

An Asymptotic Preserving 2–D Staggered Grid Method for multiscale transport equations

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December 5, 2014

Abstract

We propose a two-dimensional asymptotic-preserving scheme for linear transport equations with diffusive scalings. It is based on the time splitting developed by Jin, Pareschi and Toscani [17], but takes spatial discretizations on staggered grids. Compared with the previous methods based on regular Cartesian grids, this method preserves the discrete diffusion limit with a *more compact* stencil thus has a better spatial resolution. This scheme requires less unknowns than one could have naively expected for a method based on staggered grids. We show that the scheme is AP and we provide a stability analysis to obtain an explicit CFL condition, which couples a hyperbolic and a parabolic condition. This type of condition is common for AP schemes and guarantees uniform stability with respect to the mean free path. In addition, we obtain an upper bound on the relaxation parameter, which is the crucial parameter of the used time discretization. Several numerical examples are provided to verify the accuracy and asymptotic property of the scheme.

1 Introduction

The linear transport equation models particles interacting with a background medium (e.g. neutron transport, linear radiative transfer, . . .). In general, the model in scaled variables can be written as [17]

$$\varepsilon \partial_t f + v \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon} \left[\frac{\sigma_s}{2\pi} \int_{\Omega} f dv' - \sigma_t f \right] + \varepsilon Q, \quad (1.1)$$

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where $f(t, \mathbf{x}, v)$ denotes the probability density distribution depending on time t , position $\mathbf{x} \in \mathbb{R}^2$, and direction of velocity $v = (\xi, \eta) \in \Omega = \{(\xi, \eta) : -1 \leq \xi, \eta \leq 1, \xi^2 + \eta^2 = 1\}$ (two-dimensional flatland model – the extension to three dimensions is straightforward). Moreover, $\sigma_t = \sigma_s + \varepsilon^2 \sigma_a$ is the total transport coefficient, σ_s is the scattering coefficient, σ_a is the absorption coefficient, and Q is a v -independent source term. It is well known that the limiting equation ($\varepsilon \rightarrow 0$) of this equation (1.1) is the diffusion equation:

$$\partial_t \rho = \frac{1}{2} \nabla_{\mathbf{x}} \cdot \left(\frac{1}{\sigma_t} \nabla_{\mathbf{x}} \rho \right) - \sigma_a \rho + Q, \quad (1.2)$$

where $\rho(t, \mathbf{x}) = \frac{1}{2\pi} \int_{\Omega} f(t, \mathbf{x}, v) dv$.

In many applications, the scaling parameter of the transport equation ε (mean free path) may differ in several orders of magnitude, ranging from the rarefied kinetic regime to the hydrodynamic diffusive regime. When ε is small, in the diffusive regime, the equation becomes numerically stiff, which leads to numerical challenges: Straightforward explicit implementations lead to high computational costs in the diffusive regimes; Fully implicit schemes could be difficult to implement [6]; Multiscale multiphysics domain-decomposition approaches, which couple models at different scales, have difficulties in the transition zones, since they need to transfer data from one scale to another [12]. Thus, it is desirable to develop schemes which are suitable for all regimes (no domain-decomposition), but do not require a resolved grid in space and small time compared to the mean free path. This is the objective of AP schemes.

A scheme is called AP if it preserves the discrete analog of the asymptotic transition from the microscopic scale to the macroscopic one [11, 12], namely, in the limit $\varepsilon \rightarrow 0$, the discretization of the above transport equation (1.1) should yield a discretization of the diffusion equation (1.2). Such schemes allow mesh sizes and time steps much bigger than the mean free path/time, yet still capture the correct physical behavior. The development of such schemes started with stationary problems of linear transport equations by Larsen, Morel, and Miller [19] and for boundary value problems by Jin and Levermore [15, 13]. Uniform convergence with respect to the mean free path for an AP scheme was first established by Golse, Jin, and Levermore [9]. For time-dependent problems, AP schemes were first designed for nonlinear hyperbolic systems with relaxation by Jin and Levermore [10, 14]. There one needs to design both the time and the spatial discretization carefully [14], in particular, to overcome the stiffness of the source term. AP schemes for time-dependent transport equations with diffusive scaling started by Jin-Pareschi-Toscani [16, 17] and Klar [18]. Since then there have been many new developments in the construction of AP schemes for a large class of kinetic equations (cf. reviews by Degond [5] and Jin [12]). Time discretizations usually need an implicit-explicit (IMEX) approach [4, 10, 25, 1], exponential integration methods [7], BGK type penalty methods [8], or micro-macro decomposition based schemes [21, 22]. See also [23, 28]. One key idea of the schemes is to split the equation into a non-stiff part, which is treated explicitly, and a stiff part which will be implicit but can be implemented explicitly. The splitting should be taken in a way such that the combination preserves the AP property.

In this paper, we present a two-dimensional AP scheme for the above transport equation (1.1), based on an AP splitting using the parity equations [17] in combination with staggered grids [12]. Moreover, we present a stability analysis of the scheme in one dimension to show uniform stability. Similar to [21, 22], we obtain an explicit CFL condition, which couples a hyperbolic and a parabolic condition, and guarantees uniform stability. Beyond that, the stability of the scheme depends on the choice of the relaxation parameter. This parameter arises from the used time discretization, where the equation is split into a stiff relaxation and a non-stiff transport step and the parameter balances the terms. For the choice of this parameter, our stability analysis yields an upper bound, that is more restrictive than the one that has previously been used [17].

The advantage of the staggered approach, compared with the regular grid approach in [17], is that in the diffusion limit we approach a *compact* stencil, as pointed out by Jin [12]. To be more precise, in one-space dimension, using a regular Cartesian grid, the discrete diffusion limit of the scheme of [17] approximates the diffusion operator in (1.2) by $(\rho_{i+2} - 2\rho_i + \rho_{i-2})/(2\Delta x)^2$ (for the case $\sigma_t \equiv 1$), while the current scheme gives the compact discretization $(\rho_{i+1} - 2\rho_i + \rho_{i-1})/(\Delta x)^2$ which offers a better spatial resolution.

The remainder of the paper is organized as follows. In section 2, we first derive the parity equations for the linear transport equation. Then, we describe the numerical method. In section 3, we show the AP property. We consider the asymptotic limit of the scheme and we state and prove a stability result, which gives a CFL condition and an upper bound on the relaxation parameter. Finally, several numerical tests are presented in section 4 to confirm the AP property of the scheme.

2 The Numerical Method

First, we reformulate the transport equation into a parity equation. Then, we describe the angular, the spatial, and the time discretization of the method in detail.

We begin with the transport equation in the diffusive scaling (1.1), restricted to two spatial dimensions $\mathbf{x} = (x, y)$. Following [17], we define the even and odd parities

$$\begin{aligned} \mathbf{r}^{(1)}(\xi, \eta) &= \frac{1}{2}[f(\xi, -\eta) + f(-\xi, \eta)], & \mathbf{r}^{(2)}(\xi, \eta) &= \frac{1}{2}[f(\xi, \eta) + f(-\xi, -\eta)], \\ \mathbf{j}^{(1)}(\xi, \eta) &= \frac{1}{2\varepsilon}[f(\xi, -\eta) - f(-\xi, \eta)], & \mathbf{j}^{(2)}(\xi, \eta) &= \frac{1}{2\varepsilon}[f(\xi, \eta) - f(-\xi, -\eta)], \end{aligned} \quad (2.3)$$

and obtain the system

$$\partial_t \mathbf{r}^{(1)} + \xi \partial_x \mathbf{j}^{(1)} - \eta \partial_y \mathbf{j}^{(1)} = -\frac{\sigma_s}{\varepsilon^2}(\mathbf{r}^{(1)} - \rho) - \sigma_a \mathbf{r}^{(1)} + Q, \quad (2.4a)$$

$$\partial_t \mathbf{r}^{(2)} + \xi \partial_x \mathbf{j}^{(2)} + \eta \partial_y \mathbf{j}^{(2)} = -\frac{\sigma_s}{\varepsilon^2}(\mathbf{r}^{(2)} - \rho) - \sigma_a \mathbf{r}^{(2)} + Q, \quad (2.4b)$$

$$\partial_t \mathbf{j}^{(1)} + \frac{\xi}{\varepsilon^2} \partial_x \mathbf{r}^{(1)} - \frac{\eta}{\varepsilon^2} \partial_y \mathbf{r}^{(1)} = -\frac{\sigma_s}{\varepsilon^2} \mathbf{j}^{(1)} - \sigma_a \mathbf{j}^{(1)}, \quad (2.4c)$$

$$\partial_t \mathbf{j}^{(2)} + \frac{\xi}{\varepsilon^2} \partial_x \mathbf{r}^{(2)} + \frac{\eta}{\varepsilon^2} \partial_y \mathbf{r}^{(2)} = -\frac{\sigma_s}{\varepsilon^2} \mathbf{j}^{(2)} - \sigma_a \mathbf{j}^{(2)}, \quad (2.4d)$$

where $\rho = \frac{1}{2\pi} \int_{|v|=1} f dv$. Due to symmetry, it is sufficient to consider only positive directions $\xi, \eta \geq 0$.

The discretization of the angular variable uses Gaussian quadrature points. Let

$$\xi(\lambda) = \cos(\lambda\pi/2) \quad \text{and} \quad \eta(\lambda) = \sin(\lambda\pi/2) \quad (2.5)$$

for every $0 \leq \lambda \leq 1$. Then, the density

$$\rho = \frac{1}{2} \int_0^1 [r^{(1)}(\xi, \eta) + r^{(2)}(\xi, \eta)] d\lambda \quad (2.6)$$

can be approximated by a Gaussian quadrature, where the quadrature points $\{\lambda_i\}$ are mapped to $\{\xi_i\}$ and $\{\eta_i\}$ by (2.5).

2.1 Spatial Discretization

For simplicity, we describe the spatial discretization, while keeping the time t continuous. First, we define 2-D staggered grids

$$\begin{aligned} G_{1,1} &:= \left\{ \left((i + \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta y \right) : i, j \in \mathbb{Z} \right\}, & G_{1,2} &:= \left\{ (i\Delta x, j\Delta y) : i, j \in \mathbb{Z} \right\}, \\ G_{2,1} &:= \left\{ (i\Delta x, (j + \frac{1}{2})\Delta y) : i, j \in \mathbb{Z} \right\}, & G_{2,2} &:= \left\{ (i\Delta x, j\Delta y) : i, j \in \mathbb{Z} \right\}, \end{aligned} \quad (2.7)$$

and approximate the spatial derivatives in the equations (2.4) by half-grid centered differences. We start by placing $r^{(1)}$ on the grid $G_{1,1}$. Then the equation (2.4a) couples $r^{(1)}$ to $\partial_x j^{(1)}$, $\partial_y j^{(1)}$, and ρ . Thus, we place the unknown $j^{(1)}$ on $G_{2,1} \cup G_{1,2}$ and ρ on $G_{1,1}$. The next equation (2.4c) couples $j^{(1)}$ to $\partial_x r^{(1)}$ and $\partial_y r^{(1)}$, so the unknown $r^{(1)}$ needs to be given not only on the grid $G_{1,1}$, but also on the grid $G_{2,2}$. As a consequence, we place ρ , $r^{(1)}$, and $r^{(2)}$ on $G_{1,1} \cup G_{2,2}$, since they are coupled through the definition of ρ , and $j^{(1)}$ and $j^{(2)}$ on $G_{2,1} \cup G_{1,2}$. This leads to a closed system.

Remark 1. We obtain eight semi-discretized differential equations, and each grid point on $G_{1,1} \cup G_{1,2} \cup G_{2,1} \cup G_{2,2}$ carries two unknowns.

Let $i, j \in \mathbb{Z}$. Then, the parities $r^{(1)}$ and $r^{(2)}$ satisfy the equation (2.4a) and (2.4b), which on the grid $G_{2,2}$ and $G_{1,1}$ are given by

$$\begin{aligned} \partial_t r_{i,j}^{(1)} + \xi \frac{j_{i+\frac{1}{2},j}^{(1)} - j_{i-\frac{1}{2},j}^{(1)}}{\Delta x} - \eta \frac{j_{i,j+\frac{1}{2}}^{(1)} - j_{i,j-\frac{1}{2}}^{(1)}}{\Delta y} &= -\frac{\sigma_s}{\varepsilon^2} (r_{i,j}^{(1)} - \rho_{i,j}) - \sigma_a r_{i,j}^{(1)} + Q_{i,j}, \\ \partial_t r_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} + \xi \frac{j_{i+1,j+\frac{1}{2}}^{(1)} - j_{i,j+\frac{1}{2}}^{(1)}}{\Delta x} - \eta \frac{j_{i+\frac{1}{2},j+1}^{(1)} - j_{i+\frac{1}{2},j}^{(1)}}{\Delta y} &= -\frac{\sigma_s}{\varepsilon^2} (r_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} - \rho_{i+\frac{1}{2},j+\frac{1}{2}}) \\ &\quad - \sigma_a r_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} + Q_{i+\frac{1}{2},j+\frac{1}{2}}, \end{aligned} \quad (2.8)$$

and

$$\begin{aligned}
\partial_t \mathbf{r}_{i,j}^{(2)} + \xi \frac{j_{i+\frac{1}{2},j}^{(2)} - j_{i-\frac{1}{2},j}^{(2)}}{\Delta x} + \eta \frac{j_{i,j+\frac{1}{2}}^{(2)} - j_{i,j-\frac{1}{2}}^{(2)}}{\Delta y} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}_{i,j}^{(2)} - \rho_{i,j}) - \sigma_a \mathbf{r}_{i,j}^{(2)} + Q_{i,j}, \\
\partial_t \mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(2)} + \xi \frac{j_{i+1,j+\frac{1}{2}}^{(2)} - j_{i,j+\frac{1}{2}}^{(2)}}{\Delta x} + \eta \frac{j_{i+\frac{1}{2},j+1}^{(2)} - j_{i+\frac{1}{2},j}^{(2)}}{\Delta y} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(2)} - \rho_{i+\frac{1}{2},j+\frac{1}{2}}) \\
&\quad - \sigma_a \mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(2)} + Q_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned} \tag{2.9}$$

Similarly, the equations for the parities $\mathbf{j}^{(1)}$ and $\mathbf{j}^{(2)}$ on the grid $G_{1,2}$ and $G_{2,1}$ are given by (cf. equation (2.4c) and (2.4d))

$$\begin{aligned}
\partial_t \mathbf{j}_{i+\frac{1}{2},j}^{(1)} + \frac{\xi}{\varepsilon^2} \frac{\mathbf{r}_{i+1,j}^{(1)} - \mathbf{r}_{i,j}^{(1)}}{\Delta x} - \frac{\eta}{\varepsilon^2} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} - \mathbf{r}_{i+\frac{1}{2},j-\frac{1}{2}}^{(1)}}{\Delta y} &= -\frac{\sigma_t}{\varepsilon^2} \mathbf{j}_{i+\frac{1}{2},j}^{(1)}, \\
\partial_t \mathbf{j}_{i,j+\frac{1}{2}}^{(1)} + \frac{\xi}{\varepsilon^2} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} - \mathbf{r}_{i-\frac{1}{2},j+\frac{1}{2}}^{(1)}}{\Delta x} - \frac{\eta}{\varepsilon^2} \frac{\mathbf{r}_{i,j+1}^{(1)} - \mathbf{r}_{i,j}^{(1)}}{\Delta y} &= -\frac{\sigma_t}{\varepsilon^2} \mathbf{j}_{i,j+\frac{1}{2}}^{(1)},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\partial_t \mathbf{j}_{i+\frac{1}{2},j}^{(2)} + \frac{\xi}{\varepsilon^2} \frac{\mathbf{r}_{i+1,j}^{(2)} - \mathbf{r}_{i,j}^{(2)}}{\Delta x} + \frac{\eta}{\varepsilon^2} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(2)} - \mathbf{r}_{i+\frac{1}{2},j-\frac{1}{2}}^{(2)}}{\Delta y} &= -\frac{\sigma_t}{\varepsilon^2} \mathbf{j}_{i+\frac{1}{2},j}^{(2)}, \\
\partial_t \mathbf{j}_{i,j+\frac{1}{2}}^{(2)} + \frac{\xi}{\varepsilon^2} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(2)} - \mathbf{r}_{i-\frac{1}{2},j+\frac{1}{2}}^{(2)}}{\Delta x} + \frac{\eta}{\varepsilon^2} \frac{\mathbf{r}_{i,j+1}^{(1)} - \mathbf{r}_{i,j}^{(1)}}{\Delta y} &= -\frac{\sigma_t}{\varepsilon^2} \mathbf{j}_{i,j+\frac{1}{2}}^{(2)}.
\end{aligned} \tag{2.11}$$

2.2 Time Discretization

For simplicity, we consider again semi-discretized equations. This time, we keep the spatial variables x and y continuous and apply the time discretization technique from [17]. First, we rewrite the system of equations (2.4) as the diffusive relaxation system

$$\begin{aligned}
\partial_t \mathbf{r}^{(1)} + \xi \partial_x \mathbf{j}^{(1)} - \eta \partial_y \mathbf{j}^{(1)} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}^{(1)} - \rho) - \sigma_a \mathbf{r}^{(1)} + Q, \\
\partial_t \mathbf{r}^{(2)} + \xi \partial_x \mathbf{j}^{(2)} + \eta \partial_y \mathbf{j}^{(2)} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}^{(2)} - \rho) - \sigma_a \mathbf{r}^{(2)} + Q, \\
\partial_t \mathbf{j}^{(1)} + \phi \xi \partial_x \mathbf{r}^{(1)} - \phi \eta \partial_y \mathbf{r}^{(1)} &= -\frac{1}{\varepsilon^2} [\sigma_s \mathbf{j}^{(1)} + (1 - \varepsilon^2 \phi) \xi \partial_x \mathbf{r}^{(1)} - (1 - \varepsilon^2 \phi) \eta \partial_y \mathbf{r}^{(1)}], \\
\partial_t \mathbf{j}^{(2)} + \phi \xi \partial_x \mathbf{r}^{(2)} + \phi \eta \partial_y \mathbf{r}^{(2)} &= -\frac{1}{\varepsilon^2} [\sigma_s \mathbf{j}^{(2)} + (1 - \varepsilon^2 \phi) \xi \partial_x \mathbf{r}^{(2)} + (1 - \varepsilon^2 \phi) \eta \partial_y \mathbf{r}^{(2)}],
\end{aligned} \tag{2.12}$$

with $0 \leq \phi \leq 1/\varepsilon^2$. The condition $\phi \geq 0$ is necessary for the hyperbolicity of the left hand side, whereas the condition $\phi \leq 1/\varepsilon^2$ ensures that the bracketed term on the right hand side has a well-defined limit for $\varepsilon \rightarrow 0$.

Second, we split the equation into two parts, the transport step

$$\begin{aligned}
\partial_t \mathbf{r}^{(1)} + \xi \partial_x \mathbf{j}^{(1)} - \eta \partial_y \mathbf{j}^{(1)} &= -\sigma_a \mathbf{r}^{(1)} + Q, \\
\partial_t \mathbf{r}^{(2)} + \xi \partial_x \mathbf{j}^{(2)} + \eta \partial_y \mathbf{j}^{(2)} &= -\sigma_a \mathbf{r}^{(2)} + Q, \\
\partial_t \mathbf{j}^{(1)} + \phi \xi \partial_x \mathbf{r}^{(1)} - \phi \eta \partial_y \mathbf{r}^{(1)} &= -\sigma_a \mathbf{j}^{(1)}, \\
\partial_t \mathbf{j}^{(2)} + \phi \xi \partial_x \mathbf{r}^{(2)} + \phi \eta \partial_y \mathbf{r}^{(2)} &= -\sigma_a \mathbf{j}^{(2)},
\end{aligned} \tag{2.13}$$

and the relaxation step

$$\begin{aligned}
\partial_t \mathbf{r}^{(1)} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}^{(1)} - \rho), \\
\partial_t \mathbf{r}^{(2)} &= -\frac{\sigma_s}{\varepsilon^2} (\mathbf{r}^{(2)} - \rho), \\
\partial_t \mathbf{j}^{(1)} &= -\frac{1}{\varepsilon^2} [\sigma_s \mathbf{j}^{(1)} + (1 - \varepsilon^2 \phi) \xi \partial_x \mathbf{r}^{(1)} - (1 - \varepsilon^2 \phi) \eta \partial_y \mathbf{r}^{(1)}], \\
\partial_t \mathbf{j}^{(2)} &= -\frac{1}{\varepsilon^2} [\sigma_s \mathbf{j}^{(2)} + (1 - \varepsilon^2 \phi) \xi \partial_x \mathbf{r}^{(2)} + (1 - \varepsilon^2 \phi) \eta \partial_y \mathbf{r}^{(2)}].
\end{aligned} \tag{2.14}$$

Finally, we apply the explicit Euler method to the first step and the implicit Euler Method to the second step. Note that the implicit Euler method can be implemented explicitly, since ρ is preserved in the second step (which can be seen by adding the first two equations).

The fully discrete scheme is just splitting (2.8)-(2.11) into the two steps (2.13)-(2.14).

3 The AP property

In this section, we analyze the AP property of the above scheme in two steps. First, we derive the discrete asymptotic limit. Second, we analyze stability.

3.1 The Diffusion limit

In the same way as above, we consider the spatial and the time discretization separately. The limit, as $\varepsilon \rightarrow 0$, of the time discretization is derived in [17]. Hence, it remains to investigate the discrete limit of the spatial discretization. To this end, we consider the diffusive limit $\varepsilon \rightarrow 0$ of the semi-discretized equations (2.8)-(2.11).

First, the limit of the equations (2.8) and (2.10) for the parities $\mathbf{r}^{(1)}$ and $\mathbf{j}^{(1)}$ is given by

$$\begin{aligned}
\mathbf{r}_{i,j}^{(1)} &= \rho_{i,j}, \\
\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} &= \rho_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\mathbf{j}_{i+\frac{1}{2},j}^{(1)} &= -\frac{\xi}{\sigma_t} \frac{\mathbf{r}_{i+1,j}^{(1)} - \mathbf{r}_{i,j}^{(1)}}{\Delta x} + \frac{\eta}{\sigma_s} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} - \mathbf{r}_{i+\frac{1}{2},j-\frac{1}{2}}^{(1)}}{\Delta y}, \\
\mathbf{j}_{i,j+\frac{1}{2}}^{(1)} &= -\frac{\xi}{\sigma_t} \frac{\mathbf{r}_{i+\frac{1}{2},j+\frac{1}{2}}^{(1)} - \mathbf{r}_{i-\frac{1}{2},j+\frac{1}{2}}^{(1)}}{\Delta x} + \frac{\eta}{\sigma_s} \frac{\mathbf{r}_{i,j+1}^{(1)} - \mathbf{r}_{i,j}^{(1)}}{\Delta y}.
\end{aligned} \tag{3.15}$$

Inserting these equations in (2.8), yields

$$\begin{aligned}
\partial_t \rho_{i,j} - \frac{\xi^2}{\sigma_t} \frac{\rho_{i+1,j} - 2\rho_{i,j} + \rho_{i-1,j}}{(\Delta x)^2} + \frac{2\xi\eta}{\sigma_t} \frac{\rho_{i+\frac{1}{2},j+\frac{1}{2}} - \rho_{i+\frac{1}{2},j-\frac{1}{2}} - \rho_{i-\frac{1}{2},j+\frac{1}{2}} + \rho_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x \Delta y} \\
- \frac{\eta^2}{\sigma_t} \frac{\rho_{i,j+1} - 2\rho_{i,j} + \rho_{i,j-1}}{(\Delta y)^2} = -\sigma_a \rho_{i,j} + Q_{i,j}, \\
\partial_t \rho_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{\xi^2}{\sigma_t} \frac{\rho_{i+\frac{3}{2},j+\frac{1}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i-\frac{1}{2},j+\frac{1}{2}}}{(\Delta x)^2} + \frac{2\xi\eta}{\sigma_t} \frac{\rho_{i+1,j+1} - \rho_{i+1,j} - \rho_{i,j+1} + \rho_{i,j}}{\Delta x \Delta y} \\
- \frac{\eta^2}{\sigma_t} \frac{\rho_{i+\frac{1}{2},j+\frac{3}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i+\frac{1}{2},j-\frac{1}{2}}}{(\Delta y)^2} = -\sigma_a \rho_{i+\frac{1}{2},j+\frac{1}{2}} + Q_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Treating equations (2.9) and (2.11) in the same way as above, we additionally obtain the following differential equations for ρ :

$$\begin{aligned}
\partial_t \rho_{i,j} - \frac{\xi^2}{\sigma_s} \frac{\rho_{i+1,j} - 2\rho_{i,j} + \rho_{i-1,j}}{(\Delta x)^2} - \frac{2\xi\eta}{\sigma_s} \frac{\rho_{i+\frac{1}{2},j+\frac{1}{2}} - \rho_{i+\frac{1}{2},j-\frac{1}{2}} - \rho_{i-\frac{1}{2},j+\frac{1}{2}} + \rho_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x \Delta y} \\
- \frac{\eta^2}{\sigma_s} \frac{\rho_{i,j+1} - 2\rho_{i,j} + \rho_{i,j-1}}{(\Delta y)^2} = -\sigma_a \rho_{i,j} + Q_{i,j}, \\
\partial_t \rho_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{\xi^2}{\sigma_s} \frac{\rho_{i+\frac{3}{2},j+\frac{1}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i-\frac{1}{2},j+\frac{1}{2}}}{(\Delta x)^2} - \frac{2\xi\eta}{\sigma_s} \frac{\rho_{i+1,j+1} - \rho_{i+1,j} - \rho_{i,j+1} + \rho_{i,j}}{\Delta x \Delta y} \\
- \frac{\eta^2}{\sigma_s} \frac{\rho_{i+\frac{1}{2},j+\frac{3}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i+\frac{1}{2},j-\frac{1}{2}}}{(\Delta y)^2} = -\sigma_a \rho_{i+\frac{1}{2},j+\frac{1}{2}} + Q_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned} \tag{3.17}$$

Adding up the equations, the middle terms cancel and we obtain

$$\begin{aligned}
\partial_t \rho_{i,j} - \frac{\xi^2}{\sigma_s} \frac{\rho_{i+1,j} - 2\rho_{i,j} + \rho_{i-1,j}}{(\Delta x)^2} - \frac{\eta^2}{\sigma_s} \frac{\rho_{i,j+1} - 2\rho_{i,j} + \rho_{i,j-1}}{(\Delta y)^2} = -\sigma_a \rho_{i,j} + Q_{i,j}, \\
\partial_t \rho_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{\xi^2}{\sigma_s} \frac{\rho_{i+\frac{3}{2},j+\frac{1}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i-\frac{1}{2},j+\frac{1}{2}}}{(\Delta x)^2} - \frac{\eta^2}{\sigma_s} \frac{\rho_{i+\frac{1}{2},j+\frac{3}{2}} - 2\rho_{i+\frac{1}{2},j+\frac{1}{2}} + \rho_{i+\frac{1}{2},j-\frac{1}{2}}}{(\Delta y)^2} \\
= -\sigma_a \rho_{i+\frac{1}{2},j+\frac{1}{2}} + Q_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned} \tag{3.18}$$

Integrating over $\xi^2 + \eta^2 = 1$ yields the semi-discretized diffusion equations on the grids $G_{1,1}$ and $G_{2,2}$. Note that integrating the equations (3.16) or (3.17) over $\xi^2 + \eta^2 = 1$, the middle terms cancel as well and we get the same result.

As expected, the spatial discretization with staggered grids leads to a compact five point stencil for the diffusion equation (1.2). Together with the results [17] on the limit of the time discretization (2.13) – (2.14), this also shows that the formal limit of our scheme coincides with the diffusion equation.

3.2 Stability

We limit our discussion to the one-dimensional case (see Remark 5 for the two-dimensional case) and show uniform stability with ε using the von Neumann analysis [27, 26, 21].

In the following, we consider the transport equation in slab geometry and assume that the cross section $\sigma_t = \sigma_s + \varepsilon^2 \sigma_a > 0$ is independent of $x \in \mathbb{R}$ (see Remark 4 for space dependent scattering). Further, we consider a source-free two velocity model. Then, the even and odd parities

$$r(t, x, v) = \frac{1}{2}[f(t, x, v) + f(t, x, -v)] \quad \text{and} \quad j(t, x, v) = \frac{1}{2\varepsilon}[f(t, x, v) - f(t, x, -v)] \quad (3.19)$$

fulfill

$$\begin{aligned} \partial_t r + \partial_x j &= -\sigma_a r, \\ \partial_t j + \frac{1}{\varepsilon^2} \partial_x r &= -\frac{1}{\varepsilon^2} \sigma_s j - \sigma_a j, \end{aligned} \quad (3.20)$$

and the numerical scheme has the following update rule: For $k = 0, 1, 2, \dots$

$$\begin{aligned} r^{k+\frac{1}{2}} &= r^k - \Delta t (D_x j^k + \sigma_a r^k), \\ j^{k+\frac{1}{2}} &= j^k - \Delta t (\phi D_x r^k + \sigma_a j^k), \\ r^{k+1} &= r^{k+\frac{1}{2}}, \\ j^{k+1} &= \frac{\varepsilon^2}{\varepsilon^2 + \sigma_s \Delta t} j^{k+\frac{1}{2}} - \frac{\Delta t}{\varepsilon^2 + \sigma_s \Delta t} (1 - \varepsilon^2 \phi) D_x r^{k+1}, \end{aligned} \quad (3.21)$$

where D_x denotes the half-grid centered finite difference of the spatial derivative. We place r on the half grid points $(m + \frac{1}{2})\Delta x$ and j on the full grid points $m\Delta x$. For a von Neumann analysis of the scheme, we expand the parities in Fourier series:

$$r(x, t) = \sum_{\ell=-\infty}^{\infty} a_\ell(t) e^{i\ell x} \quad \text{and} \quad j(x, t) = \sum_{\ell=-\infty}^{\infty} b_\ell(t) e^{i\ell x}. \quad (3.22)$$

As no mixing between the Fourier modes occurs during the update of the solution, it is sufficient to consider the evolution of

$$r(x, t) = a_\ell(t) e^{i\ell x} \quad \text{and} \quad j(x, t) = b_\ell(t) e^{i\ell x} \quad (3.23)$$

for some ℓ and to determine the growth factor matrix of the Fourier coefficients. First, we note that the staggered grid derivatives can be rewritten as

$$\begin{aligned} (D_x r)(h(m + \frac{1}{2}), t) &= a_\ell(t) \frac{e^{i\ell h(m+1)} - e^{i\ell h m}}{h} = 2 \frac{i}{h} \sin\left(\frac{\ell h}{2}\right) e^{i\ell h(m+\frac{1}{2})} a_\ell(t), \\ (D_x j)(hm, t) &= b_\ell(t) \frac{e^{i\ell h(m+\frac{1}{2})} - e^{i\ell h(m-\frac{1}{2})}}{h} = 2 \frac{i}{h} \sin\left(\frac{\ell h}{2}\right) e^{i\ell h m} b_\ell(t), \end{aligned} \quad (3.24)$$

with $h := \Delta x$. To shorten the notation, we define $d_\ell := \frac{2i}{h} \sin\left(\frac{\ell h}{2}\right)$. Then, the first update step of the Fourier coefficients is given by

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} (t + \Delta t) = \underbrace{\begin{bmatrix} 1 - \sigma_a \Delta t & -\Delta t d_\ell \\ -\Delta t \phi d_\ell & 1 - \sigma_a \Delta t \end{bmatrix}}_{=: G_1} \begin{bmatrix} a_k \\ b_k \end{bmatrix} (t) \quad (3.25)$$

and in the second step is given by

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} (t + \Delta t) = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{\Delta t}{\varepsilon^2 + \sigma_s \Delta t} (1 - \varepsilon^2 \phi) d_\ell & \frac{\varepsilon^2}{\varepsilon^2 + \sigma_s \Delta t} \end{bmatrix}}_{=: G_2} \begin{bmatrix} a_k \\ b_k \end{bmatrix} (t). \quad (3.26)$$

Thus, the growth factor matrix is

$$G := G_2 \cdot G_1 = \begin{bmatrix} 1 - \sigma_a \Delta t & -\Delta t d_\ell \\ -\frac{d_\ell \Delta t}{\varepsilon^2 + \sigma_s \Delta t} (\sigma_a \Delta t (1 - \varepsilon^2 \phi) + 1) & \frac{1}{\varepsilon^2 + \sigma_s \Delta t} (\varepsilon^2 (1 - \sigma_a \Delta t) + d_\ell^2 \Delta t^2 (1 - \varepsilon^2 \phi)) \end{bmatrix}. \quad (3.27)$$

For stability, the eigenvalues of the matrix G are of main interest. They can be written as

$$\lambda_{1,2} = g \pm \sqrt{g^2 - \det(G)} \quad (3.28)$$

with g being the half trace and $\det(G)$ being the determinant of G :

$$\begin{aligned} g &= \frac{1}{2} \frac{1}{\varepsilon^2 + \sigma_s \Delta t} (\Delta t^2 d_\ell^2 (1 - \varepsilon^2 \phi) + (1 - \sigma_a \Delta t) (2\varepsilon^2 + \sigma_s \Delta t)) \quad \text{and} \\ \det(G) &= \frac{\varepsilon^2}{\varepsilon^2 + \sigma_s \Delta t} ((1 - \sigma_a \Delta t)^2 - \phi d_\ell^2 \Delta t^2). \end{aligned} \quad (3.29)$$

Proposition 1. *Let the time step Δt and the relaxation parameter ϕ satisfy*

$$\Delta t \leq \min \left\{ \frac{1}{\sigma_a}, \max \left\{ \frac{1}{2} \varepsilon h, \frac{1}{4} h^2 \sigma_t \right\} \right\}, \quad (3.30)$$

and

$$0 \leq \phi \leq \begin{cases} \frac{h \sigma_t}{2\varepsilon^3}, & h \sigma_t \leq 2\varepsilon \\ \frac{1}{\varepsilon^2}, & \text{otherwise} \end{cases}. \quad (3.31)$$

Then, the numerical scheme is L^2 -stable.

Remark 2. Note that the restriction that has previously been used $0 \leq \phi \leq \frac{1}{\varepsilon^2}$ (see above and [17]) still holds for the new choice of ϕ . Moreover, the condition $h \sigma_t \leq 2\varepsilon$ is satisfied, if and only if the hyperbolic condition $\Delta t \leq \max \left\{ \frac{1}{2} \varepsilon h, \frac{1}{4} h^2 \sigma_t \right\} = \frac{1}{2} \varepsilon h$ holds, i.e. there are the following two cases:

$$\begin{aligned} h \sigma_t \leq 2\varepsilon : \quad \Delta t &\leq \min \left\{ \frac{1}{\sigma_a}, \frac{1}{2} \varepsilon h \right\} \quad \text{and} \quad 0 \leq \phi \leq \frac{h \sigma_t}{2\varepsilon^3}, \\ h \sigma_t > 2\varepsilon : \quad \Delta t &\leq \min \left\{ \frac{1}{\sigma_a}, \frac{1}{4} h^2 \sigma_t \right\} \quad \text{and} \quad 0 \leq \phi \leq \frac{1}{\varepsilon^2}. \end{aligned} \quad (3.32)$$

In addition, as $\varepsilon \rightarrow 0$ the time step restriction becomes $\Delta t \lesssim \min \left\{ \frac{1}{\sigma_a}, \frac{1}{4} h^2 \sigma_t \right\}$, which does not vanish.

In the proof of the proposition, we use the von Neumann analysis. A complete overview of these stability conditions can be found in the lecture notes by Trefethen [27].

Proof. Stability follows from the von Neumann condition, if we can show $|\lambda_{1,2}| \leq 1$ for $\lambda_1 \neq \lambda_2$ and $|\lambda_{1,2}| < 1$ for $\lambda_1 = \lambda_2$. To show these inequalities, we consider three different cases: two complex eigenvalues; two real eigenvalues; and one eigenvalue. Since g and $\det(G)$ are real-valued (3.29), the cases are equivalent to: $g^2 < \det(G)$; $g^2 > \det(G)$; and $g^2 = \det(G)$.

Case $g^2 < \det(G)$ (two complex eigenvalues): If the eigenvalues $\lambda_{1,2}$ are complex, their real part is g and their imaginary part is $\pm\sqrt{\det(G) - g^2}$. Thus, the stability condition $|\lambda_{1,2}|^2 \leq 1$ is satisfied if $\det(G) \leq 1$. For the determinant we have the following estimate

$$\det(G) = \frac{\varepsilon^2}{\varepsilon^2 + \sigma_s \Delta t} \left((1 - \sigma_a \Delta t)^2 - \phi d_\ell^2 \Delta t^2 \right) \leq \frac{\varepsilon^2}{\varepsilon^2 + \sigma_s \Delta t} \left(1 - \sigma_a \Delta t + \phi \frac{4\Delta t^2}{h^2} \right), \quad (3.33)$$

where we used that $-d_\ell^2 = \frac{4}{h^2} \sin^2(\frac{\ell h}{2}) \leq \frac{4}{h^2}$ and the CFL-condition $\Delta t \leq \frac{1}{\sigma_a}$. It remains to show that the last term of (3.33) is bounded by 1. This is equivalent to

$$\varepsilon^2 \phi \frac{4\Delta t}{h^2} \leq \sigma_s + \varepsilon^2 \sigma_a = \sigma_t, \quad (3.34)$$

which in turn is satisfied under the condition $\Delta t \leq \max\{\frac{1}{2}\varepsilon h, \frac{1}{4}h^2\sigma_t\}$ and the assumption (3.31). This is one of the reasons for the choice of the upper bound of ϕ in the assumption (3.31).

Case $g^2 > \det(G)$ (two real eigenvalues): The determinant of G is always positive and therefore the eigenvalues are either both positive or both negative, and their sign changes with the sign of g . Thus, it is sufficient to show $\lambda_1 \leq 1$ if $g \geq 0$ and $\lambda_2 \geq -1$ if $g < 0$. In particular, one can show that this is equivalent to

$$\det(G) + 1 \mp 2g \geq 0. \quad (3.35)$$

The first inequality is generic

$$\det(G) + 1 - 2g = \frac{\Delta t^2}{\varepsilon^2 + \sigma_s \Delta t} (\sigma_a^2 \varepsilon^2 + \sigma_s \sigma_a - d_\ell) \geq 0, \quad (3.36)$$

since $d_\ell^2 \leq 0$. Whereas, the second inequality requires the CFL-condition (3.30). More precisely, under the condition $0 \leq \phi \varepsilon^2 \leq 1$ and $\Delta t \leq \frac{1}{\sigma_a}$, we obtain

$$\begin{aligned} \det(G) + 1 + 2g &> 1 + 2g = 1 + \frac{1}{\varepsilon^2 + \sigma_s \Delta t} \left(\Delta t^2 d_\ell^2 (1 - \varepsilon^2 \phi) + (1 - \sigma_a \Delta t)(2\varepsilon^2 + \sigma_s \Delta t) \right) \\ &\geq \frac{1}{\varepsilon^2 + \sigma_s \Delta t} \left(\varepsilon^2 - \frac{4\Delta t^2}{h^2} + \sigma_s \Delta t \right). \end{aligned} \quad (3.37)$$

On the one hand, this is obviously non-negative under the condition $\Delta t \leq \frac{1}{2}\varepsilon h$. On the other hand, the second term can be rewritten as

$$\varepsilon^2 - \frac{4\Delta t^2}{h^2} + \sigma_s \Delta t = \varepsilon^2 (1 - \sigma_a \Delta t) + \Delta t \left(\sigma_t - \frac{4\Delta t}{h^2} \right), \quad (3.38)$$

which is non-negative under the condition $\Delta t \leq \frac{1}{\sigma_a}$ and $\Delta t \leq \frac{1}{4}h^2\sigma_t$. Together, this yields the desired inequality $\det(G) + 1 + 2g > 0$.

Case $g^2 = \det(G)$ (one eigenvalue): The eigenvalue of G is $\lambda_1 = \lambda_2 = g$. Thus, we need to show $|g| < 1$. But as $\det(G) + 1 + 2g > 0$ and $\det(G) \leq 1$ (see above cases) already imply $g > -1$, it remains to show $g < 1$. Since $\sigma_t = \sigma_s + \varepsilon^2 \sigma_a > 0$, at least one of the terms $\sigma_a \Delta t$, $\sigma_s \Delta t$ is positive and we obtain

$$g \leq \frac{1}{2} \frac{1}{\varepsilon^2 + \sigma_s \Delta t} \left((1 - \sigma_a \Delta t)(2\varepsilon^2 + \sigma_s \Delta t) \right) < \frac{1}{2} \frac{1}{\varepsilon^2 + \sigma_s \Delta t} \left((1 - \sigma_a \Delta t + \sigma_a \Delta t)(2\varepsilon^2 + \sigma_s \Delta t + \sigma_s \Delta t) \right) = 1. \quad (3.39)$$

□

Remark 3. If there is neither scattering nor absorption and the conditions (3.30) and (3.31) hold, then the relaxation parameter satisfies $\phi = 0$ and the hyperbolic condition is always satisfied. Further, the determinant, $\det(G) = 1$, and the half trace, $g = 1 + \frac{\Delta t^2 d_\ell^2}{2\varepsilon^2}$, coincide only if $d_\ell^2 = -\frac{h^2}{4} \sin^2(\frac{\ell h}{2}) = 0$, which is equivalent to $\frac{\ell h}{2} \in \pi\mathbb{Z}$. In most cases, this does not occur and therefore the case $g^2 = \det(G)$ does not arise. Then, we obtain $g^2 < \det(G)$ and the eigenvalues are distinct and satisfy $|\lambda_{1,2}| = 1$, so that stability follows.

Remark 4. If the cross sections are space dependent, the above analysis is not valid. In practice, the CFL condition is replaced by a worst-case condition. This means that we replace σ_a and σ_t in equation (3.30) and (3.31) by its maximum and minimum,

$$\sigma_{a,\max} = \max_x \sigma_a(x) \quad \text{and} \quad \sigma_{t,\min} = \min_x \sigma_t(x), \quad (3.40)$$

respectively.

Remark 5. In two dimensions, one can expect that the stability result from Proposition 1 carries over with the following changes. We replace $h = \min(\Delta x, \Delta y)$ and add a factor of $\frac{1}{2}$ in front of the time step to account for the presence of growth rates in each of the two spatial dimensions.

4 Numerical Results

In this section, we consider different numerical test cases to demonstrate the performance of our scheme. Since we did not examine boundary conditions, we only consider examples, where the solution is compactly supported away from the boundary. We implemented periodic boundary conditions, so that there is no influence of any discretization of boundary values.

The numerical calculations are performed using the two-dimensional scheme described in Section 2 with the stability conditions from Section 3.2. This means, we first choose the number of grid points ($N \times N$) for the staggered grids corresponding to the test case. Then, we determine the maximal time step (c.f. Proposition 1, Remark 4, and Remark 5)

$$\Delta t := 0.9 \cdot \frac{1}{2} \min \left\{ \frac{1}{\sigma_{a,\max}}, \max \left\{ \frac{1}{2} \varepsilon h, \frac{1}{4} h^2 \sigma_{t,\min} \right\} \right\}, \quad (4.41)$$

and define the relaxation parameter

$$\phi := \begin{cases} h \frac{\sigma_{t,\min}}{2\varepsilon^3}, & h\sigma_t \leq 2\varepsilon \\ \frac{1}{\varepsilon^2}, & \text{otherwise} \end{cases} \quad (4.42)$$

with $h := \frac{1}{N}$, $\sigma_{a,\max} := \max_x \sigma_a(x)$, and $\sigma_{t,\min} := \min_x \sigma_t(x)$. The angular discretization uses a Gaussian quadrature with 16 points on the interval $[0, 1]$ for λ . As the quadrature points are mapped to the directions ξ and η with (2.5), we obtain 16 points per quadrant. In all test cases, we compare the numerical solution on a grid where the parameter ε is resolved, to a grid on which it is under-resolved, thus demonstrating the AP property.

In the remainder of this section, we describe the test cases and the numerical results in detail. We consider four test cases to show different aspects of the AP property. First, we focus on the ε -dependence and investigate the convergence order in different regimes. In the second and third test case, there are large spatial differences in the cross-sections. The second test case is continuous and rotationally invariant, whereas in the third test case the material cross-sections and the source term are discontinuous. These two test cases intent to demonstrate the performance in multiscale problems. The last test case investigates the stability of the scheme dependent on the choice of the relaxation parameter ϕ .

4.1 Convergence order

We examine the order of convergence with respect to the spatial variable. We expect first or second order convergence depending on the used CFL condition. If a hyperbolic condition is used, the time step is proportional to h . As the explicit Euler method is used for the time discretization, we cannot expect more than first order convergence in h . Whereas, if the parabolic condition is used, the time step is proportional to h^2 . Then, the explicit Euler method predicts $\mathcal{O}(h^2)$ convergence. Moreover, centered differences, which are used for the spatial discretization, are as well a second order approximation in h . Thus, we expect that the error is proportional to $\mathcal{O}(h)$ when the hyperbolic condition is used, and $\mathcal{O}(h^2)$, respectively, when the parabolic condition is used. To estimate the convergence order, we compute the ℓ^2 -error $E(N)$ between the solution computed on a $N \times N$ grid and a reference solution. Using two different values N_1 and N_2 , we then estimate the convergence order by

$$E_{N_1}^{N_2} = -\frac{\log(E(N_1)) - \log(E(N_2))}{\log(N_1) - \log(N_2)}. \quad (4.43)$$

4.1.1 Method of manufactured solutions

For the method of manufactured solutions (MMS), we first choose some function $f(t, x, y, \xi, \eta)$ and compute a corresponding source term and an initial condition, so that the chosen function is a solution of the transport equation. Let

$$f(t, x, y, \xi, \eta) = \exp(-t) \sin(2\pi x)^2 \sin(2\pi y)^2 (1 + \eta^2) \quad (4.44)$$

with $(x, y) \in [0, 1]^2$. Further, let the scattering cross sections be given by $\sigma_a = 0$ and $\sigma_s = 1$. Then, the corresponding source term is given by

$$Q(t, x, y, \xi, \eta) = \partial_t f + \varepsilon v \cdot \nabla_x f - \frac{1}{\varepsilon^2} \left[\frac{1}{2\pi} \int_{\Omega} f dv' \right], \quad (4.45)$$

and the initial condition is given by

$$f(t = 0, x, y, \xi, \eta) = \sin(2\pi x)^2 \sin(2\pi y)^2 (1 + \eta)^2. \quad (4.46)$$

We use the source term and the initial condition to compute a solution with the above scheme. For different grid sizes and different values of ε , we compare the computed densities with the analytic density

$$\rho(t, x, y) = \frac{1}{2\pi} \int_{\Omega} f dv' \quad (4.47)$$

at time $t = 0.1$. The results are shown in figure 1 and table 1. They confirm second order convergence in the parabolic case. In the hyperbolic case, the convergence order is even slightly higher than expected.

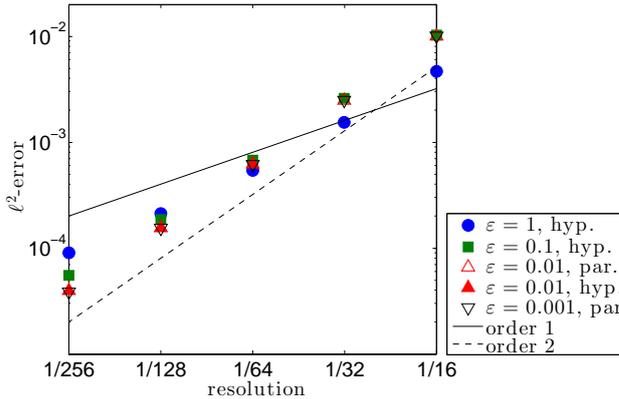


Figure 1: Convergence order (MMS): ℓ^2 -error as a function of the spatial resolution. Hyperbolic (filled markers) or parabolic (empty markers) CFL condition.

	E_{16}^{32}	E_{32}^{64}	E_{64}^{128}	E_{128}^{256}
$\varepsilon = 1$	1.60	1.50	1.35	1.23
$\varepsilon = 0.1$	1.98	1.93	1.86	1.76
$\varepsilon = 0.01$	2.02	2.00	1.99	1.97
$\varepsilon = 0.001$	2.02	2.01	2.00	2.00

Table 1: Convergence order (MMS): The term $E_{N_1}^{N_2}$ is the convergence rate when going from $N_1 \times N_1$ to $N_2 \times N_2$ grid points for a fixed mean free path ε . The dashed line indicates the switch from the hyperbolic to the parabolic condition.

4.1.2 Gauss test

We consider an example case with a smooth initial condition and isotropic scattering

$$f(t = 0, x, y, v) = \frac{1}{4\pi \cdot 10^{-2}} \exp\left(-\frac{x^2 + y^2}{4 \cdot 10^{-2}}\right) \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1], \quad (4.48)$$

$$Q = 0, \quad \sigma_t = \sigma_s = 1, \quad \sigma_a = 0, \quad \text{and } \varepsilon = 1, 10^{-1}, 10^{-2}.$$

Then, we compute the density ρ at time $t = 0.1$ for different grid sizes and different values of ε , so that the CFL condition (4.41) changes from hyperbolic to parabolic. As a

reference solution, we use a highly resolved solution with 512×512 grid points. Table 2 and Figure 2 agree with the above assertion, showing first order convergence when the hyperbolic condition holds and second order, respectively, when the parabolic condition holds.

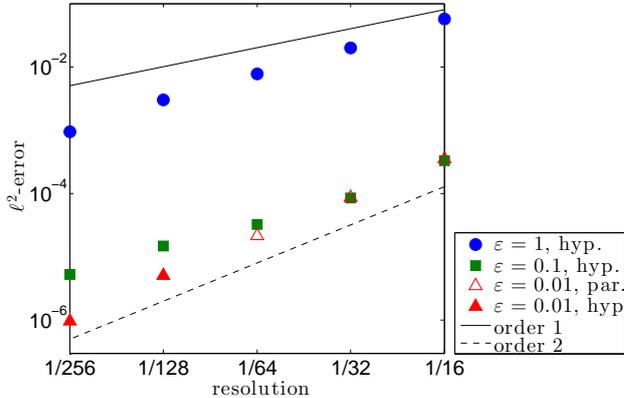


Figure 2: Convergence order (Gauss test): ℓ^2 -error as a function of the spatial resolution. Hyperbolic (filled markers) or parabolic (empty markers) CFL condition.

	E_{16}^{32}	E_{32}^{64}	E_{64}^{128}	E_{128}^{256}
$\varepsilon = 1$	1.50	1.38	1.36	1.66
$\varepsilon = 0.1$	1.96	1.37	1.15	1.50
$\varepsilon = 0.01$	2.03	2.01	2.09	2.41

Table 2: Convergence order (Gauss test): The term $E_{N_1}^{N_2}$ is the convergence rate when going from $N_1 \times N_1$ to $N_2 \times N_2$ grid points for a fixed mean free path ε . The dashed line indicates the switch from the hyperbolic to the parabolic condition.

4.2 Variable scattering

In this test case, we examine the performance of the scheme, when the scattering is space-dependent. Compared to the previous test case, we fix the scaling parameter ε and modify the scattering cross section. Let

$$\begin{aligned}
 f(t=0, x, y, v) &= \frac{1}{4\pi \cdot 10^{-2}} \exp\left(-\frac{x^2+y^2}{4 \cdot 10^{-2}}\right) \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1], \\
 \varepsilon &= \frac{1}{100}, \quad Q = 0, \quad \sigma_a = 0, \quad \text{and} \\
 \sigma_t(x, y) = \sigma_s(x, y) &= \begin{cases} c^A (c + \sqrt{2})^2 (c - \sqrt{2})^2, & c = \sqrt{x^2 + y^2} < 1 \\ 1, & \text{otherwise} \end{cases}.
 \end{aligned} \tag{4.49}$$

Note that, the total cross section $\sigma_t(x, y)$ can be periodically extended to a C^2 -function and $\frac{\sigma_t(x, y)}{\varepsilon}$ ranges from 0 to 100. This wide range compared to the size of the domain, causes strong variations of the solution, which are a challenge for numerical schemes.

We compute the solution up to time $t = \varepsilon$ on two different grids. One of the grids under-resolves the length scale $\varepsilon = \frac{1}{100}$ (32×32 grid points) and the other one resolves it (512×512 grid points). Comparing the solution at different times ($t = \frac{1}{10}\varepsilon, \frac{1}{2}\varepsilon, \varepsilon$, see Figure 3), we observe that the density, computed on the under-resolved grid, matches the behavior of the density, computed on the resolved grid.

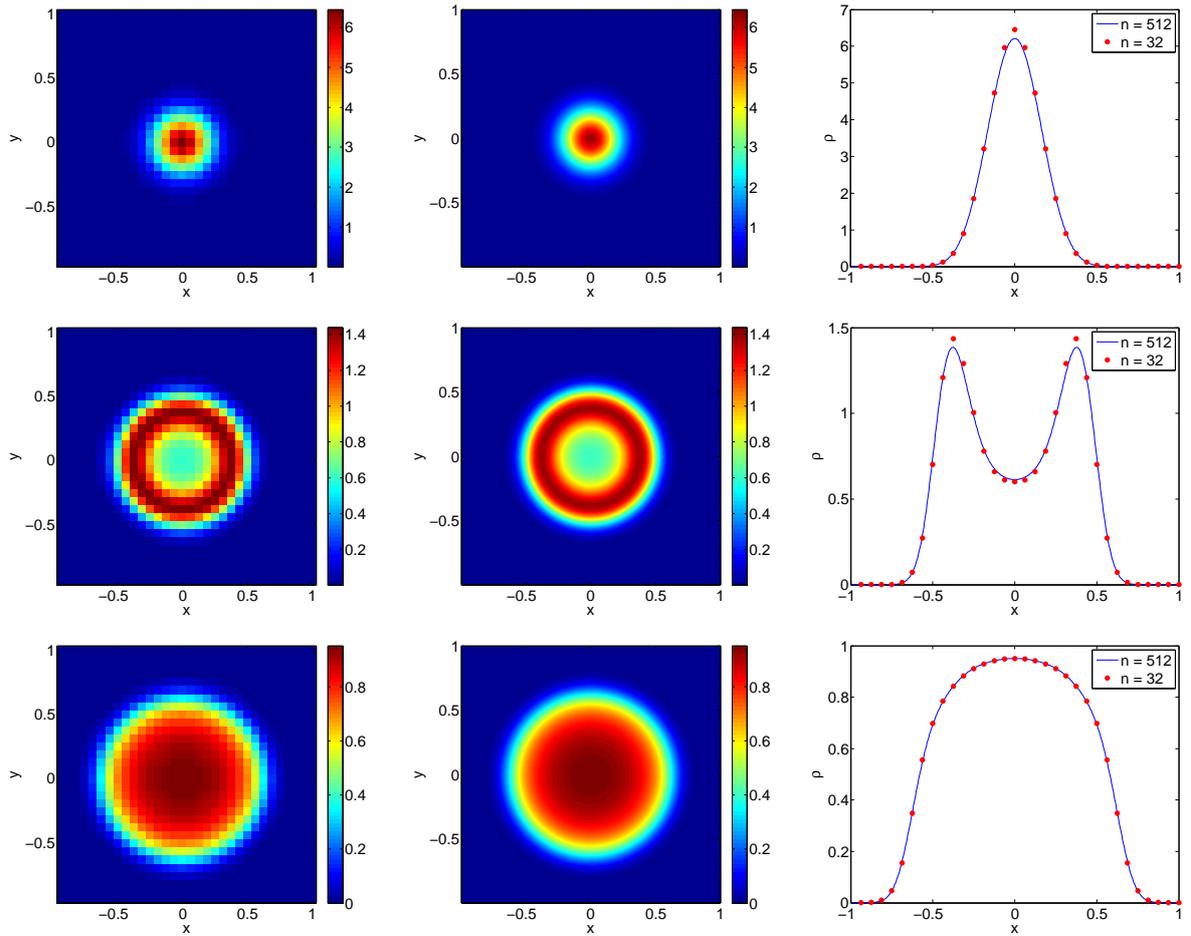


Figure 3: Variable scattering: Density at $\frac{t}{\varepsilon} = 0.1$ (first row), $\frac{t}{\varepsilon} = 0.5$ (second row), and $\frac{t}{\varepsilon} = 1.0$ (third row), computed on a 32×32 grid (first column) or a 512×512 grid (second column). The third column shows the density on a cut along $y = 0$.

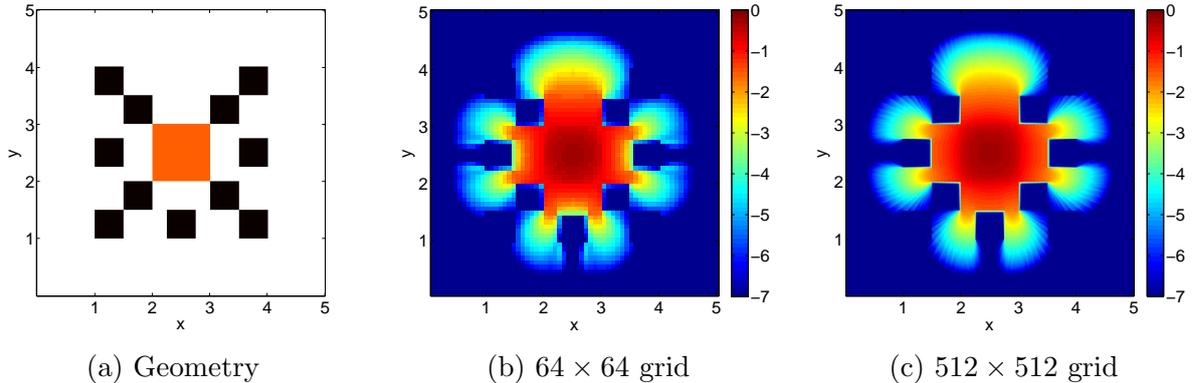


Figure 4: Two material test: (a): Geometry – source (orange), purely scattering $\sigma_t = \sigma_s = 1$ (white and orange), purely absorbing $\sigma_t = \sigma_a = 100$ (black). (b) and (c): Density ρ at $t = 1.7$, computed on a 64×64 grid (b), or 512×512 grid (c). Logarithmic scaling, values are limited to seven orders of magnitude.

4.3 Two material test

The two material test case is a slight modification of the lattice test, which was proposed in [3]. It models a domain with different materials by discontinuous material cross-sections and a discontinuous source term in space.

In this problem, the computational domain is a 5×5 square. Most of the domain is purely scattering, except for some purely absorbing squares of size 0.5, which are distributed around an isotropic source in the middle of the domain

$$Q(x, y) = \begin{cases} 1, & (x, y) \in [2, 3]^2 \\ 0, & \text{otherwise} \end{cases}. \quad (4.50)$$

In the absorbing spots (c.f. Figure 4a), the absorption coefficient jumps from 0 to 100, while the scattering coefficient jumps from 1 to 0. Thus, there are diffusive and kinetic regimes, although the scaling parameter satisfies $\varepsilon = 1$. We obtain a rapid change of the solution at the transition zones, which may cause difficulties in the numerics.

We compute the density up to time $t = 1.7$ on a coarse grid (64×64) and on a fine grid (512×512). The solutions are shown in Figure 4. Again, we observe that the solution on the under-resolved grid resembles the solution on a grid that is resolved. In the case of the resolved solution, the oscillations near the beam edges are due to the angular discretization. They are the well-known ray effects for finite discrete velocity models (cf. [2] and references therein, as well as [20, 24]).

4.4 Relaxation parameters and stability

In our final test, we consider different relaxation parameters. Proposition 1 suggests an upper bound on the relaxation parameter ϕ , which in the hyperbolic case is more restrictive

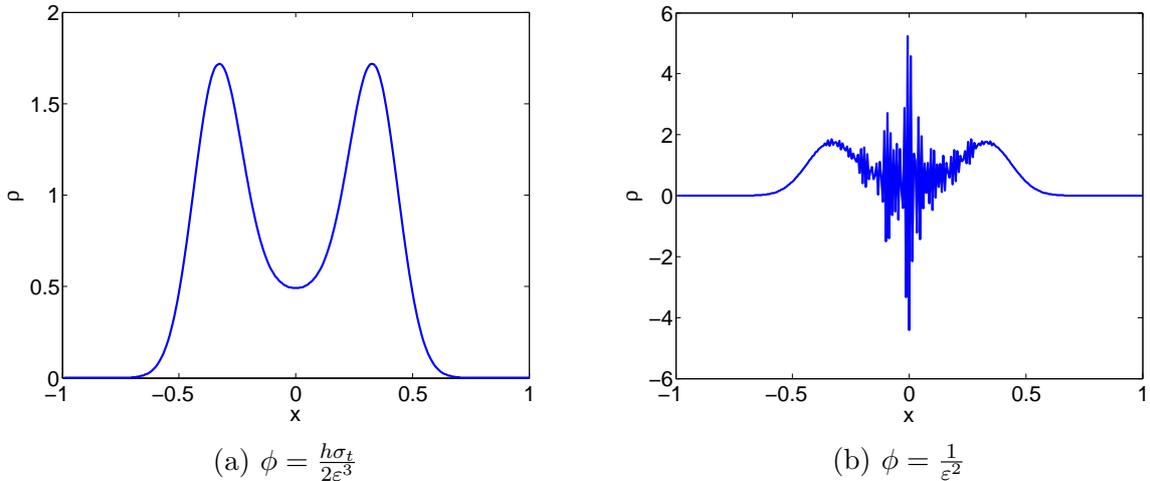


Figure 5: Stability: Density on a cut along $y = 0$, computed on a 300×300 grid up to time $t = 0.36$ using different values of ϕ .

than in the parabolic case. We expect that in certain example cases our scheme becomes unstable if ϕ is too large.

Similar to the Gauss test, let

$$f(t=0, x, y, v) = \frac{1}{4\pi \cdot 5 \times 10^{-3}} \exp\left(-\frac{x^2+y^2}{4 \cdot 5 \times 10^{-3}}\right) \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1], \quad (4.51)$$

$$\varepsilon = 1, \quad Q = 0, \quad \sigma_t = \sigma_s = 1, \quad \sigma_a = 0, \quad N = 300, \quad \text{and } t = 0.36.$$

Then, we compute the density on a $N \times N = 300 \times 300$ grid up to time $t = 0.36$ using different relaxation parameters

$$\phi_1 = \frac{h\sigma_t}{2\varepsilon^3} = \frac{10}{3} \times 10^{-4} \quad \text{and} \quad \phi_2 = \frac{1}{\varepsilon^2} = 1. \quad (4.52)$$

In the first case, the relaxation parameter satisfies the assumption of Proposition 1 and the solution is stable (see Figure 5a). Whereas in the second case, the assumption is violated and the solution starts to blow up (see Figure 5b). As a consequence, the upper bound on the relaxation parameter in Proposition 1 can in general not be substituted by the less restrictive upper bound $\phi \leq \frac{1}{\varepsilon^2}$.

5 Conclusions

In this paper, we have introduced a two-dimensional AP scheme for the linear transport equation. The linear transport equation has the diffusion equation as an analytic asymptotic limit. For AP schemes the discretization has to be chosen, such that the analytic limit is preserved at a discrete level and the scheme is uniformly stable with respect to the mean free path. Here, we used a parity-based time discretization combined with a staggered-grid spatial discretization.

We have shown for the spatial discretization has the desired asymptotic-preserving property. In particular, due to the use of staggered grids, a compact five point stencil can be achieved in the limiting discrete diffusion limit. Furthermore, the parity-based time discretization is suitable for the use of staggered grids, as the coupling between the even and odd parities reduces the number of the required unknowns. In addition, we have presented a rigorous stability analysis for the same scheme in one-dimension. This provides a condition on the relaxation parameter and a CFL condition. Finally, we have performed several numerical tests for the two-dimensional scheme, which demonstrate the AP property. Since staggered grids can easily be extended to three dimensions, there is a straightforward generalization of our method to three spatial-dimensions. Although we did not test the method, we expect that it has similar properties.

In the future, it would be worthwhile to investigate the time discretization. Since our method uses a simple time-integration method (explicit Euler method), the convergence order is in general limited to one. To maintain a second order scheme, one could use some higher order implicit-explicit (IMEX) time integration method. Another possible scope of future work is to apply staggered grids in combination with a parity-based time discretization to other kinetic equations.

Acknowledgments

S. Jin was supported partially by NSF DMS grant no. 1107291: RNMS "KI-Net" and NSFC grant no. 91330203. K. Küpper and M. Frank were funded by the Excellence Initiative of the German Federal and State Governments.

References

- [1] S. Boscarino and G. Russo. On a class of uniformly accurate IMEX Runge-Kutta schemes and applications to hyperbolic systems with relaxation. *SIAM Journal on Scientific Computing*, 31(3):1926–1945, 2009.
- [2] T. Brunner. Forms of approximate radiation transport. *SAND2002-1778*, Sandia National Laboratory, 2002.
- [3] T. Brunner and J. Holloway. Two-dimensional time dependent Riemann solvers for neutron transport. *Journal of Computational Physics*, 210(1):386–399, 2005.
- [4] R. E. Caflisch, S. Jin, and G. Russo. Uniformly accurate schemes for hyperbolic systems with relaxation. *SIAM J. Numer. Anal.*, 34(1):246–281, 1997.
- [5] P. Degond. Asymptotic-preserving schemes for fluid models of plasmas. *arXiv preprint arXiv:1104.1869*, 2011.
- [6] J. Deng. Implicit asymptotic preserving schemes for semiconductor Boltzmann equation in the diffusive regime. *Int. J. Numer. Anal. Model.*, 11(1):1–23, 2014.

- [7] G. Dimarco and L. Pareschi. Exponential Runge-Kutta methods for stiff kinetic equations. *SIAM Journal on Numerical Analysis*, 49(5):2057–2077, 2011.
- [8] F. Filbet and S. Jin. A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources. *Journal of Computational Physics*, 229(20):7625–7648, 2010.
- [9] F. Golse, S. Jin, and C. Levermore. The convergence of numerical transfer schemes in diffusive regimes I: Discrete-ordinate method. *SIAM journal on numerical analysis*, 36(5):1333–1369, 1999.
- [10] S. Jin. Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms. *J. Comput. Phys.*, 122(1):51–67, 1995.
- [11] S. Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM Journal on Scientific Computing*, 21(2):441–454, 1999.
- [12] S. Jin. Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: A review. In *Lecture Notes for Summer School on Methods and Models of Kinetic Theory(M&MKT), Porto Ercole (Grosseto, Italy), June 2010*, pages 177–216. Riv. Mat. Univ. Parma, 2012.
- [13] S. Jin and C. D. Levermore. Fully discrete numerical transfer in diffusive regimes. *Transport Theory Statist. Phys.*, 22(6):739–791, 1993.
- [14] S. Jin and C. D. Levermore. Numerical schemes for hyperbolic conservation laws with stiff relaxation terms. *J. Comput. Phys.*, 126(2):449–467, 1996.
- [15] S. Jin and D. Levermore. The discrete-ordinate method in diffusive regimes. *Transport Theory Statist. Phys.*, 20(5-6):413–439, 1991.
- [16] S. Jin, L. Pareschi, and G. Toscani. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM Journal on Numerical Analysis*, 35(6):2405–2439, 1998.
- [17] S. Jin, L. Pareschi, and G. Toscani. Uniformly accurate diffusive relaxation schemes for multiscale transport equations. *SIAM Journal on Numerical Analysis*, 38(3):913–936, 2000.
- [18] A. Klar. An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit. *SIAM journal on numerical analysis*, 35(3):1073–1094, 1998.
- [19] E. Larsen, J. Morel, and W. Miller Jr. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. *Journal of Computational Physics*, 69(2):283–324, 1987.
- [20] K. Lathrop. Ray effects in discrete ordinates equations. *Nucl. Sci. Eng.*, 32(3):357–369, 1968.
- [21] M. Lemou and L. Mieussens. A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit. *SIAM Journal on Scientific Computing*, 31(1):334–368, 2008.

- [22] J.-G. Liu and L. Mieussens. Analysis of an asymptotic preserving scheme for linear kinetic equations in the diffusion limit. *SIAM Journal on Numerical Analysis*, 48(4):1474–1491, 2010.
- [23] L. Mieussens. On the asymptotic preserving property of the unified gas kinetic scheme for the diffusion limit of linear kinetic models. *J. Comput. Phys.*, 253:138–156, 2013.
- [24] W. Miller and W. Reed. Ray-effect mitigation methods for two-dimensional neutron transport theory. *Nucl. Sci. Eng.*, 62:391–411, 1977.
- [25] L. Pareschi and G. Russo. Implicit-explicit Runge-Kutta schemes for stiff systems of differential equations. *Recent trends in numerical analysis*, 3:269–289, 2000.
- [26] B. Seibold and M. Frank. StaRMAP—A second order staggered grid method for spherical harmonics moment equations of radiative transfer. *ACM Transactions on Mathematical Software*, 41, 2014.
- [27] L. Trefethen. Finite difference and spectral methods for ordinary and partial differential equations, unpublished text, 1996. available at <http://people.maths.ox.ac.uk/trefethen/pdetext.html>.
- [28] K. Xu and J.-C. Huang. A unified gas-kinetic scheme for continuum and rarefied flows. *J. Comput. Phys.*, 229(20):7747–7764, 2010.