > 0$. Then (2.9) has a unique solution $f = \nu(w + q)$ satisfying

$$\sup_t |w|_{0,s}^2 + \frac{1}{2}\int_0^T |\nabla_m w|_{0,s}^2 dt \leq e^{F(|v|_{s+1})}\left(|w_0|_{0,s}^2 + \|q\|_{1,1,s+1}^2\right), \tag{4.2}$$

where F was defined in (2.4).

The proof of Theorem 19 consists of two parts: first we show the existence of the solution f to problem (2.9) by using the flow map, followed by proving regularity in x inductively such that $w \in C_t H_x^s L_\mu^2 \cap L_t^2 H_x^s H_\mu^1$ with v, f_0 and q given in (4.1). In the second step, we derive estimate (4.2) directly from (2.9) to control f in terms of the given data. The uniqueness can be obtained from the estimation (4.2) as performed in the proof of Theorem 13.

First, we state a technical lemma.

Lemma 20. *Suppose that $\psi \in H_\mu^1$ and $\phi \in \overset{\circ}{H}{}_\mu^1$. Then for the trace map $\mathcal{T} : W^{1,1}(B) \to L^1(\partial B)$*

$$\mathcal{T}(\psi\phi\mu) = 0. \tag{4.3}$$

Proof. Since C_c^∞ is a dense subset of $\overset{\circ}{H}{}_\mu^1$, it suffices to show that for a fixed $\psi \in H_\mu^1$ and any $\phi \in C_c^\infty$

$$\|\psi\phi\mu\|_{W^{1,1}} \leq C\|\phi\|_{H_\mu^1}. \tag{4.4}$$

Then, the standard trace theorem asserts that $\mathcal{T}(\psi\phi\mu)$ is well-defined in $L^1(\partial B)$ and it vanishes, also $\mathcal{T}$ is a continuous map with respect to ϕ, we can thus conclude (4.3) for any $\phi \in \overset{\circ}{H}{}_\mu^1$ by passing to the limit of sequence $\phi_n \in C_c^\infty$ such that $\phi_n \to \phi$.

(4.4) is indeed the case. It is obvious that $\psi\phi\mu, \nabla_m\psi\phi\mu$ and $\psi\nabla_m\phi\mu$ are integrable. For $b \neq 2$, $|\nabla_m\mu| \leq C\sqrt{\mu\mu^*}$ and (3.11) yield

$$\int |\psi\phi\nabla_m\mu dm| \leq C\|\psi\|_{L_\mu^2}\|\phi\|_{H_\mu^1}.$$

For $b = 2$,

$$|\nabla_m \mu| \le C(\ln^2 \frac{e}{\rho} + \ln \frac{e}{\rho}) \le C(\ln^2 \frac{e}{\rho} + \sqrt{\mu\mu^*}).$$

Using (3.12) and $\psi \in H^1_\mu$, we obtain $\psi \in L^2_{-1+\delta}$ for any $\delta > 0$. Hence

$$\left| \int \psi\phi \ln^2 \frac{e}{\rho} dm \right| \le C \left(\sqrt{\int \psi^2 \rho^{-1+\delta} dm} \sqrt{\int \phi^2 \rho^{1-\delta} \ln^4 (\frac{e}{\rho}) dm} \right).$$

It follows that for any $b > 0$

$$\int |\psi\phi\mu| + |\nabla_m(\psi\phi\mu)| dm < C\|\psi\|_{H^1_\mu} \|\phi\|_{H^1_\mu}$$

as we desired. □

The main ingredient for the proof of Theorem 19 is to use the calculus inequalities in the Sobolev spaces, see Appendix 3.5 of [40]: for any positive integer $r > 0$ and $u, v \in L^\infty_x \cap H^r_x$,

$$\sum_{|\gamma| \le r} \|\partial^\gamma(uv) - u\partial^\gamma v\|_{L^2} \le C \left(\|\nabla u\|_{L^\infty} \|v\|_{H^{r-1}} + \|u\|_{H^r} \|v\|_{L^\infty} \right), \tag{4.5}$$

$$\|uv\|_{H^r} \le C(\|u\|_{L^\infty} \|v\|_{H^r} + \|u\|_{H^r} \|v\|_{L^\infty}). \tag{4.6}$$

Note that (4.5) remains valid when ∂^γ on the left hand is replaced by the corresponding difference operator.

Proof of Theorem 19.
Step1 (well-posedness) Let a particle path be defined by

$$\partial_t x(t, y) = v(t, x(t, y)), \quad x(0, y) = y,$$

along which the distribution function $\tilde{f}(t, y, m) := f(t, x(t, y), m)$ solves

$$\partial_t \tilde{f} + \mathcal{L}[\tilde{f}] = 0, \tag{4.7a}$$

$$\tilde{f}(0, y, m) = f_0(y, m), \tag{4.7b}$$

$$\tilde{f}(t, y, m)\nu^{-1}|_{\partial B} = \tilde{q}(t, y, m)|_{\partial B}. \tag{4.7c}$$

Here $\mathcal{L}$ is defined in (3.2) with κ replaced by $\tilde{\kappa}(t, y) = \nabla v(t, x(t, y))$, and $\tilde{q}(t, y, m) := q(t, x(t, y), m)$.

In order to show existence of a solution to (2.9) under the conditions $v \in C_t H^s_x \cap L^2_t H^{s+1}_x$ and $\nabla \cdot v = 0$, it suffices to show that (4.7) has a solution $\tilde{f} = \nu(\tilde{w} + \tilde{q})$ such that

$$\tilde{w} := w(t, x(t, y), m) \in C_t H^s_y L^2_\mu \cap L^2_t H^s_y H^1_\mu,$$

assuming that

$$\tilde{\kappa} \in L^2_t H^s_y, \quad w_0 \in H^s_y L^2_\mu, \quad \tilde{q} \in C^1_t H^s_y H^1_\mu. \tag{4.8}$$

These follow from (4.1) since $|\tilde{\kappa}(t)|_s \le C|v(t)|_{s+1}$ for $t > 0$, $w_0(x, m) = f_0\nu^{-1} - \tilde{q}(t = 0) = w_0(y, m)$, and $\|\tilde{q}\|_{1,1,s} \le C\|q\|_{1,1,s+1}$, for which we have used $\partial_t \tilde{q} = \partial_t q + v \cdot \nabla q$.

Using Theorem 13 for each y, there exists a unique solution $\tilde{f}$ such that

$$\tilde{f} = \nu(\tilde{w} + \tilde{q})$$

with $\tilde{w}$ satisfying (3.27), i.e.,

$$\sup_t \|\tilde{w}(t,y,\cdot)\|^2_{L^2_\mu} + \frac{1}{2}\|\tilde{w}(\cdot,y,\cdot)\|^2_{L^2_t H^1_\mu} \leq e^{F(|\tilde{\kappa}(\cdot,y)|)} \Big(\|w_0(y,\cdot)\|^2_{L^2_\mu} + \int_0^T (1+|\tilde{\kappa}(\cdot,y)|^2)\|\tilde{q}(t,y,\cdot)\|^2 dt \Big). \tag{4.9}$$

Integration of (4.9) with respect to y, upon exchanging the order of integration in y and m and using the Sobolev inequality, $\sup_y |\tilde{\kappa}| \leq C|\tilde{\kappa}|_{s-1}$, gives

$$\sup_t |\tilde{w}|^2_{0,0} + \frac{1}{2}\int_0^T |\tilde{w}|^2_{1,0} dt \leq e^{F(|\tilde{\kappa}|_{s-1})} \left(|w_0|^2_{0,0} + \|\tilde{q}\|^2_{1,1,0} \right). \tag{4.10}$$

Hence $\tilde{w} \in C_t L^2_y L^2_\mu \cap L^2_t L^2_y H^1_\mu$. On the other hand, the right hand side of (4.9) is uniformly bounded in y, taking $\sup_y$ of (4.9) gives

$$\sup_{t,y} \|\tilde{w}(t,y,\cdot)\|^2_{L^2_\mu} \leq e^{F(|\tilde{\kappa}|_{s-1})}(|w_0|^2_{0,s-1} + \|\tilde{q}\|^2_{1,1,s-1}). \tag{4.11}$$

We now use an induction argument to prove that $\tilde{w} \in C_t H^r_y L^2_\mu \cap L^2_t H^r_y H^1_\mu$ for $0 \leq r \leq s$, and

$$\sup_t |\tilde{w}|^2_{0,r} + \frac{1}{2}\int_0^T |\tilde{w}|^2_{1,r} dt \leq e^{F(|\tilde{\kappa}|_s)}(|w_0|^2_{0,s} + \|\tilde{q}\|^2_{1,1,s}). \tag{4.12}$$

The case $r = 0$ has been proved as shown in (4.10). Suppose (4.12) holds for $r = k$, we only need to show (4.12) for $r = k+1 \leq s$.

To prove regularity of $\tilde{f}$ in the y variable, we use difference quotients. Define the difference operator in the y variable as

$$\delta^\gamma := \delta_1^{\gamma_1} \cdots \delta_N^{\gamma_N}, \quad \delta_i u(y) := \frac{1}{\eta}[u(y+\eta e_i) - u(y)].$$

Apply δ^γ to (4.7) with $|\gamma| \leq s$, then

$$\partial_t \delta^\gamma \tilde{f} + \mathcal{L}[\delta^\gamma \tilde{f}] = \nabla_m \cdot J, \tag{4.13a}$$

$$\delta^\gamma \tilde{f}(0,y,m) = \delta^\gamma f_0(y,m), \tag{4.13b}$$

$$\delta^\gamma \tilde{f}(t,y,m)\nu^{-1}|_{\partial B} = \delta^\gamma \tilde{q}(t,y,m)|_{\partial B}, \tag{4.13c}$$

where

$$J = \tilde{\kappa} m \delta^\gamma \tilde{f} - \delta^\gamma(\tilde{\kappa} m \tilde{f}). \tag{4.14}$$

This when transformed into the w-problem of form (3.3) involves the following non-homogeneous term

$$h = -\partial_t \delta^\gamma \tilde{q} - \mu^{-1} L[\delta^\gamma \tilde{q}] + \nabla_m \cdot J\nu^{-1}. \tag{4.15}$$

Using Theorem 13 again for each y, $\delta^\gamma \tilde{f}$ is the unique solution to (4.13) as long as $h \in L^2_t(\overset{\circ}{H}{}^1_\mu)^*$. Moreover,

$$\delta^\gamma \tilde{f} = \nu(\delta^\gamma \tilde{w} + \delta^\gamma \tilde{q}),$$

where $\delta^\gamma \tilde{w}$, using (3.19), satisfies

$$\sup_t \|\delta^\gamma \tilde{w}(t,y,\cdot)\|^2_{L^2_\mu} + \frac{1}{2}\|\delta^\gamma \tilde{w}(\cdot,y,\cdot)\|^2_{L^2_t H^1_\mu} \leq e^{F(|\tilde{\kappa}(\cdot,y)|)} \left(\|\delta^\gamma w_0\|^2_{L^2_\mu} + \|h\|^2_{L^2_t(\overset{\circ}{H}{}^1_\mu)^*} \right).$$

Integration in y gives

$$\sup_t |\delta^\gamma \tilde{w}|^2_{0,0} + \frac{1}{2}\int_0^T |\delta^\gamma \tilde{w}|^2_{1,0} dt \le e^{F(\sup_y |\tilde{\kappa}(\cdot,y)|)} \left(|\delta^\gamma w_0|^2_{0,0} + \|h\|^2_{L^2_t L^2_y (\overset{\circ}{H}{}^1_\mu)^*} \right)$$

$$\le e^{F(|\tilde{\kappa}|_{s-1})} \left(|w_0|^2_{0,s} + \|h\|^2_{L^2_t L^2_y (\overset{\circ}{H}{}^1_\mu)^*} \right). \tag{4.16}$$

We now turn to bound the last term in the above inequality. For any $\phi \in \overset{\circ}{H}{}^1_\mu$ and J defined in (4.14), Lemma 20 allows the use of integration by parts. Hence,

$$\begin{aligned} \left| \int \nabla_m \cdot J\nu^{-1} \phi \mu dm \right| &\le \left(\int |J\nu^{-1}| |\nu \nabla_m \frac{\mu}{\nu}| |\phi| dm + \int |J\nu^{-1}| |\nabla_m \phi| \mu dm \right) \\ &\le C \|J\nu^{-1}\|_{L^2_\mu} (\|\phi\|_{L^2_{\mu^*}} + \|\nabla_m \phi\|_{L^2_\mu}) \\ &\le C \|J\nu^{-1}\|_{L^2_\mu} \|\phi\|_{H^1_\mu}. \end{aligned}$$

Here we have used $|\nu \nabla_m \frac{\mu}{\nu}| \le C\sqrt{\mu^* \mu}$ and the embedding theorem (3.11). This together with Lemma 12 and (4.15) yields

(4.17)

$$\|h\|^2_{L^2_t L^2_y (\overset{\circ}{H}{}^1_\mu)^*} \le C \int_0^T (1 + \sup_y |\tilde{\kappa}(t,y)|^2) \int \|\delta^\gamma \tilde{q}(t,y,\cdot)\|^2 dy dt + C \int_0^T |J\nu^{-1}|^2_{0,0} dt.$$

For $|\gamma| \le s$, the first term on the right side is bounded by

$$F(|\tilde{\kappa}|_{s-1}) \|\delta^\gamma \tilde{q}\|^2_{1,1,0} \le F(|\tilde{\kappa}|_{s-1}) \|\tilde{q}\|^2_{1,1,s}. \tag{4.18}$$

To obtain (4.12) for $r = k+1 \le s$, it remains to estimate the last term in (4.17) with $|\gamma| = k+1$. In fact,

$$\begin{aligned} |J\nu^{-1}|^2_{0,0} &= |(\delta^\gamma (\tilde{\kappa} m \tilde{f}) - \tilde{\kappa} m \delta^\gamma \tilde{f})\nu^{-1}|^2_{0,0} \\ &\le C(\sup_y |\nabla_y \tilde{\kappa}|^2 |\tilde{f}\nu^{-1}|^2_{0,k} + |\tilde{\kappa}|^2_{k+1} \sup_y \|\tilde{f}\nu^{-1}\|^2_{L^2_\mu}) \\ &\le C|\tilde{\kappa}|^2_s (|\tilde{w}|^2_{0,k} + \sup_y \|\tilde{w}\|^2_{L^2_\mu} + \|\tilde{q}\|^2_{1,1,s}), \end{aligned}$$

where we have used (4.5) with ∂^γ replaced by δ^γ.
Using (4.12) for $r = k$ and (4.11) we have

$$\int_0^T |J\nu^{-1}|^2_{0,0} dt \le e^{F(|\tilde{\kappa}|_s)} (|w_0|^2_{0,s} + \|\tilde{q}\|^2_{1,1,s}).$$

This and (4.18) when inserted into (4.17) give a bound for $\|h\|^2_{L^2_t L^2_y (\overset{\circ}{H}{}^1_\mu)^*}$. That bound combined with (4.16) yields

$$\sup_t |\delta^\gamma \tilde{w}|^2_{0,0} + \frac{1}{2}\int_0^T |\delta^\gamma \tilde{w}|^2_{1,0} dt \le e^{F(|\tilde{\kappa}|_s)} \left(|w_0|^2_{0,s} + \|\tilde{q}\|^2_{1,1,s} \right) < \infty, \quad |\gamma| = k+1.$$

Sending $\eta \to 0$ we obtain (4.12) with $r = k+1$. Hence, (4.12) holds for any $r \le s$, and thus the solution f to (2.9) exists, and

$$\sup_t |w|^2_{0,s} + \frac{1}{2}\int_0^T |w|^2_{1,s} dt < \infty.$$

One may obtain an upper bound from (4.12) with $r = s$ using the inverse map of $x = x(t, y)$. Nevertheless, the next step gives the claimed bound in (4.2).

Step2 (a priori estimate) For a priori estimate, we consider the w-problem (2.11)

$$\mu(\partial_t + v\cdot\nabla)w + L[w] \;=\; -\mu(\partial_t + v\cdot\nabla)q - L[q]. \tag{4.19}$$

Recall that

$$L[w] = -\frac{1}{2}\nabla_m\cdot(\nabla_m w\mu) + \nabla_m\cdot(\kappa m w\mu) - Kw.$$

Take γ derivative in x-variable. Then, the left and right hand side of (4.19) will be

$$\begin{aligned}
I \;&=\; \mu(\partial_t + v\cdot\nabla)\partial^\gamma w - \frac{1}{2}\nabla_m\cdot(\nabla_m\partial^\gamma w\mu) && (4.20)\\
&+\; \mu[\partial^\gamma((v\cdot\nabla)w) - (v\cdot\nabla)\partial^\gamma w] && (4.21)\\
&+\; \nabla_m\cdot(\partial^\gamma(\kappa m w\mu)) && (4.22)\\
&-\; \partial^\gamma(Kw), && (4.23)\\
II \;&=\; -\mu\partial_t\partial^\gamma q + \frac{1}{2}\nabla_m\cdot(\nabla_m\partial^\gamma q\mu) && (4.24)\\
&-\; \mu\partial^\gamma((v\cdot\nabla)q) && (4.25)\\
&-\; \nabla_m\cdot(\partial^\gamma(\kappa m q\mu)) && (4.26)\\
&+\; \partial^\gamma(Kq). && (4.27)
\end{aligned}$$

We now estimate term by term of

$$\sum_{|\gamma|\le s}\int\!\!\int I\partial^\gamma w\,dm\,dx = \sum_{|\gamma|\le s}\int\!\!\int II\partial^\gamma w\,dm\,dx. \tag{4.28}$$

Since v is divergence free, the first two terms on the left hand side will be

$$\frac{1}{2}\frac{d}{dt}|w|_{0,s}^2 + \frac{1}{2}|\nabla_m w|_{0,s}^2.$$

Indeed, the Cauchy inequality shows that the term related to (4.21) is bounded by

$$\varepsilon|w|_{0,s}^2 + C_\varepsilon\sum_{|\gamma|\le s}\int\int|\partial^\gamma((v\cdot\nabla)w) - (v\cdot\nabla)\partial^\gamma w|^2\mu\,dm\,dx.$$

Now, we exchange the order of integration in x and m, and apply (4.5) to obtain

$$\begin{aligned}
&\varepsilon|w|_{0,s}^2 + C_\varepsilon\int\left(\|\nabla v\|_{L_x^\infty}^2\|\nabla w(\cdot,m)\|_{H_x^{s-1}}^2 + \|v\|_{H_x^s}^2\|\nabla w(\cdot,m)\|_{L_x^\infty}^2\right)\mu\,dm\\
\le\;& \varepsilon|w|_{0,s}^2 + C_\varepsilon|v|_s^2|w|_{0,s}^2,
\end{aligned}$$

where the Sobolev inequality, $|u|_0 \le C|u|_{s-1}$ for any $u\in H_x^{s-1}$, is invoked in the last inequality. Similarly, the term with (4.22) will be estimated as follows due to (4.6);

$$\begin{aligned}
&\varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon\sum_{|\gamma|\le s}\int\int|\partial^\gamma(\kappa m w)|^2\mu\,dm\,dx\\
\le\;& \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon\int\left(|\kappa|_{L_x^\infty}^2|w(\cdot,m)|_s^2 + |\kappa|_s^2|w(\cdot,m)|_{L_x^\infty}^2\right)\mu\,dm\\
\le\;& \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon|v|_{s+1}^2|w|_{0,s}^2.
\end{aligned}$$

Recall that

$$K = \begin{cases} 0, & 0 < b < 2, \\ (N + 2\kappa m \cdot m) \ln \dfrac{e}{\rho}, & b = 2, \\ (N + 2\kappa m \cdot m)(b/2 - 1)\rho^{1-b/2}, & b > 2. \end{cases}$$

Thus, we can express K as

$$K = c_1\sqrt{\mu\mu^*} + c_2\kappa m \cdot m\sqrt{\mu\mu^*} \tag{4.29}$$

for some positive constat c_i depending on N and b. We now estimate the last term on the left hand side, by using

$$\partial^\gamma(Kw)\partial^\gamma w = c_1|\partial^\gamma w|^2\sqrt{\mu\mu^*} + c_2\partial^\gamma(\kappa m \cdot mw)\partial^\gamma w\sqrt{\mu\mu^*}.$$

The Cauchy inequality and the embedding theorem (3.11) give

$$\begin{aligned} c_1 \sum_{|\gamma|\le s} \iint |\partial^\gamma w|^2\sqrt{\mu\mu^*}dmdx &= c_1 \int |w(t,\cdot,m)|_s^2\sqrt{\mu\mu^*}dm \\ &\le \epsilon \int |w(t,\cdot,m)|_s^2\mu^* dm + C_\epsilon \int |w(t,\cdot,m)|_s^2 \mu dm \\ &\le \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon|w|_{0,s}^2. \end{aligned}$$

Similarly,

$$c_2 \sum_{|\gamma|\le s} \int\int |\partial^\gamma(\kappa m \cdot mw)\partial^\gamma w|\sqrt{\mu\mu^*}dmdx \le \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon \iint |\partial^\gamma(\kappa m \cdot mw)|^2\mu dmdx.$$

The last term, using (4.6) and the Sobolev inequality for $\kappa = \nabla v$, is then bounded by

$$C_\epsilon|v|_{s+1}^2|w|_{0,s}^2.$$

Hence,

$$\left|\sum_{|\gamma|\le s} \iint \partial^\gamma(Kw)\partial^\gamma w dmdx\right| \le \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon(|v|_{s+1}^2 + 1)|w|_{0,s}^2.$$

Now we turn to the right hand side, related to (4.24)-(4.27). The estimation is similar to that for the left hand side. Except that here we have to assume higher regularity of q in x than that of w since $\int v \cdot \int \nabla\partial^\gamma q \partial^\gamma w\mu dmdx$ does not vanish as $\int v \cdot \int \nabla\partial^\gamma w\partial^\gamma w\mu dmdx$. Indeed, the first two terms, related to (4.24) are bounded by

$$\varepsilon|w|_{0,s}^2 + C_\varepsilon|\partial_t q|_{0,s}^2 + \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon|\nabla_m q|_{0,s}^2,$$

and the other terms are estimated as follows;

$$\begin{aligned}
\sum_{|\gamma|\le s}\left|\iint \partial^\gamma(v\cdot\nabla q)\partial^\gamma w\mu dmdx\right| &\le \varepsilon|w|_{0,s}^2 + C_\varepsilon |v|_s^2|q|_{0,s+1}^2,\\
\sum_{|\gamma|\le s}\left|\iint \partial^\gamma(\kappa mq)\nabla_m\partial^\gamma w\mu dmdx\right| &\le \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2|q|_{0,s}^2,\\
\sum_{|\gamma|\le s}\left|\iint \partial^\gamma(Kq)\partial^\gamma w dmdx\right| &\le \varepsilon|\nabla_m w|_{0,s}^2 + C_\varepsilon|q|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2|q|_{0,s}^2.
\end{aligned}$$

We combine all estimates for sufficiently small ε to obtain

$$\partial_t|w|_{0,s}^2 + \frac{1}{2}|\nabla_m w|_{0,s}^2 \le C(|v|_{s+1}^2+1)\left(|w|_{0,s}^2 + (|q|_{1,s+1}^2 + |\partial_t q|_{1,s+1}^2)\right). \tag{4.30}$$

We deduce that

$$|w|_{0,s}^2 + \frac{1}{2}\int_0^t |\nabla_m w|_{0,s}^2 dt \le e^{F(|v|_{s+1})}\left(|w_0|_{0,s}^2 + F(|v|_{s+1})\|q\|_{1,1,s+1}^2\right).$$

Replacing Fe^F by e^F leads to (4.2). □

5. Coupled system

In this section, we prove Theorem 1 by the fixed point argument as described in Section 2.

We begin with a key lemma, which will be used to estimate the stress τ.

Lemma 21. *Suppose that $\phi \in \overset{\circ}{H}{}^1_\mu$. For any $\varepsilon > 0$ there exists C_ε such that*

$$\left|\int \phi\nu\rho^{-1}dm\right|^2 \le C_\varepsilon \int |\phi|^2\mu dm + \varepsilon\int |\nabla_m\phi|^2\mu dm. \tag{5.1}$$

Proof. For $b > 2$, the Cauchy-Schwartz inequality yields

$$\left|\int \phi dm\right|^2 \le \int |\phi|^2\mu dm \int \mu^{-1}dm.$$

For any $\varepsilon > 0$, taking $C_\epsilon = \int \mu^{-1}dm < \infty$, we obtain (5.1) for $b > 2$.

For $b \le 2$, we define for fixed M,

$$G = \{\phi \in \overset{\circ}{H}{}^1_\mu : \int \phi\nu\rho^{-1}dm = 1, \|\phi\|_{H^1_\mu} \le M\}.$$

It suffices to prove

$$l := \inf_{\phi\in G}\int |\phi|^2\mu dm > 0.$$

Let $\{\phi_n\} \subset G$ be a sequence such that

$$\lim_{n\to\infty}\int |\phi_n|^2\mu dm = \inf_{\phi\in G}\int |\phi|^2\mu dm.$$

Since $\{\phi_n\}$ is bounded in H^1_μ, by embedding theorem (3.11), there exists a subsequence $\{\phi_{n_k}\}$ such that

$$\begin{aligned} \phi_{n_k} &\rightharpoonup \phi^* && \text{in } H^1_\mu, \\ \phi_{n_k} &\rightharpoonup \phi^* && \text{in } L^2_\mu, \\ \phi_{n_k} &\rightharpoonup \phi^* && \text{in } L^2_{\mu^*}. \end{aligned}$$

Furthermore, since $\sqrt{\dfrac{\mu}{\mu^*}} \in L^2_{\mu^*}$ for $b \le 2$

$$\begin{aligned} \int \phi^* \nu \rho^{-1} dm &= \int \phi^* \sqrt{\frac{\mu}{\mu^*}} \mu^* dm \\ &= \lim_{n_k \to \infty} \int \phi_{n_k} \sqrt{\frac{\mu}{\mu^*}} \mu^* dm = 1. \end{aligned}$$

This shows that $\phi^* \in G$. On the other hand,

$$\int |\phi^*|^2 \mu dm \le \lim_{n_k \to \infty} \int |\phi_{n_k}|^2 \mu dm = l.$$

If $l = 0$, then $\phi^* = 0$ which is a contradiction to $\phi^* \in G$. □

The zero trace of ϕ is essential for the estimate (5.1). For the general case, i.e., for $\phi \in H^1_\mu$, one can only have a weaker estimate.

Lemma 22. *If $\phi \in H^1_\mu$, then there exists C such that*

$$\left| \int \phi \nu \rho^{-1} dm \right|^2 \le C \|\phi\|^2_{H^1_\mu}. \tag{5.2}$$

Proof. For $b > 2$, we have

$$\left| \int \phi \nu \rho^{-1} dm \right|^2 \le C \int |\phi|^2 \mu dm, \quad C := \int \mu^{-1} dm < \infty.$$

For $b \le 2$,

$$\left| \int \phi \nu \rho^{-1} dm \right|^2 \le C_\delta \int |\phi|^2 \rho^{-1+\delta} dm, \quad C_\delta := \left(\int \nu^2 \rho^{-1-\delta} dm \right).$$

We choose $\delta > 0$ small enough so that C_δ is bounded. On the other hand, by (3.12) in Lemma 9 we have

$$\begin{aligned} \int |\phi|^2 \rho^{-1+\delta} dm &\le C \int (|\phi|^2 + |\nabla_m \phi|^2) \rho^{b/2} dm = C \int (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b < 2 \\ \int |\phi|^2 \rho^{-1+\delta} dm &\le C \int (|\phi|^2 + |\nabla_m \phi|^2) \rho dm \le C \int (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b = 2. \end{aligned}$$

This completes the proof. □

We now turn to the map

$$\begin{aligned} \mathcal{F} : \quad \mathbf{M} &\to \mathbf{M} \\ (u, \varpi) &\mapsto (v, w), \end{aligned}$$

and

$$\mathbf{M} = \left\{ (v,w) : \sup_{0\le t\le T} |v|_s^2 \le A_1, \sup_{0\le t\le T} |w|_{0,s}^2 + \frac{1}{2}\int_0^T |\nabla_m w|_{0,s}^2 dt \le A_2 \right\}.$$

We first prove that, given $v_0 \in H_x^s, f_0\nu^{-1} \in H_x^s L_\mu^2$ and $q \in C_t^1 H_x^{s+1} H_\mu^1$, the map $\mathcal{F}$ is well defined, i.e., $\mathcal{F}(\mathbf{M}) \subset \mathbf{M}$ for some A_1, A_2, T.

Let $(u,\varpi) \in \mathbf{M}$. It is now well known that (2.8) has a unique solution v such that

$$\sup_t |v|_s^2 + \int_0^T |v|_{s+1}^2 dt \le |v_0|_s^2 + C\int_0^T |u|_s |v|_s^2 dt + \int_0^T |\tau|_s^2 dt, \quad s > N/2+1. \tag{5.3}$$

By Gronwall's inequality and $\sup_{0\le t\le T} |u|_s^2 \le A_1$, we have

$$\sup_t |v|_s^2 + \int_0^T |v|_{s+1}^2 dt \le \left(|v_0|_s^2 + \int_0^T |\tau|_s^2 dt \right) e^{C\sqrt{A_1}T}. \tag{5.4}$$

We proceed to estimate the stress term

$$\int_0^T |\tau|_s^2 dt = \int_0^T \sum_{|\gamma|\le s} \int |\partial^\gamma \tau|^2 dx dt,$$

where using Lemma 21,

$$\begin{aligned} |\partial^\gamma \tau|^2 &= b^2 \left| \int_B m\otimes m \partial^\gamma(\varpi + q)\nu\rho^{-1} dm \right|^2 \\ &\le C_\varepsilon \int |\partial^\gamma \varpi|^2 \mu dm + \frac{\varepsilon}{2}\int |\partial^\gamma \nabla_m \varpi|^2 \mu dm + 2b^4 \left| \int \partial^\gamma q \nu\rho^{-1} dm \right|^2. \end{aligned}$$

Using (5.2) the last term is uniformly bounded by

$$C\|\partial^\gamma q(t,x,\cdot)\|_{H_\mu^1}^2 \le C\|q\|_{1,1,s+1}^2.$$

Hence for $(u,\varpi) \in \mathbf{M}$ we obtain

$$\int_0^T |\tau|_s^2 dt \le C_\varepsilon T A_2 + \epsilon A_2 + CT|q|_{1,1,s+1}^2 \le CT(A_2 + \|q\|_{1,1,s+1}^2) + \epsilon A_2, \tag{5.5}$$

where we have used the assumption $q \in C_t^1 H_x^{s+1} H_\mu^1$.

We choose A_1 as

$$A_1 = 2|v_0|_s^2 e, \tag{5.6}$$

A_2 as

$$A_2 = (|w_0|_{0,s}^2 + \|q\|_{1,1,s+1}^2) e^{C(T+A_1)} \tag{5.7}$$

for $T \le 1/(C\sqrt{A_1})$.

Hence, if T and ε are chosen small enough so that

$$CT(A_2 + \|q\|_{1,1,s+1}^2) + \varepsilon A_2 \le \frac{1}{2e} A_1,$$

we get

$$e^{C\sqrt{A_1}T}\left(|v_0|_s^2 + CT(A_2 + |q|_{1,1,s+1}^2) + \epsilon A_2 \right) \le e(|v_0|_s^2 + \frac{1}{2e}A_1) \le A_1. \tag{5.8}$$

This together with (5.4), (5.5) gives

$$\sup_t |v|_s^2 + \int_0^{T_1} |v|_{s+1}^2 dt \le A_1. \tag{5.9}$$

Estimate (4.2) in Theorem 19, (5.7) and (5.9) yield

$$\sup_t |w|_{0,s}^2 + \frac{1}{2}\int_0^{T} |\nabla_m w|_{0,s}^2 dt \le A_2. \tag{5.10}$$

So the map $\mathcal{F}$ is well defined in $\mathbf{M}$.

Next, we show that $\mathcal{F}$ is a contraction mapping for small enough T using a weak norm on $\mathbf{M}$, i.e.

$$\|(v,w)\|_{\mathbf{M}}^2 := \sup_t |v|_0^2 + \sup_t |w|_{0,0}^2 + \frac{1}{2}\int_0^{T} |\nabla_m w|_{0,0}^2 dt. \tag{5.11}$$

Suppose that $v_i (i = 1, 2)$ are solutions of the NSE (2.8) with $u_i(i = 1, 2)$ and $\tau_i(i = 1, 2)$ computed from $\varpi_i(i = 1, 2)$ respectively. Then we obtain

$$\partial_t v + (u_2 \cdot \nabla)v + (u \cdot \nabla)v_1 + \nabla p = \nabla \cdot \tau + \Delta v, \quad v(0, \cdot) = 0, \tag{5.12}$$

where $v = v_2-v_1, u = u_2-u_1, p = p_2-p_1, \tau = \tau_2-\tau_1$ and $\varpi = \varpi_2-\varpi_1$. Multiplication by v to (5.12) and integration with respect to x yield

$$\frac{1}{2}\frac{d}{dt}|v|_0^2 + \int (u \cdot \nabla v_1) v dx = -\int \tau \nabla v dx - \int |\nabla v|^2 dx.$$

Hence

$$\begin{aligned} \frac{d}{dt}|v|_0^2 + |\nabla v|_0^2 &\le |u|_0^2 + |\tau|_0^2 + \sup_x |\nabla v_1|^2 |v|_0^2 \\ &\le |u|_0^2 + |\tau|_0^2 + A_1|v|_0^2. \end{aligned} \tag{5.13}$$

Let f_i be the solutions to (2.9) associated with $v_i(i = 1, 2)$. Then

$$w = (f_2 - f_1)\nu^{-1} =: w_2 - w_1$$

solves

$$\partial_t w\mu + v_2 \cdot \nabla w\mu + L_2[w] = -v \cdot \nabla w_1 \mu - \nabla_m \cdot (\nabla v m \tilde{w}_1 \nu)\frac{\mu}{\nu}, \tag{5.14a}$$

$$w(0, x, m) = 0, \tag{5.14b}$$

$$w(t, x, m)|_{\partial B} = 0, \tag{5.14c}$$

where $L_2[w] = L[w]$ defined in (3.5) with $\kappa = \nabla v_2$. Note that $w_i|_{\partial B} = q|_{\partial B}$, i.e. $w_i(t, x, \cdot) \in H^1_\mu$, so $w(t, x, \cdot) \in \mathring{H}^1_\mu$.

We deduce from (5.14a) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w|_{0,0}^2 + \frac{1}{2}|\nabla_m w|_{0,0}^2 \le& \int\int |\nabla v_2 m \cdot \nabla_m w \nu w| dm dx + \int\int |Kw^2| dm dx \\ &+ \int |v \cdot \int \nabla w_1 w \mu dm| dx + \int \left| \int \nabla_m \cdot (\nabla v m w_1 \nu)\frac{\mu}{\nu} w dm \right| dx. \end{aligned}$$

Similar to that led to (4.2), first two terms on the right hand side are bounded by

$$C_\varepsilon(|v_2|_s^2 + 1)|w|_{0,0}^2 + \varepsilon|\nabla_m w|_{0,0}^2,$$

and the third term

$$\begin{aligned}\int |v \cdot \int \nabla w_1 w \mu dm| dx &\leq C \int |v|^2 \int |\nabla w_1|^2 \mu dm dx + \int\int |w|^2 \mu dm dx \\ &\leq C|v|_0^2 |w_1|_{0,s}^2 + |w|_{0,0}^2.\end{aligned}$$

The last term, using integration by parts with vanished boundary term due to Lemma 20, is bounded by

$$\begin{aligned}\int \left| \int \nabla_m \cdot (\nabla v m w_1 \nu) \frac{\mu}{\nu} w dm \right| dx &= \int \left| \int \nabla v m w_1 \nu \cdot \nabla_m \left(\frac{\mu}{\nu} w \right) dm \right| dx \\ &\leq C_\varepsilon |\nabla v|_0^2 |w_1|_{0,s}^2 + \varepsilon |\nabla_m w|_{0,0}^2.\end{aligned}$$

Putting all together we have

$$\begin{aligned}\frac{d}{dt}|w|_{0,0}^2 + \frac{1}{2}|\nabla_m w|_{0,0}^2 &\leq C(|v_2|_s^2 + 1)|w|_{0,0}^2 + C|w_1|_{0,s}^2 (|v|_0^2 + |\nabla v|_0^2) \\ &\leq C(A_1 + 1)|w|_{0,0}^2 + CA_2(|v|_0^2 + |\nabla v|_0^2).\end{aligned}$$

Substitution of the estimates of $|\nabla v|_0^2$ and $\frac{d}{dt}|v|_0^2$ in (5.13) gives

$$\frac{d}{dt}(|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2}|\nabla_m w|_{0,0}^2 \leq D(|v|_0^2 + |w|_{0,0}^2) + D|u|_0^2 + D|\tau|_0^2, \tag{5.15}$$

where D is a large constant depending on C, A_1, A_2, for example we may choose

$$D = C(A_1 + 1)(A_2 + 1).$$

The Gronwall inequality gives

$$\sup_t (|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2}\int_0^{T^*} |\nabla_m w|_{0,0}^2 dt \leq De^{DT} \int_0^{T^*} |u|_0^2 + |\tau|_0^2 dt$$

for any $0 < T^* \leq T$. Due to the similar estimate for τ as (5.5), the right hand side is bounded by

$$De^{DT} \left(T^* \sup_t |u|_0^2 + C_\varepsilon T^* \sup_t |\varpi|_{0,0}^2 + \varepsilon \int_0^{T^*} |\nabla_m \varpi|_{0,0}^2 dt \right).$$

We choose $\varepsilon = \dfrac{1}{4De^{DT}}$, $T^* = \dfrac{1}{2}\min\left\{ T, \dfrac{1}{(C_\varepsilon + 1)De^{DT}} \right\}$ and redefine $T = T^*$ to obtain

$$\|(v_2, w_2) - (v_1, w_1)\|_{\mathbf{M}}^2 = \|(v, w)\|_{\mathbf{M}}^2 \leq \frac{1}{2}\|(u_2, \varpi_2) - (u_1, \varpi_1)\|_{\mathbf{M}}^2. \tag{5.16}$$

This shows that $\mathcal{F}$ has a fixed point (v, w) in $\mathbf{M}$, which is a solution to the coupled problem (2.1). Since $\mathcal{F}(v, w) = (v, w)$, (5.3) and Theorem 13 imply that $(v, w) \in \mathbf{X}_\mu$.

The uniqueness follows from the same computation of estimates for the contraction mapping. Let $(v_i, f_i \nu^{-1})(i = 1, 2)$ be solutions of the coupled problem (2.1). Then $v = v_2 - v_1$ solves (5.12) with $u_i = v_i$, $u = v$, and $\tau = \tau_2 - \tau_1$ computed from f_i. $w = (f_2 - f_1)\nu^{-1}$ also solves (5.14) with $w_1 = f_1 \nu^{-1}$. Similar to (5.15), we obtain

$$\frac{d}{dt}(|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2}|\nabla_m w|_{0,0}^2 \leq D(|v|_0^2 + |w|_{0,0}^2 + |\tau|_0^2).$$

It follows from the estimate for τ and Gronwall inequality that $(v, w) \equiv (0, 0)$, which gives the uniqueness of problem (2.1).

6. A further look at $b \geq 6$

In this section, we sketch proofs of Theorem 2 and Theorem 4 for the case of $\mu = \mu_0$.

Consider (2.9) when x is not involved, i.e., (3.1). The corresponding w-problem for $w = f\nu^{-1} - q$ with $\mu = \mu_0$ solves (3.3) with the operator L replaced by

$$L_0[w] = -\frac{1}{2}\nabla \cdot (\nabla w \mu_0) + \left(2 - \frac{1}{2}b - \theta\right) m \cdot \nabla w \rho^{\theta-1} + \nabla \cdot (\kappa m w \mu_0) - K_0 w, \tag{6.1}$$

where

$$K_0 = [N(b/2-1) + 2\kappa m \cdot m(1-\theta)]\, \rho^{\theta-1}. \tag{6.2}$$

Define the conjugate of μ_0 as (3.10), $\mu_0^* = \rho^{\theta-2}$, then K_0 can be rewritten as

$$K_0 = [N(b/2-1) + 2\kappa m \cdot m(1-\theta)] \sqrt{\mu_0 \mu_0^*}. \tag{6.3}$$

To ensure well-posedness of (3.3), we need to check the coercivity of $\mathcal{B}_0[w, w; t]$, which is defined as

$$\frac{1}{2}\int |\nabla w|^2 \mu_0 dm = \mathcal{B}_0[w, w; t] - \left(2 - \frac{1}{2}b - \theta\right) \int m \cdot \nabla w w \rho^{\theta-1} dm$$
$$- \int \nabla \cdot (\kappa m w \mu_0) w dm + \int K_0 w^2 dm.$$

From the proof of Lemma 10, the last two terms are bounded by

$$C_\varepsilon \int w^2 \mu_0 dm + \varepsilon \int |\nabla w|^2 \mu_0 dm,$$

where the embedding theorem (3.11) has been used. For small enough ε, this estimate yields

$$\frac{1}{4}\int |\nabla w|^2 \mu_0 dm \leq \mathcal{B}_0[w, w; t] + C \int w^2 \mu_0 dm,$$

as long as

$$\int \left(2 - \frac{1}{2}b - \theta\right) m \cdot \nabla w w \rho^{\theta-1} dm \geq 0,$$

for $w \in \overset{\circ}{H}{}^1_{\mu_0}$. This is indeed the case, as shown below.

Lemma 23. *Let $w \in \overset{\circ}{H}{}^1_{\mu_0}$. Then*

$$\int (2 - \frac{1}{2}b - \theta) m \cdot \nabla w w \rho^{\theta-1} dm \geq 0. \tag{6.4}$$

Proof. From $-1 < \theta < 1$ and $b \geq 6$, we see that $(2 - b/2 - \theta) < 0$. It suffices to show

$$\int m \cdot \nabla w w \rho^{\theta-1} dm = \frac{1}{2}\int m \cdot \nabla w^2 \rho^{\theta-1} dm \leq 0.$$

Integration by parts gives

$$\int m \cdot \nabla w^2 \rho^{\theta-1} dm = -\int w^2 (N\rho^{\theta-1} + 2(1-\theta)|m|^2 \rho^{\theta-2}) dm + \int_{\partial B} w^2 \rho^{\theta-1} m \cdot \frac{m}{|m|} dS$$
$$\leq \sqrt{b} \int_{\partial B} w^2 \rho^{\theta-1} dS = 0.$$

Here we use the fact that $w^2\rho^{\theta-1} \in W^{1,1}$ and $w^2\rho^{\theta-1}|_{\partial B} = 0$. To see this, for any $w \in \overset{\circ}{H}{}^1_{\mu_0}$, we estimate

$$\begin{aligned}\int w^2\rho^{\theta-1} + |\nabla(w^2\rho^{\theta-1})|dm &\leq \int w^2\rho^{\theta-1} + 2|w\nabla w|\rho^{\theta-1} + 2(1-\theta)|mw^2|\rho^{\theta-2}dm \\ &\leq C\int w^2\sqrt{\mu_0\mu_0^*} + |w||\nabla w|\sqrt{\mu_0\mu_0^*} + w^2\mu_0^*dm \\ &\leq C\|w\|^2_{H^1_{\mu_0}},\end{aligned}$$

due to the embedding theorem (3.11). Thus $w^2\rho^{\theta-1}|_{\partial B} \in L^1(\partial B)$ from the trace theorem and it is zero from the fact that C_c^∞ is a dense subset of $\overset{\circ}{H}{}^1_{\mu_0}$. Thus (6.4) follows. □

We now turn to the FPE problem including x-variable. The first step in the proof of Theorem 19 remains valid for $\mu = \mu_0$. To check the second part of the proof, we need only look at two extra terms beyond those in (4.28).

$$-\left(2-\frac{1}{2}b-\theta\right)\int m\cdot\nabla_m\partial^\gamma w\rho^{\theta-1}\partial^\gamma w dm, \quad -\left(2-\frac{1}{2}b-\theta\right)\int m\cdot\nabla_m\partial^\gamma q\rho^{\theta-1}\partial^\gamma w dm.$$

The first term is non-positive from Lemma 23, and the second term is bounded by

$$C\left|\iint m\cdot\nabla_m\partial^\gamma q\rho^{\theta-1}\partial^\gamma w dm\right| \leq C_\varepsilon\int|\nabla_m\partial^\gamma q|^2\mu_0 dm + \varepsilon\int|\partial^\gamma w|^2\mu_0^* dm.$$

These ensure the same estimate (4.30) and thus (4.2).

For the well-posedness for the coupled problem, we utilize $\theta < 1$ and Lemma 23. For example, for the proof of Lemma 21 with μ_0

$$|\int\phi\nu\rho^{-1}dm|^2 = |\int\phi dm|^2 \leq \int\phi^2\mu_0 dm\int\mu_0^{-1}dm.$$

Since $\theta < 1$ we have $\int\mu_0^{-1}dm < \infty$, hence (5.1). Verification of other terms is omitted.

The remaining is to show Theorem 4, the solution f is a probability distribution if and only if $q|_{\partial B} = 0$ for $\mu = \mu_0$, Positivity of f follows as in Proposition 16. For the conservation of mass, as in Proposition 17, we only have to check (3.30),

$$\int_{B^\varepsilon}(w\kappa m - \nabla w\cdot\nabla\phi_\varepsilon\nu)dm - \int_{B^\varepsilon}w\rho^{b/2}\nabla\phi_\varepsilon\cdot\nabla(\nu\rho^{-b/2})dm.$$

Since $\nu^2/\mu_0 = \rho^{2-\theta}$ and $2-\theta > 1$

$$\frac{\varepsilon}{2}\int_{\partial B_r}|\nabla\phi_\varepsilon|^2\rho^{2-\theta}dS$$

converges to 0 as $\varepsilon \to 0$. Thus the first term converges to 0 as well. On the other hand, the same argument shows that the second term converges to $C\int_{\partial B}qdS$ for some nonzero constant C. Hence, we conclude Theorem 4 under the assumption of Theorem 2.

7. Conclusion

In this paper, we have analyzed the FENE dumbbell model which is of bead-spring type Navier-Stokes-Fokker-Planck models for dilute polymeric fluids, with our focus on developing a local well-posedness theory subject to a class of Dirichlet-type boundary conditions

$$f\nu^{-1} = q \quad \text{on } \partial B$$

for the polymer distribution f, where ν depends on $b > 0$ through the distance function, and q is a given smooth function measuring the relative ratio of f/ν near boundary. We have thus identified a sharp Dirichlet-type boundary requirement for each $b > 0$, while the sharpness of the boundary requirement is a consequence of the existence result for each specification of the boundary behavior. It has been shown that the probability density governed by the Fokker-Planck equation approaches zero near boundary, necessarily faster than the distance function d for $b > 2$, faster than $d|lnd|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Moreover, the sharp boundary requirement for $b \geq 2$ is also sufficient for the distribution to remain a probability density.

Acknowledgments

Liu's research was partially supported by the National Science Foundation under Kinetic FRG grant DMS07-57227 and grant DMS09-07963. The authors would like to thank Robert Pego for helpful discussions.

References

[1] A. Arnold, J. A. Carrillo, and C. Manzini. Refined long-time asymptotics for some polymeric fluid flow models. Preprint, 2009.
[2] J. W. Barrett, C. Schwab, and E. Süli. Existence of global weak solutions for some polymeric flow models. *Math. Models Methods Appl. Sci.*, 15(6):939–983, 2005.
[3] J. W. Barrett, and E. Süli. Existence of global weak solutions to kinetic models of dilute polymers. *Multiscale Model. Simul.*, 6:506–546, 2007.
[4] J. W. Barrett, and E. Süli. Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off. *Math. Mod. Meth. Appl. Sci.* , 18:935–971, 2008.
[5] J. W. Barrett, and E. Süli. Existence of equilibration of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers. *Preprint* , 2010.
[6] R. B. Bird, C. Curtiss, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids, Volume 2: Kinetic Theory.* Wiley Interscience, New York, 1987.
[7] L. Chupin. The FENE model for viscoelastic thin film flows. *Methods Appl. Anal.*, 16(2):217-261, 2009.
[8] C. Chauvière, and A. Lozinski. Simulation of complex viscoelastic flows using Fokker-Planck equation: 3D FENE model. *J. Non-Newtonian Fluid Mech.* , 122:201–214, 2004.
[9] C. Chauvière, and A. Lozinski. Simulation of dilute polymer solutions using a Fokker-Planck equation. *J. Comput. Fluids.*, 33:687–696, 2004.
[10] P. Constantin, C. Fefferman, E. S. Titi and A. Zarnescu. Regularity of coupled two-dimensional nonlinear Fokker-Planck and Navier-Stokes systems. *Comm. Math. Phys.*, 270(3):789811, 2007.
[11] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84112, 2001.
[12] P. Constantin and N. Masmoudi. Global well-posedness for a Smoluchowski equation coupled with Navier- Stokes equations in 2D. *Comm. Math. Phys.*, 278(1):179191, 2008.

[13] M. Doi and S. F. Edwards. *The Theory of Polymer Dynamics*. Oxford University Press, Oxford, 1986.

[14] P. Degond and H. Liu. Kinetic models for polymers with inertial effects. *Netw. Heterog. Media*, 4(4):625647, 2009.

[15] P. Degond, M. Lemou and M. Picasso. Viscoelastic fluid models derived from kinetic equations for polymers. *SIAM J. Appl. Math.*, 62(5):15011519, 2002.

[16] Q. Du, C. Liu, and P. Yu. FENE dumbbell model and its several linear and nonlinear closure approximations. *Multiscale Model. Simul.*, 4(3):709–731, 2005.

[17] W. E, T. Li, and P. Zhang. Well-posedness for the dumbbell model of polymeric fluids. *Comm. Math. Phys.*, 248(2):409–427, 2004.

[18] C. Guillope and J. C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849869, 1990.

[19] C. Guillope and J. C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO Model. Math. Anal. Numer.*, 24(3):369401, 1990.

[20] L. He and P. Zhang. L2 decay of solutions to a micro-macro model for polymeric fluids near equilibrium. *SIAM J. Math. Anal.*, 40(5):19051922, 2008/09.

[21] B. Jourdain and T. Lelièvre. Mathematical analysis of a stochastic differential equation arising in the micro-macro modelling of polymeric fluids. In *Probabilistic methods in fluids*, pages 205–223. World Sci. Publ., River Edge, NJ, 2003.

[22] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.*, 209(1):162–193, 2004.

[23] B. Jourdain, T. Lelièvre, C. Le Bris, and F. Otto. Long-time asymptotics of amultiscale model for polymeric fluid flows. *Arch. Ration. Mech. Anal.*, 181:97–148, 2006.

[24] D. J. Knezevic, and E. Süli. Spectral Galerkin approximation of Fokker-Planck equations with unbounded drift. *ESAIM: M2AN*, 42(3):445–485, 2009.

[25] A. Kufner. *Weighted Sobolev Spaces*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1985. Translated from the Czech.

[26] A. Kufner and L.-E. Persson. Weighted inequalities of Hardy type. World Scientific Publishing Co. Inc., River Edge, NJ, 2003

[27] F. Lin, P. Zhang, and Z. Zhang. On the global existence of smooth solution to the 2-D FENE dumbbell model. *Comm. Math. Phys.*, 277(2):531–553, 2008.

[28] F.-H. Lin, C. Liu, and P. Zhang. On a micro-macro model for polymeric fluids near equilibrium. *Comm. Pure Appl. Math.*, 60(6):838–866, 2007.

[29] F.-H. Lin, C. Liu and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, 58(11):14371471, 2005.

[30] Z. Lei, C. Liu and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.

[31] P.-L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131146, 2000.

[32] Z. Lei and Y. Zhou. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.*, 37(3):797814, 2005.

[33] F. H. Lin and P. Zhang. The FENE dumbbell model near equilibrium. *Acta Math. Sin. (Engl. Ser.)*, 24(4):529–538, 2008.

[34] P.-L. Lions and N. Masmoudi. Global existence of weak solutions to some micro-macro models. *C. R. Math. Acad. Sci. Paris*, 345(1):15–20, 2007.

[35] C. Liu and H. Liu. Boundary conditions for the microscopic FENE models. *SIAM J. Appl. Math.*, 68(5):1304–1315, 2008.

[36] H. Liu and J. Shin. Global well-posedness for the microscopic FENE model with a sharp boundary condition to appear in Journal of Diff Equ., 2011.

[37] H. Liu and H. Yu. An entropy satisfying finite volume method for the Fokker-Planck equation of FENE dumbbell model for polymers. Preprint, 2010.

[38] N. Masmoudi. Well-posedness for the FENE dumbbell model of polymeric flows. *Comm. Pure Appl. Math.*, 61(12):1685–1714, 2008.

[39] N. Masmoudi. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Preprint*, 2010.
[40] A. J. Majda and A. L. Bertozzi. Vorticity and incompressible flow. *Cambridge University Press*, Cambridge, 2002.
[41] N. Masmoudi, P. Zhang and Z. Zhang. Global well-posedness for 2D polymeric fluid models and growth estimate. *Phys. D*, 237(10-12):16631675, 2008.
[42] R.G. Owens and T.N. Phillips. *Computational Rheology.* Imperial College Press, London, 2002.
[43] H. Öttinger. *Stochastic Processes in Polymeric Liquids.* Springer-Verlag, Berlin and New York, 1996.
[44] M. Renardy. An existence theorem for model equations resulting from kinetic theories of polymer solutions. *SIAM J. Math. Anal.*, 22(2):313–327, 1991.
[45] M. E. Schonbek. Existence and decay of polymeric flows. *SIAM J. Math. Anal.*, 41(2):564587,2009.
[46] J. Shen and H. J. Yu. On the approximation of the Fokker-Planck equation of FENE dumbbell model, I: A new weighted formulation and an optimal Spectral-Galerkin algorithm in 2-D. *Preprint*, 2010.
[47] H. Zhang and P. Zhang. Local existence for the FENE-dumbbell model of polymeric fluids. *Arch. Ration. Mech. Anal.*, 181(2):373–400, 2006.
[48] L. Zhang, H. Zhang and P. Zhang. Global existence of weak solutions to the regularized Hookean dumbbell model. *Commun. Math. Sci.*, 6(1):85–124, 2008.

Iowa State University, Mathematics Department, Ames, IA 50011
E-mail address: hliu@iastate.edu

Institute for Mathematics and its applications, University of Minnesota, Minneapolis, MN 55455,
E-mail address: shin@ima.umn.edu