TOPOLOGICAL MODELS FOR EMERGENT DYNAMICS WITH SHORT-RANGE INTERACTIONS

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Abstract. We introduce a new class of models for emergent dynamics. It is based on a new communication protocol which incorporates two main features: short-range kernels which restrict the communication to local geometric balls, and anisotropic communication kernels, adapted to the local density in these balls, which form *topological neighborhoods*. We prove flocking behavior — the emergence of global alignment for regular, non-vacuous solutions of such models. The (global) regularity and hence unconditional flocking of the one-dimensional model is proved via an application of a De Giorgi-type method. To handle the singular kernels used for geometric and topological communication, we develop a new analysis for *local* fractional elliptic operators, interesting for its own sake, encountered in the construction of our class of models.

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1. Introduction and statement of main results

1.1. **A brief overview on emergent dynamics.** A fascinating aspect of collective dynamics is *self-organization*: long-range, higher order organized patterns emerge from an underlying dynamics driven by short-range interactions. This type of emergent dynamics is found in a wide variety of biological, social, and technological contexts. We investigate this phenomena in the context of canonical models for flocking and swarming. The key feature in these models is 'environmental averaging', where N agents — of birds, insects, fish etc., align their velocities $\{\mathbf{v}_i(t)\}_{i=1}^N \in \mathbb{R}^n$ (or their orientations $\{s\boldsymbol{\omega}_i(t)\}_{i=1}^N$ with fixed speed s), by averaging local gradients over the environment of neighboring agents [39, 1, 49, 60, 3, 22, 23, 43]:

$$\mathbf{v}_i(t+\Delta t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)\mathbf{v}_j(t), \qquad \sum_j a_{ij}(t) = 1.$$

Rearranging terms using a frequency $\lambda = 1/\Delta t$ which is being kept fixed while letting $\Delta t \downarrow 0$ for the rest, we arrive at the dynamical system

$$\dot{\mathbf{v}}_i = \lambda \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\mathbf{v}_j - \mathbf{v}_i), \qquad \sum_j a_{ij}(t) = 1. \tag{1.1}$$

The essential notion of a 'neighborhood' depends on the active weights $\mathcal{N}_i = \{j : a_{ij}(t) > 0\}$. A general class of such models utilize a pairwise interaction, $\phi(\mathbf{x}_i, \mathbf{x}_j)$, using an communication kernel $\phi(\cdot,\cdot) \geqslant 0$,

$$\dot{\mathbf{v}}_i = \lambda \sum_{j\in\mathcal{N}_i} \phi\big(\mathbf{x}_i(t), \mathbf{x}_j(t)\big)(\mathbf{v}_j - \mathbf{v}_i), \qquad \dot{\mathbf{x}}_i = \mathbf{v}_i \tag{1.2}$$

Different models distinguish themselves with different choices of communication kernels. The most notable examples found in the literature employ *radial* communication kernels, $\psi(|\mathbf{x}_j - \mathbf{x}_j|)$ which are normalized by proper scaling factor $deg_i(t)$,

$$a_{ij}(t) = \frac{\psi_{ij}(t)}{deg_i(t)}, \qquad \psi_{ij}(t) := \psi(|\mathbf{x}_j(t) - \mathbf{x}_j(t)|). \tag{1.3a}$$

Thus, the collective dynamics in these models is driven by *geometric neighborhoods*, where agent "i" interacts with those neighbors dictated by the support of ψ,

$$\mathcal{N}_i = \{j \mid |\mathbf{x}_j(t) - \mathbf{x}_i(t)| \in \operatorname{supp}\psi\}. \tag{1.3b}$$

Observe that the geometric neighborhoods are, in general, time dependent, $\mathcal{N}_i = \mathcal{N}_i(t)$. The collective dynamics (1.3) is driven by pairwise distances taken in one of two generic configurations — the whole space $\{\mathbf{x}_i\} \in \mathbb{R}^n$ or over the torus $\{\mathbf{x}_i\} \in \mathbb{T}^n$. The problem of handing boundaries is mostly open.

The communication kernels $\phi(\cdot,\cdot)$ or $\psi(\cdot)$ are in general unknown: their approximate shape is either derived empirically [18, 2, 17, 16, 21, 11], or learned from the data [8, 41], or postulated based on phenomenological arguments, combining the first two, [61, 5, 4]. Here are three prototype radial examples. A first example is a simple averaging over finite-range ball corresponding to $\psi(r) = \chi_{R_0}(r)$ as in Vicsek model [60, 61],

$$\begin{cases} \mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t+\Delta t), \\ \mathbf{v}_i(t+\Delta t) = s\dfrac{\sum_{j:|\mathbf{x}_j-\mathbf{x}_i|<R_0} \mathbf{v}_j(t)}{\left|\sum_{j:|\mathbf{x}_j-\mathbf{x}_i|<R_0} \mathbf{v}_j(t)\right|} + \text{perturbation}. \end{cases} \tag{1.4}$$

A second example is of a communication kernel with a decreasing profile, found in Cucker-Smale model [22, 23],

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\lambda}{N}\displaystyle\sum_{j=1}^N \psi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i), \qquad \psi(r) = \dfrac{1}{(1+r^2)^\beta}. \end{cases} \tag{1.5}$$

We remark that setting a uniform degree $deg_i = \dfrac{N}{\lambda}$ with $\lambda = \dfrac{N}{max_i \sum_k \psi_{ik}}$ yields the Cucker-Smale scaling in (1.5),

$$a_{ij} = \frac{\psi_{ij}}{deg_i} = \frac{\psi(|\mathbf{x}_i - \mathbf{x}_j|)}{N}. \tag{1.6}$$

Note that $\sum_{j\neq i} a_{ij} \leqslant 1$ can be complemented $a_{ii} := 1 - \sum_{j\neq i} a_{ij} \geqslant 0$ to form the convex combination sought in (1.1). This uniform scaling has the drawback that it distorts the dynamics far from equilibrium. Motsch and Tadmor advocated in [43] the more general normalization based on a graph degree,

$$deg_i(t) = \sum_{k=1}^N \psi(|\mathbf{x}_i(t) - \mathbf{x}_k(t)|), \qquad \sum_j \frac{\psi_{ij}}{deg_i} = 1, \tag{1.7}$$

as appropriate scaling adapted to both scenarios — close to equilibrium where $\psi_{ij} \approx \psi_0$, as well as far from equilibrium where deg_i adapts itself to the number of significant near-by neighbors. The Motsch-Tadmor scaling amounts to non-radial interactions

$$a_{ij} = \phi(\mathbf{x}_i, \mathbf{x}_j), \qquad \phi(\mathbf{x}_i, \mathbf{x}_j) = \frac{\psi(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_k \psi(|\mathbf{x}_i - \mathbf{x}_k|)}. \tag{1.8}$$

Finally, we mention the example of a Coulomb-like singular interaction kernels $\psi(r) = r^{-\beta}$,

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\lambda}{N}\displaystyle\sum_{j=1}^N \dfrac{\mathbf{v}_j - \mathbf{v}_i}{|\mathbf{x}_i - \mathbf{x}_j|^\beta}. \end{cases} \tag{1.9}$$

The role of the singularity here is to emphasize interactions with nearby neighbors over those farther away. Peszek and co-workers [15, 46, 47] proved that with strong enough singularity $\beta \geqslant 1$, no collisions occur in finite time, proving that classical solutions of $(1.9)_\beta$ exist globally in time, while for $\beta < 1$ collisions are possible.

Since the precise form of ψ is in general not known, it is therefore imperative to understand how general classes of ψ's affect the large-time, large-crowd dynamics. It is here that we make a distinction between *long-range* and *short-range* interactions.

Long-range interactions. Here, all agents are located inside the support of ψ so that every agent is in direct communication with every other agent. In particular, if

$$\int^\infty \psi(r)\,\mathrm{d}r = \infty \tag{1.10}$$

then all agents cannot escape a finite ball, thus forming a 'flock':

$$\text{there exists a finite } D_\infty \text{ such that } \operatorname{diam}\{\mathbf{x}_i(t)\}_{i=1}^N < D_\infty \text{ for all } t > 0, \tag{1.11a}$$

and consequently, the alignment dynamics (1.1),(1.3) enforces the agents to 'aggregate' around a limiting velocity, $\mathbf{u}_\infty \in \mathbb{R}^n$, yielding the following flocking behavior:

$$\text{there exists } \mathbf{u}_\infty \in \mathbb{R}^n \text{ such that } \max_i |\mathbf{v}_i(t) - \mathbf{u}_\infty| \stackrel{t\to\infty}{\longrightarrow} 0. \tag{1.11b}$$

The unconditional flocking stated in (1.11) for long-range interactions (1.10) holds for all initial configurations. It goes back to Cucker-Smale [22, 23] in their study of *symmetric* interactions (1.6) with the special influence function $\psi(r) = (1+r^2)^{-\beta}$ for $\beta \leqslant 1/2$. It was later extended by Ha and Tadmor [31] for general long-range ψ's (1.6),(1.10), by Ha and Liu [30] (see also [29]) based on a Lyapunov functional for the symmetric scaling (1.6),(1.10) and by Motsch and Tadmor [44] for general, possibly *non-symmetric* interactions (1.2).

Short-range interactions. In more realistic scenarios, the communication among agents is driven by local influence function, so that ψ vanishes when two agents which are far apart try to communicate: $\operatorname{supp}\psi \subset B_{2R_0}$ and $2R_0 < diam\{\mathbf{x}_i(t)\}_{i=1}^N$. Thus, in *short-range interactions* agent "i" interacts with $N_i \ll N$ neighbors inside the local ball $\mathcal{N}_i = \{j \,\big|\, |\mathbf{x}_j - \mathbf{x}_i| \leqslant 2R_0\}$. In this case agents do not necessarily communicate with every other agent and flocking behavior requires their configuration to be at least connected. Indeed, in [44] it was proved that the flocking behavior (1.11) of (1.1) follows from *propagation of connectivity* of the underlying graph associated with the symmetric adjacency matrix, $\{a_{ij}\}$

$$\int^\infty \kappa_2(\Delta a)(t)\,\mathrm{d}t = \infty, \qquad (\Delta a)_{ij}(t) := -(1-\delta_{ij})a_{ij}(t) + \delta_{ij}\sum_{k\neq i} a_{ik}(t). \tag{1.12}$$

Here, κ_2 is the Fiedler number — the second eigenvalue associated with the graph Laplacian Δa. In the case of geometric-based neighborhoods (1.6), $Na_{ij} = \psi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)$, we have $\kappa_2(\Delta a)(t) \gtrsim \psi(V_0 t)$ for $t \gg 1$, and we recover the flocking behavior of long-range interactions (1.10): they maintain uniform connectivity since the neighborhoods $\mathcal{N}_i$ involve the whole crowd and remain independent of time. The question of flocking behavior with short range interactions is more subtle, however, since the graph connectivity associated with $\{\psi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)_{ij}\}$ may break down at a finite time e.g., [44]: in this case, the *time-dependent covering* $\cup_{i=1}^N \mathcal{N}_i(t)$ may be unstable for a finite N, and one needs to *assume* the persistence of connectivity for all time [34] or at least close enough to a constant state, so close that it does not allow connectivity to be lost [59, 35]. We therefore turn to study the mean-field behavior associated with large crowds, $N \gg 1$.

Large crowd dynamics. Assuming that the empirical measure associated with (1.2), $\mu_N(t,\mathbf{x},\mathbf{v}) = \sum_i m_i \delta_{\mathbf{x}_i(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_i(t)}(\mathbf{v})$, admits a mean-field limit $\mu_N \to f(t,\mathbf{x},\mathbf{v})$ for 'large-crowds' as $N \to \infty$, the latter is governed by the Vlasov-type kinetic equation, consult appendix 5.1 below

$$f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot Q(f,f) = 0, \tag{1.13a}$$

where $Q(\cdot,\cdot)$ is the bi-linear form which captures all the underlying binary interactions with a general pairwise communication kernel $\phi = \phi(\mathbf{x},\mathbf{y})$,

$$Q(f,f)(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^{2n}} \phi(\mathbf{x},\mathbf{y})(\mathbf{w}-\mathbf{v}) f(t,\mathbf{y},\mathbf{w}) f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{y}. \tag{1.13b}$$

In the radial case, $\phi(\mathbf{x},\mathbf{y}) = \dfrac{\psi(|\mathbf{x}-\mathbf{y}|)}{deg(t,\mathbf{x})}$, where $deg(t,\mathbf{x}) \equiv 1/\lambda$ corresponds to uniform Cucker-Smale scaling (1.6), and $deg(t,\mathbf{x}) := \displaystyle\int_{\mathbb{R}^{2n}} \psi(|\mathbf{x}-\mathbf{y}|) f(t,\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{y}$ realizes the Motsch-Tadmor scaling (1.7).

The passage to the limit for bounded radial kernels was justified by Ha and Tadmor [31] using a formal BBGKY limiting procedure, by Ha and Liu [30] in the sense of a proper weak limit and was established in Carrillo et al [12] using a kinetic variant of exponential flocking. The mean field limit for singular kernels $\psi(r) = r^{-\beta}$ for $\beta < \frac{1}{2}$ was proved in [45].

Seeking a monokinetic ansatz, $f(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\delta(\mathbf{v}-\mathbf{u}(t,\mathbf{x}))$, one recovers the macroscopic density and momentum $(\rho,\rho\mathbf{u})$ from the first two $\mathbf{v}$-moments of (1.13): the large crowd alignment-based dynamics (1.1),(1.6) or (1.1),(1.8) is captured by the following *hydrodynamic description*, [31, 13],

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \lambda(t,\mathbf{x}) \displaystyle\int_{\mathbb{R}^n} \psi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(t,\mathbf{y})-\mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{cases} \tag{1.14a}$$

Here $\lambda(t,\mathbf{x})$ captures both scaling:

$$\lambda(t,\mathbf{x}) = \begin{cases} \lambda, & \lambda \equiv Const. \text{ corresponding to (1.6)}, \\ \dfrac{1}{\mathrm{deg}(t,\mathbf{x})}, & deg(t,\mathbf{x}) = \psi \star \rho(t,\mathbf{x}) \text{ corresponding to (1.8)}. \end{cases} \tag{1.14b}$$

The particular case of uniform scaling of Cucker-Smale (which we set $\lambda \equiv 1$) leads to symmetric interactions that will be at the center of attention in this present paper,

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{\mathbb{R}^n} \psi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(t,\mathbf{y})-\mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{cases} \tag{1.15}$$

Its rigorous derivation was established by Figalli and Kang in [26].
The dynamics (1.15) is subject to prescribed initial conditions, $(\rho_0,\mathbf{u}_0)$, with two main configurations: either compactly supported density $diam\,\{supp\,\rho_0\} \leqslant D_0$ in $\mathbb{R}^n$ or over the torus $\mathbb{T}^n$.

Remark 1.1 (**beyond the pressureless equations**). System (1.15) is governed by the competition between nonlinear convection and alignment based on symmetric interactions. Symmetry is derived from the uniform Cucker-Smale scaling and additional pressure term is avoided due to the mono-kinetic ansatz. As noted earlier, a more realistic scaling is offered by Motsch-Tadmor (1.8). Karper, Mellet and Trivisa [37, 38] combined a long range CS scaling $\mathrm{deg}(\mathbf{x}) = 1/\lambda$ with short range MT scaling (1.7), while passing to hydrodynamic limit in the presence of strong local alignment and strong noise: the resulting monokinetic ansatz is then replaced by a Maxwellian distribution $f(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\exp\{-\frac{1}{2}|\mathbf{v}-\mathbf{u}(t,\mathbf{x})|^2\}$, resulting in the additional forcing of *macroscopic pressure* $-\nabla\rho$,

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\rho = \int_{\mathbb{R}^n} \psi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(t,\mathbf{y})-\mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

We shall focus on the the pressures-less model (1.15), leaving the treatment of pressure to future work. As in the agent-based description, the hydrodynamic behavior of flocking depends on the features of the communication kernel, ψ, where we distinguish between two main classes — long-range and short-range communication kernels.

Hydrodynamic flocking — long range interactions. We quantify the long range interactions, assuming a communication kernel ϕ with enough tail mass such that,

$$V_0 < M_0 \int_{D_0}^{\infty} \min_{s\leqslant r} \psi(s)\,\mathrm{d}r, \qquad M_0 := \int \rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{1.16}$$

where D_0 and V_0 are the initial finite diameters of non-vacuum density and velocity, $(\rho_0, \mathbf{u}_0)$. The flocking behavior of (1.14) is captured by the statement "smooth solutions must flock".

Theorem 1.2 ([57, 33])**.** *If $(\rho(t,\cdot), \mathbf{u}(t,\cdot)) \in (L^\infty \cap L^1)\times W^{1,\infty}$ is a global strong solution of (1.14) subject to initial data $(\rho_0, \mathbf{u}_0)$ such that (1.16) holds, then $(\rho, \mathbf{u})$ converges to a flock at an exponential rate, namely — (compare (1.11)),*

$$\text{the support of } \rho(t,\cdot) \text{ remains within a finite diameter } D_\infty, \tag{1.17a}$$

and there exist a limiting velocity $\mathbf{u}_\infty$ and $\eta > 0$ such that

$$\max_x |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}_\infty| \leqslant 2V_0 e^{-\eta t} \to 0. \tag{1.17b}$$

In the particular case that ψ satisfies the 'fat tail' condition (1.10) then (1.16) holds for all V_0's, and unconditional flocking follows for all finite initial diameters.

Remark 1.3 (**on the limiting velocity $\mathbf{u}_\infty$**). In the symmetric case, (1.15), the limiting velocity is given in terms of the conserved total mass and momentum

$$\mathbf{u}_\infty = \frac{\mathbf{P}_0}{M_0}, \qquad M = \int \rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv M_0, \quad \mathbf{P} = \int \rho\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv \mathbf{P}_0.$$

The characterization of $\mathbf{u}_\infty$ in the general non-symmetric case (1.14a), is wide open.

The conditional statement for long range interactions shifts the burden of proving their flocking behavior to the regularity theory. Here we make a further distinction between bounded and singular ψ's.

For *bounded kernels*, global regularity in dimension $n = 1$ holds if and only if the initial configuration satisfies the threshold condition, $u_0' \geqslant -\psi \star \rho_0$, [14]. A sufficient threshold condition in dimension $n = 2$ was given by He and Tadmor [33] (see also [57]) via spectral condition on the 2×2 symmetric tensor $S := \frac{1}{2}(\nabla\mathbf{u} + \nabla^\top \mathbf{u})$: its initial trace $\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 \geqslant -\psi \star \rho_0$ (—a precise extension of the 1D case), and the spectral gap $\kappa_2(S_0) - \kappa_1(S_0) \lesssim M_0$. The question of threshold regularity for dimension $n \geqslant 3$ is wide open. Global regularity (and hence flocking behavior) of (1.15) for any dimension but for small data was proved in [28] in higher order Sobolev spaces[1], $|\mathbf{u}|_{H^{s+1}} < \varepsilon_0(|\rho_0|_{H^s})$.
The regularity and flocking behavior of the hydrodynamic limit (1.15) with *singular kernels* was studied by Poyato and Soler [48] for weakly singular kernels in the range $0 < \beta < n$, and by the authors [55, 53, 54] and Do et. al. [25] for strongly singular kernels, $\beta = n + \alpha$,

[1]Throughout the paper we denote by $H^s(\mathbb{T}^n)$ the L^2-based Sobolev space of regularity s, and by $H^s_0(\mathbb{T}^n)$ the space of mean-zero functions. We use $|\cdot|_X$ to denote classical norms, and a shorter notation for the Lebesgue spaces, $|\cdot|_p = |\cdot|_{L^p}$.

$0 < \alpha < 2$. In the latter case, the system (1.15) is endowed with a fractional parabolic diffusion structure, $-(-\Delta)^{\alpha/2}$, associated with the periodic kernel

$$\psi(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^n} \frac{1}{|\mathbf{x}+2\pi\mathbf{k}|^{n+\alpha}}, \qquad 0<\alpha<2. \tag{1.18}$$

Strong singularity made it possible to prove, at least in the one-dimensional case, *unconditional flocking behavior*, independent of any initial threshold. We quote here the main result of [55, 54] as it will also be echoed in the statements of this present paper.

Theorem 1.4. *Consider the system* (1.15), (1.18), $0<\alpha<2$ *on the 1-dimensional torus* $\mathbb{T}$, *subject to* $(\rho_0, u_0) \in H^{s-1+\alpha} \times H^s$, $s \geqslant 4$ *away from the vacuum. It admits a unique global solution,* $(\rho, u) \in L^\infty([0,\infty); H^{s-1+\alpha} \times H^s)$, *which converges exponentially fast to a flocking traveling wave,* $\bar\rho = \rho_\infty(x - tu_\infty) \in H^{s-1+\alpha}$ *with a speed* u_∞, *so that*

$$|u(t,\cdot) - u_\infty|_{H^s} + |\rho(t,\cdot) - \bar\rho(t)|_{H^{s-1}} \leqslant Ce^{-\eta t}, \quad t>0, \qquad u_\infty := \frac{P_0}{M_0}. \tag{1.19}$$

The question of regularity for strongly singular kernels $\phi(r) = r^{-(n+\alpha)}$ in $n>1$ dimensions is wide open, with the only exceptions of recent small initial data results in [52] for Hölder spaces, $V_0 \lesssim (1 + |\rho_0|_{W^{3,\infty}} + |\mathbf{u}_0|_{W^{3,\infty}})^{-n}$ with $0<\alpha<2$, and in [24] for small Besov data $|\mathbf{u}_0|_{B^{2-\alpha}_{n,1}} + |\rho_0 - 1|_{B^1_{n,1}} \leqslant \varepsilon$ with $\alpha \in (1,2)$.

Hydrodynamic flocking — short range interactions. The class of singular kernels $\psi = r^{-\beta}$ offers a communication framework which emphasizes short-range interactions over long-range interactions, yet their global support reflects global communication. In fact, weakly singular kernels, $\beta < n$, satisfy the fat tail condition (1.10) which characterizes global communication, and unconditional flocking follows by considering their cut-off $\min\{M, r^{-\beta}\}$. The class of strongly singular kernels, however, $\beta = n+\alpha$, $0<\alpha<2$, demonstrates hydrodynamic flocking for thinner tails, beyond the framework of (1.10). Still, the infinite support of this class of strongly singular kernels reflects global communication which in turn is responsible for hydrodynamic flocking.

This brings us back to the original question alluded to at the beginning, namely — understanding self-organization driven by a *purely local communication protocol*. This is the question we address in our present work, in the context of unconditional alignment for the hydrodynamics system (1.15),

$$\psi(r) = \frac{h(r)}{r^{n+\alpha}} \quad \text{with a smooth cutoff } \frac{1}{\Lambda}\chi_{R_0}(r) \leqslant h(r) \leqslant \Lambda\chi_{2R_0}(r). \tag{1.20}$$

It provides a first fundamental step in our understanding of emergent phenomena in collective dynamics driven by local communication kernels, where $2R_0 \ll supp\, \rho_0$.

Let us first point out that the regularity of (1.15) with the localized kernel (1.20) holds in exact same cases as for the full kernel due to the fact that the difference $(1-h(r))r^{-(n+\alpha)}$ contributes a smooth source term. In particular, global regularity of the $n=1$-dimensional case follows from theorem 1.4, see [40]. It has been however an open question whether the emergence of hydrodynamic flocking survives this kind of localization. The situation is analogous to the scenario of discrete crowd with short range communication, (1.2), which may fail to flock due to finite-time loss of graph connectivity, (1.12). At the level of hydrodynamic description, lack of connectivity manifests itself as 'thinning' of crowd in low density sub-regions of *supp* $\rho(t,\cdot)$, which in turn may prevent hydrodynamic flocking (1.17). Indeed, in the extreme case of a vacuous sub-region, it does not exert any alignment on its neighborhood hence the dynamics is reduced to inviscid Burgers-type blowup [58], thereby demonstrating necessity of the no-vacuum assumption. This brings us to the following local version of the statement "smooth solutions must flock" for non-vacuous solutions of alignment dynamics associated with a general class of *local* singular communication kernels.

Proposition 1.5 (**Smooth solutions must flock — local kernels).** *Consider the class of local singular kernels,* $\psi(|\mathbf{x}-\mathbf{y}|)$*, such that*

$$\psi(r) = \frac{h(r)}{r^{n+\alpha}}, \quad 0<\alpha<2, \quad \frac{1}{\Lambda}\chi_{R_0}(r) \leqslant h(r) \leqslant \Lambda \chi_{2R_0}(r). \tag{1.21}$$

Let $(\rho(t,\cdot),\mathbf{u}(t,\cdot))$ *be a global strong solution of the corresponding alignment dynamics over the torus* $\mathbb{T}^n$

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{\mathbb{T}^n} \psi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{cases}$$

Assume that

$$\frac{c}{\sqrt{1+t}} \leqslant \rho(t,\cdot) \leqslant C, \qquad C>c>0. \tag{1.22}$$

Then the solution converges to a flock at algebraic rate, namely — there exist a limiting velocity $\mathbf{u}_\infty$ *and* $\eta>0$ *such that*

$$\int_{\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{1}{2M_0\,t^\eta}, \qquad \mathbf{u}_\infty = \frac{\mathbf{P}_0}{M_0}. \tag{1.23}$$

Proposition 1.5 follows from a flocking statement for a general local symmetric singular kernels[2], $\phi(\mathbf{x},\mathbf{y}) = \phi(\mathbf{y},\mathbf{x}) \simeq |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\chi_{R_0}(|\mathbf{x}-\mathbf{y}|)$, consult theorem 3.1 below. It provides a general framework for the flocking behavior of smooth, *non-vacuous* solutions for alignment dynamics driven by local, singular communication kernels. Here, the precise decay rate of the density $\min\rho(t,\cdot)$ is at the heart of matter: our requirement for an *apriori* bound $\rho(t) \geqslant \dfrac{c}{\sqrt{1+t}}$ is too restrictive to verify the best available one-dimensional 'thinning' rate one can get on the density is, [53], $\rho(t) \geqslant \dfrac{c}{1+t}$.
To address this difficulty, we now introduce a new local communication protocol, interesting for its own sake, which tames the required decay rate of the density by adapting itself to thin sub-regions.

1.2. **A new paradigm for collective dynamics – topological kernels.**

We introduce a new communication protocol based on the principle that

information between agents spreads faster in regions of lower density

(see Section 2.1 for more detailed discussion). To realize this principle we begin by revisiting the underlying agent-based description.

Agent-based dynamics with local topological interactions. Given a pair of agents located at $\mathbf{x}_i,\mathbf{x}_j\in\mathbb{R}^n$, we fix their intermediate *region of communication* $\Omega(\mathbf{x}_i,\mathbf{x}_j)\subset\mathbb{R}^n$.

[2] We denote $A\simeq B$ if there exist constants $0<c<C$ such that $cB\leqslant A\leqslant CB$.

In the one-dimensional case, it is taken simply as the closed interval $\Omega(x,y)=[x,y]$; in the multi-dimensional case, we choose a conical region outlined in section 2.2. The communication between agents now depends on both — their geometric distance in $\mathbb{R}^n$ (and respectively in $\mathbb{T}^n$),

$$r(\mathbf{x}_i,\mathbf{x}_j)=|\mathbf{x}_i-\mathbf{x}_j|, \tag{1.24a}$$

and by probing the environment — agents react to the average density in their region of communication

$$\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j):=\left(\frac{\#\{\mathbf{x}_k\,|\,\mathbf{x}_k\in\Omega(\mathbf{x}_i,\mathbf{x}_j)\}}{N}\right)^{\frac{1}{n}}. \tag{1.24b}$$

We end up with alignment dynamics based on an anisotropic (yet symmetric) communication $\phi(\mathbf{x}_i,\mathbf{x}_j)=\psi_1(r(\mathbf{x}_i,\mathbf{x}_j))\times\psi_2(\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j))$

$$\begin{cases}\dot{\mathbf{x}}_i=\mathbf{v}_i,\\ \dot{\mathbf{v}}_i=\dfrac{\lambda}{N}\displaystyle\sum_{j=1}^N\phi(\mathbf{x}_i,\mathbf{x}_j)(\mathbf{v}_j-\mathbf{v}_i), \qquad \phi(\mathbf{x}_i,\mathbf{x}_j)=\psi_1(r(\mathbf{x}_i,\mathbf{x}_j))\times\psi_2(\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j)).\end{cases} \tag{1.25}$$

Remark 1.6. (**Topological neighborhoods**). We have at our disposal the choices of kernels ψ_1 and ψ_2. For the geometric part, we use the singular kernel (1.20),

$$\psi_1(r)=\frac{h(r)}{r^{n+\alpha-\tau}}, \qquad \frac{1}{\Lambda}\chi_{R_0}(r)\leqslant h(r)\leqslant\Lambda\chi_{2R_0}(r). \tag{1.26a}$$

The smooth cut-off $h(r)$ guarantees that communication is localized within isotropic balls of radius $\leqslant 2R_0$. For the density-probing part we also use a singular kernel which enhances communication in regions of low density,

$$\psi_2(\mathrm{d})=\frac{1}{\mathrm{d}^\tau}, \tag{1.26b}$$

Indeed, the singularity of ψ_2 means that agent $\mathbf{x}_i$ gives strong preference for the communication with its *nearest* agents, $\{\mathbf{x}_j\,|\,\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j)\sim N^{-\frac{1}{n}}\}$, over the increased interference in communication with agents farther away, $\{\mathbf{x}_j\,|\,\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j)\lesssim 1\}$. The net effect of probing low density using such singular kernels is that communication is dictated by the *number* of nearest agents rather than geometric proximity, [32, 6, 7]. Accordingly, we refer to $\mathrm{d}(\mathbf{x}_i,\mathbf{x}_j)$ as *topological (quasi-)distance*. This is consistent with the established terminology in experimental literature, which refers to such topological communication in flocking birds [18, 2, 16] and in human interaction in pedestrian dynamics [50].

Hydrodynamic alignment with local topological interactions. We consider the alignment dynamics

$$\begin{cases}\rho_t+\nabla\cdot(\rho\mathbf{u})=0,\\ \mathbf{u}_t+\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u}=\displaystyle\int_{\mathbb{R}^n}\phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \phi(\mathbf{x},\mathbf{y})=\psi_1(r)\times\psi_2(\mathrm{d}_\rho).\end{cases} \tag{1.27a}$$

Here, the communication kernel $\phi(\mathbf{x},\mathbf{y})=\psi_1(r)\times\psi_2(\mathrm{d}_\rho)$ depends on two main features: $\psi_1(r(\mathbf{x},\mathbf{y}))$ reflects the dependence on *geometric distance* in $\mathbb{R}^n$ (and respectively in $\mathbb{T}^n$),

$r(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$, and $\psi_2(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))$ reflects the use of "mass" as a *topological measure of distance* between agents

$$\mathrm{d}_\rho(\mathbf{x},\mathbf{y}) := \left[\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}\right]^{\frac{1}{n}} \quad \text{with } \Omega(\mathbf{x},\mathbf{y}) \text{ given in (2.3).} \tag{1.27b}$$

Formally, the passage from the agent-based description (1.25) to (1.27) can be accomplished by the topological distance (1.24b) in terms of a proper limit of the empirical distribution $\mu_t^N(\mathbf{x},\mathbf{v}) = \frac{1}{N}\sum_j \delta_{\mathbf{x}_j(t)}(\mathbf{x})\otimes\delta_{\mathbf{v}_j(t)}(\mathbf{v})$

$$\mathrm{d}(\mathbf{x},\mathbf{y}) = \left(\mu_t^N(\Omega(\mathbf{x},\mathbf{y}))\right)^{\frac{1}{n}} \mapsto \mathrm{d}_\rho(\mathbf{x},\mathbf{y}) \ \text{ as } \ N\to\infty,$$

which in turn, recovers the full communication kernel $\phi(\mathbf{x},\mathbf{y}) = \psi_1(r(\mathbf{x},\mathbf{y}))\psi_2(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))$. Taking the $\mathbf{v}$-moments of the kinetic formulation (1.13) we arrive at the alignment dynamics (1.27a); consult appendix 5.1. With the choice of singular kernels (1.26) we finally end up with the hybrid geometric-topological kernel given by

$$\phi(\mathbf{x},\mathbf{y}) = \frac{h(r(\mathbf{x},\mathbf{y}))}{(r(\mathbf{x},\mathbf{y}))^{n+\alpha-\tau}} \times \frac{1}{(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))^{\tau}}, \qquad \tau>0, \quad 0<\alpha<2. \tag{1.27c}$$

Here τ represents the strength of the topological component within the kernel with 'total' singularity of order $n+\alpha$: $\phi(\mathbf{x},\mathbf{y}) \simeq |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}$. Note that the kernel is properly local, non-convolutive, and though ϕ is symmetric $\phi(\mathbf{x},\mathbf{y}) = \phi(\mathbf{y},\mathbf{x})$, the total action of $K(\mathbf{x},\mathbf{y},t) := \phi(\mathbf{x},\mathbf{y})\rho(\mathbf{y})$ is not. The proper notion of the non-symmetric singular alignment action, $\mathscr{L}_\phi(f) = \int \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$, is discussed in section 2.4. As before, the flocking behavior of the so-called (τ,α)-model (1.27) will proceed in two main steps.

STEP 1: **Smooth solutions must flock**. Consider the topological model based on the singular (τ,α)-kernel (1.27) on $\mathbb{T}^n$. We prove unconditional alignment, $|\mathbf{u}(t,\cdot)-\mathbf{u}_\infty|_\infty \to 0$, of any global smooth solution $(\rho,\mathbf{u})$ with non-vacuous lower-density bound[3]

$$\rho(t,\mathbf{x}) \geqslant \frac{c}{(1+t)^{\beta_0}}, \qquad \beta_0 := \min\left\{1, \frac{n}{2n-\tau}\right\}.$$

Observe that the case of purely geometric interactions, $\tau = 0$ 'recovers' the restricted lower-bound $\rho(t,\cdot) \gtrsim 1/\sqrt{1+t}$ encountered before in (1.22). But with the presence of topological kernel of order $\tau \geqslant n$, unconditional flocking for smooth solutions follows from the relaxed lower-bound $\rho(t,\cdot) \gtrsim 1/(1+t)$.

The proof, given in theorem 3.2 below, traces the propagation of information between the extreme values of (the components of) $\mathbf{u}(t,\cdot)$, which are most susceptible to breakup since they can no longer rely on distant communication. Instead, we introduce a new method of sliding averages, in which we measure how far $\mathbf{u}(t,\mathbf{x})$ deviates from its average over the *local* balls $B(\mathbf{x},r)$, $r\leqslant R_0$, using a density-weighted Campanato class. For some algebraic sequence of times $t_n\to\infty$, these deviations are proved to be small. At the same time, we show that overwhelmingly, $\mathbf{u}(t,\mathbf{x})$ stays close to its extreme values near the critical points where these values are attained. To achieve this, we estimate, in terms of the mass-measure $\mathrm{d}m_t = \rho\,\mathrm{d}\mathbf{x}$, the conditional probability of an unlikely event of $\mathbf{u}$ being far from its extremes: it is here that the topological-based alignment in the (τ,α) interaction kernel (1.27c) plays a key role. We end up with a (finite) overlapping chain of non-vacuous balls to connect any two points and by chain estimates, the fluctuations of $\mathbf{u}(t,\cdot)$ are shown to decay uniformly in

[3] And with an additional integrability condition, consult theorem 3.2 for precise details.

time. This explains the emergence of global alignment from short-range interactions which, to the best of our knowledge, is the first result of its kind.

STEP 2: **Global regularity: drift-diffusion beyond symmetric kernels**. It remains to show that (1.27) admit global smooth solutions. In section 4 we prove the global regularity of the one-dimensional (τ,α) model over $\mathbb{T}$ with topological interaction kernel of any order $0 \leqslant \tau \leqslant \alpha$ (and with a small initial data condition if $\tau > \alpha$). Once such result is established the lower bound on the density $\rho(t,\cdot) \gtrsim 1/(1+t)$ follows automatically in 1D, consult Lemma 4.6, and hence we obtain unconditional flocking for a range of (τ,α)-models.

To elaborate in more detail, we note that both density and momentum equations in (1.27a) fall under a general class of *parabolic drift-diffusion* equations,

$$u_t + \mathbf{b}\cdot\nabla_{\mathbf{x}} u = \int K(\mathbf{x},\mathbf{y},t)(u(\mathbf{y})-u(\mathbf{x}))\,\mathrm{d}\mathbf{y} + f,$$

with (a priori) rough coefficients, $\mathbf{b}$, and with a proper singular local kernels

$$\frac{1}{\Lambda|\mathbf{x}-\mathbf{y}|^{1+\alpha}}\chi_{R_0}(|\mathbf{x}-\mathbf{y}|) \leqslant K(\mathbf{x},\mathbf{y},t) \leqslant \frac{\Lambda}{|\mathbf{x}-\mathbf{y}|^{1+\alpha}}\chi_{2R_0}(|\mathbf{x}-\mathbf{y}|)$$

Regularity theory for equations of this type had a rapid development in recent years due to breakthroughs in understanding of the non-local structure of the fractional Laplacian, see Caffarelli et al [9, 10], Silverstre et al [56, 51], Mikulevicius and Pragarauskas [42], and local jump processes in Chen et. al. [19] and the references therein. Any of these regularity results requires, however, the symmetry of the kernel $K(\cdot,\cdot,t)$ which we lack in the present framework: thus, the velocity $\mathbf{u}$ in our topological model (1.27a) is governed by drift-diffusion associated with kernel $K(\mathbf{x},\mathbf{y}) = \phi(\mathbf{x},\mathbf{y})\rho(\mathbf{y})$; while $\phi(\cdot,\cdot)$ is symmetric, K is not. Similarly, the same dynamics expressed in terms of the momentum, $\mathbf{m} := \rho\mathbf{u}$ or the density, consult (4.17) and respectively (4.16), encounters the non-symmetric kernel $K(\mathbf{x},\mathbf{y}) = \phi(\mathbf{x},\mathbf{y})\rho(\mathbf{x})$.

Lack of symmetry in the K- kernels associated with the (τ,α)-topological model (1.27) poses a fundamental difficulty which prevents us from using the known results about the regularizing effect in such transport-diffusion. Instead, we adapt the De Giorigi method to settle the critical case $\alpha = 1$, employ fractional Schauder estimates to address the $\alpha > 1$ case, and apply Silvestre's result [56] to handle the case $0 < \alpha < 1$. Our most general regularity result is stated as follows, see Theorem 4.1 for the full statement.

Theorem 1.7 (**Regularity of the 1D** (τ,α)**-model.**). *Given non-vacuous initial data* $(\rho_0,u_0) \in H^{3+\alpha/2}\times H^4$*, then the 1D* (τ,α)*-model* (1.27) *admits a unique global in time solution,* (ρ,u)*, in the class*

$$\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{3+\frac{\alpha}{2}}), \quad u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^4) \cap L^2_{\mathrm{loc}}(R^+; H^{4+\frac{\alpha}{2}}), \quad \tau \leqslant \alpha, \quad 0 < \alpha < 2.$$

Combined with the general alignment result at Step 1 we obtain unconditional regularity and alignment of the (τ,α)-system in the range of parameters $1 \leqslant \tau \leqslant \alpha < 2$.

Theorem 1.8 (**Unconditional flocking for 1D local topological kernels**). *Consider the one-dimensional system* (1.27) *on* $\mathbb{T}$ *with local* (τ,α)*-kernel with topological singularity of order* $1 \leqslant \tau \leqslant \alpha < 2$*. Then any non-vacuous smooth initial data* $\rho_0 > 0, u_0$ *gives rise to a unique global solution which aligns,* $|u(t,\cdot) - u_\infty|_\infty \to 0$.

2. New protocol. Topological models

In this section we outline basic principles behind our new class of models and introduce the corresponding agent-based, kinetic, and hydrodynamic descriptions.

2.1. **Agent-based description.** In the standard agent-based models (1.2) 'all agents are made equal'. In our new protocol, the dynamics depends on the density of agents and therefore the "mass" of agents needs to be taken into account. We assume that each agent $(\mathbf{x}_i, \mathbf{v}_i)$ has a intrinsic fixed "mass" m_i. The meaning of mass depends on the context: one can think of m_i as an intrinsic parameter that quantifies the alignment power of agent at $\mathbf{x}_i$ to influence others. Thus, the bigger m_i is, the more direct influence agent "i" has on others. At the same time it is natural to assume that "massive" agents are more resistant to the influence by others. This latter property will be encoded into our new communication protocol $\phi(\mathbf{x}_i, \mathbf{x}_j)$

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \lambda \sum_{j \in \mathcal{N}_i} m_j \phi(\mathbf{x}_i, \mathbf{x}_j)(\mathbf{v}_j - \mathbf{v}_i). \end{cases} \tag{2.1}$$

Here, $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, denote positions of agents, $\mathbf{v}_i \in \mathbb{R}^n$ their velocities, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, and m_j is the "mass" of agent "j". The model we have in mind is based on the following two principles:

1. **Geometric neighborhoods**. Every agent has a finite range of communication — the communication of agent located at $\mathbf{x}_i$ is limited within a Euclidean ball of radius $2R_0$ centered at $\mathbf{x}_i$ and denoted $B(\mathbf{x}_i, 2R_0)$.

2. **Topological neighborhoods**. The communication between every two agents is influenced by the level of interference between them — the more crowded it is, leads to a decreasing level of communication. To this end, we argue that two agents located at $\mathbf{x}_i$ and $\mathbf{x}_j$ probe how crowded is an intermediate (closed) region of communication[4] $\Omega(\mathbf{x}_i, \mathbf{x}_j)$, see e.g., figure 1,

$$m_{ij} = \sum_{k:\, \mathbf{x}_k \in \Omega(\mathbf{x}_i, \mathbf{x}_j)} m_k.$$

The new protocol then results in a communication kernel, $\phi(\mathbf{x}_i, \mathbf{x}_j) = \psi_1(r(\mathbf{x}_i, \mathbf{x}_j)) \times \psi_2(\mathrm{d}(\mathbf{x}_i, \mathbf{x}_j))$ which involves both — the geometric distance, $r = |\mathbf{x}_i - \mathbf{x}_j|$, and the topological distance $\mathrm{d}(\mathbf{x}_i, \mathbf{x}_j) := m_{ij}^{1/n}$. Our model now reads

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \lambda \sum_{j=1}^{N} m_j \psi_1(|\mathbf{x}_i - \mathbf{x}_j|) \times \psi_2(\mathrm{d}(\mathbf{x}_i, \mathbf{x}_j))(\mathbf{v}_j - \mathbf{v}_i), \ \mathrm{d}(\mathbf{x}_i, \mathbf{x}_j) = \Big(\sum_{\mathbf{x}_k \in \Omega(\mathbf{x}_i, \mathbf{x}_j)} m_k \Big)^{\frac{1}{n}}. \end{cases} \tag{2.2}$$

The special case of equi-distributed mass, $m_j = 1/N$, is recorded in the Introduction (1.24),(1.25).

[4]In particular, $\mathbf{x}_i, \mathbf{x}_j \in \partial\Omega(\mathbf{x}_i, \mathbf{x}_j)$.

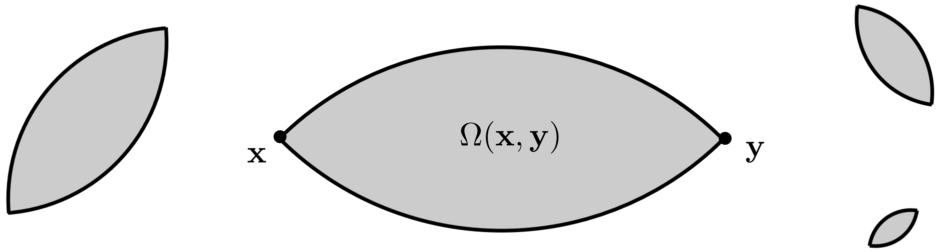

FIGURE 1. Communication domains between agents

2.2. **Region of communication.** The topological distance $\mathrm{d}(\mathbf{x},\mathbf{y})$ requires us to specify a domain of communication, $\Omega(\mathbf{x},\mathbf{y})$, which is probed by agents located at $\mathbf{x}$ and $\mathbf{y}$. In the one-dimensional case, it is simply the closed interval, $\Omega(x,y)=[x,y]$. In the multi-dimensional case, it is reasonably argued that the 'intermediate environment' between agents could be an n-dimensional region inside the ball enclosed by $\mathbf{x}$ and $\mathbf{y}$, namely $B(\frac{\mathbf{x}+\mathbf{y}}{2},r)$ with radius $r:=\frac{|\mathbf{x}-\mathbf{y}|}{2}$. For example, one can simply set $\Omega(\mathbf{x},\mathbf{y})$ to be that ball. As we shall see below, however, the fine structure of the local regions of communication, $\Omega(\mathbf{x}_i,\mathbf{x}_j)$, is important in order to retain unconditional flocking. To this end, we set a more restrictive *conical* region $\Omega(\mathbf{x},\mathbf{y})$, see Figure 1. First, we consider two basic locations $\mathbf{x}=(-1,0,...,0)$ and $\mathbf{y}=(1,0,...,0)$ and set the region of revolution generated by a parabolic arch connecting $\mathbf{x}$ and $\mathbf{y}$:

$$\Omega_0:=\{\mathbf{z}=(t,\mathbf{z}_-)\ \big|\ |\mathbf{z}_-|<1-t^2,-1\leqslant t\leqslant 1\}.$$

For an arbitrary pair of points $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, let $\Omega(\mathbf{x},\mathbf{y})$ denote the region scaled and translated from Ω_0:

$$\Omega(\mathbf{x},\mathbf{y}):=\{\mathbf{z}\ \big|\ |\mathbf{z}-\mathbf{z}_-|<1-r^2t_-^2\},\qquad r=\frac{|\mathbf{x}-\mathbf{y}|}{2}, \tag{2.3}$$

where $\mathbf{z}_-:=\mathbf{z}(t_-)$ is the projection of $\mathbf{z}$ on the diameter $\{\mathbf{z}_-(t)=\frac{\mathbf{x}+\mathbf{y}}{2}+\frac{t}{2}(\mathbf{y}-\mathbf{x}),\ -1\leqslant t\leqslant 1\}$ connecting $\mathbf{x}$ and $\mathbf{y}$.

Observe that at the tips, $\Omega(\mathbf{x},\mathbf{y})$ has the opening of $\frac{\pi}{2}$. For subsequent analysis,it can be replaced by any angle $<\pi$, calibrated according to a particular application[5]. It is crucial, however, that the region of communication is not locally smooth near the tips $\mathbf{x},\mathbf{y}$, see Claim 3.7 below, which excludes the ball $B(\frac{\mathbf{x}+\mathbf{y}}{2},r)$ with conical opening of $90°$.

2.3. **From kinetic to hydrodynamic description.** The large crowd dynamics of our geometric-topological agent-based model (2.2) is captured by the kinetic formulation (1.13)

$$f_t+\mathbf{v}\cdot\nabla_{\mathbf{x}}f+\nabla_{\mathbf{v}}\cdot Q(f,f)=0, \tag{2.4a}$$

with a general *symmetric* communication kernel $\phi(\mathbf{x},\mathbf{y})$

$$Q(f,f)(t,\mathbf{x},\mathbf{v})=\int_{\mathbb{R}^{2n}}\psi_1(r(\mathbf{x},\mathbf{y}))\times\psi_2(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))(\mathbf{w}-\mathbf{v})f(t,\mathbf{y},\mathbf{w})f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{y}. \tag{2.4b}$$

The passage (2.2) $\rightsquigarrow$ (2.4) is outlined in the Appnedix 5.1. By taking its $\mathbf{v}$-moments, we can read off the system of equations for macrolocal density $\rho(t,\mathbf{x})=\int_{\mathbb{R}^n}f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$ and

[5]Thus, for example, (2.3)can be enlarged to $\Omega(\mathbf{x},\mathbf{y}):=\{\mathbf{z}\ \big|\ |\mathbf{z}-\mathbf{z}_-|^\alpha<1-r^2t_-^2\}$ for any $0<\alpha<2$.

momentum $\rho\mathbf{u} = \int \mathbf{v} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$,

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + R) = \displaystyle\int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \end{cases} \tag{2.5}$$

Here ϕ is a *symmetric* communication kernel and R is the second-order Reynolds stress tensor $R(t,\mathbf{x}) = \displaystyle\int_{\mathbb{R}^n} (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$. The formal closure of the system at the first-order moments is achieved by considering a mono-kinetic ansatz, $f(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\delta(\mathbf{v}-\mathbf{u}(t,\mathbf{x}))$, concentrated at the macroscopic velocity $\mathbf{u}$: Such an ansatz creates zero stress $R = 0$, and we end up with the hydrodynamic description expressed in terms of the $(\rho,\mathbf{u})$-pair

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{cases} \tag{2.6}$$

A distinctive feature of the right hand side of the $\mathbf{u}$-equation is that it has a commutator structure

$$\int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = [\mathscr{L}_\phi,\mathbf{u}](\rho) := \mathscr{L}_\phi(\rho\mathbf{u}) - \mathscr{L}_\phi(\rho)\mathbf{u},$$

where $\mathscr{L}_\phi$ is the integral operator given by

$$\mathscr{L}_\phi(f) := \int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))\,\mathrm{d}\mathbf{y}. \tag{2.7}$$

Strong solutions to the system (2.6) satisfy energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x} = -\int \phi(\mathbf{x},\mathbf{y})|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{2.8a}$$

which will be a key component in establishing alignment. We note on passing that in view of the symmetry of the kernel ϕ, we have conservation of mass and momentum:

$$M = \int_{\mathbb{R}^n} \rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv M_0, \quad \mathbf{P} = \int_{\mathbb{R}^n} \rho\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv \mathbf{P}_0.$$

Hence, the rate of decay of the energy of the left of (2.8a) is the same rate of decay of the fluctuations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = 2cM_0\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}. \tag{2.8b}$$

Observe that for all global regular solutions $u \in L^1_{\mathrm{loc}}W^{1,\infty}$ the density retains sign,

$$\rho_0(x) > 0 \Rightarrow \rho(t,\mathbf{x}) > 0 \text{ for all } t \geqslant 0.$$

and in absence of pressure, each component u of $\mathbf{u}$ satisfies the maximum principle

$$\max u(\cdot,t) \leqslant \max u_0, \quad \min u(\cdot,t) \geqslant \min u_0. \tag{2.9}$$

Finally, we have the Galilean invariance

$$\mathbf{u} \to \mathbf{u}(\mathbf{x}+t\mathbf{U},t) - \mathbf{U}, \quad \rho \to \rho(\mathbf{x}+t\mathbf{U},t). \tag{2.10}$$

2.4. **Topological kernels and operators they define.** In what follows we restrict ourselves to the periodic domain $\mathbb{T}^n$. This choice is motivated by the fact that the density in (2.6) defines parabolicity of the equation. With finite mass $M < \infty$ such parabolicity cannot be controlled uniformly on the open space.

In the definitions below we assume that ρ is a bounded density with no vacuum

$$0 < c \leqslant \rho(t, \mathbf{x}) \leqslant C < \infty, \quad \mathbf{x} \in \mathbb{T}^n. \tag{2.11}$$

Let us fix a range scale $R_0 < \pi/4$. Let $|\mathbf{x} - \mathbf{y}| < 2R_0$. We define the topological distance between x and y, in accordance with how much mass is located in the communication domain between the points:

$$\mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) = \left[\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t, \mathbf{z}) \,\mathrm{d}\mathbf{z}\right]^{\frac{1}{n}}.$$

Here $\Omega(\mathbf{x}, \mathbf{y})$ is the region of communication enclosed between $\mathbf{x}$ and $\mathbf{y}$. Let us note that although the distance function d defines an equivalent topology on $\mathbb{T}^n$, it is not a proper metric, except for the one-dimensional case where it accumulates the mass along the interval $\Omega(x, y) = [x, y]$,

$$\mathrm{d}_\rho(x, y) = \left|\int_x^y \rho(t, z) \,\mathrm{d}z\right|, \tag{2.12}$$

Also note that all the distances are bounded by the total mass M, and $\mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) \geqslant c|\mathbf{x} - \mathbf{y}|$. Moreover, since $\Omega(\mathbf{x}, \mathbf{y}) = \Omega(\mathbf{y}, \mathbf{x})$, the distance is symmetric $\mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) = \mathrm{d}_\rho(\mathbf{y}, \mathbf{x})$.

Let us now define a class of *singular* topological kernels we will be studying:

$$\phi(\mathbf{x}, \mathbf{y}) = \psi_1(r(\mathbf{x}, \mathbf{y})) \times \psi_2(\mathrm{d}_\rho(\mathbf{x}, \mathbf{y})), \quad \psi_1(r) = r^{-(n+\alpha-\tau)} h(r), \quad \psi_2(\mathrm{d}_\rho) = \mathrm{d}_\rho^{-\tau}. \tag{2.13}$$

Here $h(r)$ is a smooth cutoff function supported on $[0, 2R_0]$ which coincides with $\chi_{R_0}(r)$ on $[0, R_0)$; since $R_0 < \frac{\pi}{4}$ the kernel is properly supported on the cube $[-\pi, \pi]^n$ and is viewed as a function on the torus $\mathbb{T}^n$ (no need to periodization as in (1.18)). We refer to (2.13) as (τ, α)-kernel: the exponent $\tau > 0$ measures a portion of the topological part in the kernel with overall singularity of order $n + \alpha$, $0 < \alpha < 2$. Note that we allow τ to be larger than $n + \alpha$, and thus $n + \alpha - \tau < 0$, which corresponds to the case of an overwhelming dominance of the topological component over the Euclidean one.

The corresponding alignment operator is given (formally) by the commutator form

$$\mathscr{C}_\phi(\zeta, f) = [\mathscr{L}_\phi, f](\zeta) := \mathscr{L}_\phi(\zeta f) - \mathscr{L}_\phi(\zeta) f = \int_{\mathbb{T}^n} \phi(\mathbf{x}, \mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}))\zeta(\mathbf{y}) \,\mathrm{d}\mathbf{y}. \tag{2.14}$$

We note that a proper care has to be given in order to properly define these operators for strongly singular case $\alpha \geqslant 1$. Our immediate goal below is therefore to develop formal definitions and initial facts about the operator $\mathscr{L}_\phi$ in multi-D settings (more details specific for 1D situation will follow in Section 4.1). Due to the non-convolutive and anisotropic nature of the kernel, most of the standard facts do not apply and will need to be readdressed. Our plan is to define $\mathscr{L}_\phi f$ as a distribution first. Then we state a formal justification of pointwise evaluations of $\mathscr{L}_\phi f(\mathbf{x})$ and $\mathscr{C}_\phi(\zeta, f)(\mathbf{x})$. Technicalities of the proofs will be collected in section 5.2 in the Appendix.

Definition 2.1. We define an operator $\mathscr{L}_\phi : H^{\alpha/2} \to H^{-\alpha/2}$ by the following action: for any $f \in H^{\alpha/2}$ and $g \in H^{\alpha/2}$

$$\langle \mathscr{L}_\phi f, g\rangle = -\frac{1}{2}\int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))(g(\mathbf{x}) - g(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}. \tag{2.15}$$

Note that formally such action could be obtained from (2.7), if (2.7) made sense pointwise, by the usual symmetrization. Clearly, from the Gagliardo-Sobolevskii definition of $H^{\alpha/2}$, we have

$$|\langle \mathscr{L}_\phi f, g\rangle| \lesssim |f|_{H^{\alpha/2}} |g|_{H^{\alpha/2}}.$$

Due to the symmetry of the kernel, the operator $\mathscr{L}_\phi$ is clearly self-adjoint, and its range is in $H_0^{-\alpha/2}$. By the standard operator theory this implies the following statement.

Lemma 2.2. *The restricted operator $\mathscr{L}_\phi : H_0^{\alpha/2} \to H_0^{-\alpha/2}$ is invertible.*

Proof. Clearly, $-\langle \mathscr{L}_\phi f, f\rangle \sim |f|^2_{H_0^{\alpha/2}}$. Hence $|\mathscr{L}_\phi f|_{H^{-\alpha/2}} \geqslant |f|_{H^{\alpha/2}}$ which shows that the operator has closed range and is injective. If the range is not all of $H_0^{-\alpha/2}$, then there is a $g \in H_0^{\alpha/2}$ for which $\langle \mathscr{L}_\phi f, g\rangle = 0$ for all $f \in H^{\alpha/2}$. Taking $f = g$ we arrive at a contradiction. Thus, $\mathscr{L}_\phi$ is invertible. ☐

First, let us consider the case $0 < \alpha < 1$. In this weakly singular case, pointwise evaluation of the integral expressions in (2.7) and (2.14) presents no problem as long as $f \in C^1$. The rigorous argument goes by "unwinding" the symmetric defining formula (2.15). To demonstrate it, let us denote by $L_\phi f(\mathbf{x})$ the integral on the right hand side of (2.7). Clearly, $L_\phi f \in C(\mathbb{T}^n)$. Let us fix a point $\mathbf{x}_0 \in \mathbb{T}$. Let g be the standard non-negative Friedrichs' mollifier supported on the ball of radius 1. Denote $g_\varepsilon = \frac{1}{\varepsilon^n} g((\mathbf{x} - \mathbf{x}_0)/\varepsilon)$. It suffices to show that

$$\langle \mathscr{L}_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0).$$

Since for $0 < \alpha < 1$, $L_\phi f(x)$ is a continuous function we can break up the integral without ambiguity:

$$\begin{aligned}\langle \mathscr{L}_\phi f, g_\varepsilon\rangle &= -\frac{1}{2}\int_{\mathbb{T}^{2n}} (f(\mathbf{x}) - f(\mathbf{y}))(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{T}^{2n}} (f(\mathbf{y}) - f(\mathbf{x}))g_\varepsilon(\mathbf{x})\phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = \langle L_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0).\end{aligned}$$

The higher case $1 \leqslant \alpha < 2$ is more subtle. Let us show that when ρ and f are smooth, the element $\mathscr{L}_\phi f \in H^{-\alpha/2}$ gains regularity. Formally, this first step is necessary to even discuss pointwise values $\mathscr{L}_\phi f(\mathbf{x})$. So, let us make the following observation:

$$\begin{aligned}\nabla_{\mathbf{x}} \mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) &= \frac{1}{n(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}))^{n-1}} \int_{\Omega(\mathbf{x}+\mathbf{z},\mathbf{x})} \nabla\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &= \frac{1}{n(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}))^{n-1}} \int_{\partial\Omega(\mathbf{x}+\mathbf{z},\mathbf{x})} \vec{\nu}\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}.\end{aligned} \tag{2.16}$$

Clearly, if $|\nabla\rho|_\infty < \infty$, then $|\nabla_{\mathbf{x}} \mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})| \lesssim |\mathbf{z}|$. Next, we rewrite the defining formula (2.15) in terms of the difference operator $\delta_{\mathbf{z}} f(\mathbf{x}) := f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})$,

$$\begin{aligned}\langle \mathscr{L}_\phi f, g\rangle &= -\frac{1}{2}\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x})\delta_{\mathbf{z}} g(\mathbf{x})\psi_1(|\mathbf{z}|)\psi_2(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\\ &= -\frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x})\nabla g(\mathbf{x}+\theta\mathbf{z})\cdot\mathbf{z}\ \psi_1(|\mathbf{z}|)\psi_2(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\langle \mathscr{L}_\phi f, g\rangle = &\frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}}\nabla f(\mathbf{x})\cdot\mathbf{z} g(\mathbf{x}+\theta\mathbf{z})\psi_1(|\mathbf{z}|)\psi_2(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta\\ &+\frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x}) g(\mathbf{x}+\theta\mathbf{z})\delta_{\mathbf{z}}\rho(\mathbf{x})\psi_1(|\mathbf{z}|)\psi_2'(\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}))\nabla \mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})\cdot\mathbf{z}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta.\end{aligned}$$

Note that the singularity of $\psi_1\,\psi_2$ (of order $n+\alpha$) is now masked by the second-order vanishing terms of of $\delta_{\mathbf{z}} f, \delta_{\mathbf{z}}\rho$ and $\nabla \mathrm{d}_\rho$, so we end up with an integrable singularity of order $\alpha - 1$. Consequently,

$$|\langle \mathscr{L}_\phi f, g\rangle| \lesssim (|f|_{C^2} + |f|_{C^1}|\rho|_{C^1})|g|_{L^\infty}.$$

This is of course not an optimal bound, but it shows that the regularity of $\mathscr{L}_\phi f$ improves. One can continue in similar fashion. Assuming $g = \partial_x^k h$, for some $h \in L^\infty$, one obtains

$$|\langle \mathscr{L}_\phi f, \partial_x^k h\rangle| \lesssim (|f|_{C^{k+2}}, |\rho|_{C^{k+1}})|h|_{L^\infty}.$$

Thus, $\mathscr{L}_\phi f \in (C^{-k})^* \subset C^{k-\varepsilon}$, for any $\varepsilon > 0$.

Lemmas 5.1 and 5.2 stated in the Appendix make a formal justification for representation formulas (2.7) and (2.14) which are to be understood in the principal value sense. They come with estimates that will be crucial in the proof of the global regularity in 1D, see Section 4.

3. Smooth solutions must flock

The goal of this section will be to prove that any global, non-vacuous smooth solution to the topological model (2.6) aligns to its average velocity vector $\mathbf{u}_\infty$ which can be determined from the conservation of momentum and mass: $\mathbf{u}_\infty = \mathbf{P}_0/M_0$.

3.1. **Flocking of local symmetric kernels.** We begin with a general class of local symmetric singular communication kernels, $\phi = \phi(\mathbf{x},\mathbf{y})$ such that

$$\phi(\mathbf{x},\mathbf{y}) \gtrsim |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\chi_{R_0}(|\mathbf{x}-\mathbf{y}|), \qquad 0<\alpha<2, \tag{3.1a}$$

and consider the corresponding alignment dynamics

$$\begin{cases}\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{\mathbb{R}^n}\phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y},\end{cases} \tag{3.1b}$$

The two primary examples we have in mind are the the topological kernel (2.13), $\phi(\mathbf{x},\mathbf{y}) = \psi_1(r(\mathbf{x},\mathbf{y}))\times\psi_2(\mathrm{d}(\mathbf{x},\mathbf{y}))$, and the particular case of the geometric kernel (1.20), $\phi(\mathbf{x},\mathbf{y}) = \psi_1(r(\mathbf{x},\mathbf{y}))$. We emphasize, however, that (3.1) allows for a rather general class of symmetric ϕ's, localized along the diagonal with geometric singularity of order $n+\alpha$: we do not dwell on the fine structure originating with the topological portion of the singular kernel.

This generality comes with the subtlety of making a proper sense of the integral operator,

$\mathscr{L}_\phi(f) := \displaystyle\int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))\,\mathrm{d}\mathbf{y}$, which in turn defines the alignment term through its commutator structure, [53] $\displaystyle\int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} := \mathscr{L}_\phi(\rho\mathbf{u})-\mathscr{L}_\phi(\rho)\mathbf{u}$. As before, this is achieved by symmetrization, defining $\mathscr{L}_\phi : H^{\alpha/2} \to H^{-\alpha/2}$ by

$$\langle \mathscr{L}_\phi f, g\rangle = -\frac{1}{2}\int_{\mathbb{T}^{2n}} (f(\mathbf{x})-f(\mathbf{y}))(g(\mathbf{x})-g(\mathbf{y}))\phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}, \qquad f,g\in H^{\alpha/2}, \tag{3.2}$$

and noting that the restricted operator $\mathscr{L}_\phi : H_0^{\alpha/2} \to H_0^{-\alpha/2}$ is invertible.

At this point we record the fundamental statement, balancing the decay rate of fluctuations in (3.1b), driven by alignment $\mathscr{L}_\phi$ associated with *any* symmetric kernel $\phi(\cdot,\cdot)$: using the definition (2.15) we recast (2.8) in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^{2n}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} &= -2M_0\int_{\mathbb{T}^{2n}} \langle[\mathscr{L}_\phi,\mathbf{u}](\rho),\mathbf{u}\rangle_\rho\,\mathrm{d}\mathbf{y} \\ &= -2M_0\int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.\end{aligned} \tag{3.3}$$

The main aspect of our investigation in this section is the derivation of lower-bounds of the *enstrophy* on the right of (3.3) for short-range ϕ's.

It is clear that a *necessary condition* for flocking $|\mathbf{u}(t,\cdot)-\mathbf{u}_\infty| \to 0$ requires the density to be bounded away from vacuum, or else the flow may break apart into two or more separate 'islands', traveling in their own velocity which is disconnected from the influence of others. Indeed, when $\rho(\cdot,t)$ vanishes on a compact set, the momentum equation (3.1b) is reduced to the pressureless Burgers system $\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = 0$ which in turn leads to a finite-time blow-up, see [58]. Precisely how far from vacuum the density must be in order to fulfill an alignment dynamics for general local kernels ϕ is investigated below.

Theorem 3.1 (**General local singular kernels).** *Let ϕ be a symmetric, local, singular kernel* (3.1a) *and let $(\rho(t,\cdot),\mathbf{u}(t,\cdot))$ be a global strong solution of the corresponding alignment dynamics,* (3.1b)*, with a properly defined alignment term* (3.2). *Assume that*

$$C \geqslant \rho(t,\cdot) \geqslant \frac{c}{\sqrt{1+t}}, \qquad C > c > 0.$$

Then the solution converges to a flock at algebraic rate, namely — there exist a limiting velocity $\mathbf{u}_\infty$ and $\eta > 0$ such that

$$\int_{\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{1}{2M_0\,t^\eta}, \qquad \mathbf{u}_\infty = \frac{\mathbf{P}_0}{M_0}. \tag{3.4}$$

Proof. We begin by setting up the general Hilbert structure for a variational formulation of the problem. Let us assume that

$$0 < c(t) < \rho = \rho(t,\mathbf{x}) < C(t).$$

Let us denote by L^2_ρ the space of $L^2(\mathbb{T}^n)$-fields $\mathbf{u}$ with scalar product given by

$$\langle \mathbf{u},\mathbf{v}\rangle_\rho = \int_{\mathbb{T}^n} \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Note that the metric of the space L^2_ρ changes in time.

Next, we consider the family of eigenvalue problems parametrized by time: we seek eigenpairs, $\kappa(t)$ and $\mathbf{u}(t,\cdot) \in \mathscr{U}^\alpha_{\rho(t,\cdot)}$,

$$\int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y} = \kappa(t)\mathbf{u}(\mathbf{x}), \qquad \mathbf{u} \in \mathscr{U}^\alpha_\rho := L^2_\rho \cap H^{\alpha/2}. \tag{3.5}$$

Note that the left hand side is precisely the action of the commutator $\mathscr{C}_\phi(\rho,\mathbf{u})$. For a fixed smooth ρ, and any symmetric singular singular kernel ϕ, the corresponding alignment operator

$$\mathbf{u} \to \mathscr{C}_\phi(\rho,\mathbf{u}) := [\mathscr{L}_\phi, \mathbf{u}](\rho) = \int_{\mathbb{R}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

maps $H^{\alpha/2}$ into $H^{-\alpha/2}$. Moreover, the symmetric definition of $\mathscr{L}_\phi$ (2.15) yields that $-\mathscr{C}_\phi(\rho,\mathbf{u})$ is non-negative, $-(\mathscr{C}_\phi(\rho,\mathbf{u}),\mathbf{u}) \geqslant 0$. Hence $\kappa_1 = 0$ is the minimal eigenevalue corresponding to the constant solution $\mathbf{u} \equiv \mathbf{1}$, and this allows us to seek the *second* minimal eigenvalue as a solution to the variational problem[6]

$$\kappa_2(t) = \inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho} \frac{-\langle \mathscr{C}_\phi(\rho,\mathbf{u}-\overline{\mathbf{u}}),\mathbf{u}-\overline{\mathbf{u}}\rangle_\rho}{|\mathbf{u}-\overline{\mathbf{u}}|^2_{L^2_\rho}}, \quad \overline{\mathbf{u}} := \frac{\int \mathbf{u}\rho}{\int \rho} \ \text{ so that } \ \langle \mathbf{u}-\overline{\mathbf{u}}, \mathbf{1}\rangle_\rho = 0 \tag{3.6}$$

or — stated explicitly in terms of $|\mathbf{u}-\overline{\mathbf{u}}|^2_{L^2_\rho} = \dfrac{1}{2M_0}\displaystyle\int_{\mathbb{T}^{2n}} |\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$,

$$\kappa_2(t) = 2M_0 \times \inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho} \frac{\displaystyle\int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(t,\mathbf{y})\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\displaystyle\int_{\mathbb{T}^{2n}} |\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}. \tag{3.7}$$

Since the numerator with $\phi(\mathbf{x},\mathbf{y}) \simeq |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\chi_{R_0}(|\mathbf{x}-\mathbf{y}|)$ is equivalent for the $H^{\alpha/2}$-norm, the existence follows classically by compactness. This links the enstrophy on the right of (3.3) to the Fiedler number, $\kappa_2(t)$, in complete analogy to the discrete case indicated in (1.12) (consult [44, sec 2.2]).

We can now state an alignment estimate in terms of the shrinking L^2_ρ-diameter of the velocity, given by

$$V_2[\mathbf{u},\rho](t) := \int_{\mathbb{T}^{2n}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{3.8}$$

By (3.3), (3.7) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2[\mathbf{u},\rho](t) \leqslant -\kappa_2(t)V_2[\mathbf{u},\rho](t). \tag{3.9}$$

The implication of (3.9) is of course the bound

$$2M_0\int_{\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = V_2[\mathbf{u},\rho](t) \leqslant V_2[\mathbf{u}_0,\rho_0]\exp\left\{-\int_0^t \kappa_2(s)\,\mathrm{d}s\right\}. \tag{3.10}$$

Consequently, the solution aligns in the L^2_ρ-distance sense if $\displaystyle\int_0^\infty \kappa_2(s)\,\mathrm{d}s = \infty$ (again, in complete analogy with the discrete setup (1.12)).

[6] By symmetry $\overline{\mathbf{u}} = \mathbf{u}_\infty := \mathbf{P}_0/M_0$ but we keep the separate notation of $\overline{\mathbf{u}}$ to signify orthogonality to the 0-eigen-space spanned by $\mathbf{1}$.

We will derive the particular critical power law $\kappa_2(t) \geqslant c/(1+t)$ which clearly fulfills this requirement. It is here that we use the assumed lower-bound on the density, $\rho(t,\cdot) \gtrsim 1/\sqrt{1+t}$, the assumed singularity of our kernel $\phi(\mathbf{x},\mathbf{y}) \gtrsim |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\chi_{|\mathbf{x}-\mathbf{y}|<R_0}$ and by the uniform upper-bound of the density, $|\mathbf{u}-\overline{\mathbf{u}}|_{L^2_\rho} \lesssim |\mathbf{u}|_{L^2}$, in order to bound the spectral gap

$$\kappa_2(t) \geqslant \frac{c}{t} \inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho} \frac{\displaystyle\int_{|\mathbf{x}-\mathbf{y}|<R_0} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{|\mathbf{u}|_2^2}. \tag{3.11}$$

Technically, the infimum still depends on time since it is taken over the orthogonal complement of the line spanned by $\rho(t)$, denoted $[\rho(t)]$, in the classical $L^2(\mathbb{T}^n)$. We now have to show that this infimum still stays bounded away from zero. Geometrically this is due to the fact that the space $[\rho(t)]^\perp$ does not come close to the span of constants $\mathbb{R}^n$ in the sense of Hausdorff distance. It is more straightforward to argue by contradiction, however.

Suppose there is a sequence of times $t_k \in R^+$, and $\mathbf{u}_k \in L^2_{\rho(t_k)} \cap H^{\alpha/2}$ such that $|\mathbf{u}_k|_2 = 1$ yet the homogeneous local $H^{\alpha/2}$-norm tends to zero:

$$\int_{|\mathbf{x}-\mathbf{y}|<R_0} \frac{|\mathbf{u}_k(\mathbf{x})-\mathbf{u}_k(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \to 0. \tag{3.12}$$

Note that the latter, in particular, implies compactness of the sequence $\{\mathbf{u}_k\}_k$ in L^2. Hence, up to a subsequence, $\mathbf{u}_k \to \mathbf{u}_*$ strongly in L^2 and weakly in $H^{\alpha/2}$. By the lower-weak-semi-continuity, and (3.12), we conclude that $\mathbf{u}_* \in \mathbb{R}^n$ is a constant field, with $|\mathbf{u}_*| = 1$ due to $|\mathbf{u}_k|_2 \to |\mathbf{u}_*|_2$.

At the same time, since $\rho(t) > 0$ and $\int \rho(t_k,\mathbf{x})\,\mathrm{d}\mathbf{x} = M_0$, there exists a weak* limit of a further subsequence $\rho(t_k) \to \mu$, where μ is a positive Radon measure on $\mathbb{T}^n$ with non-trivial total mass $\mu(\mathbb{T}^n) = M_0$. We now reach a contradiction by claiming the limit

$$0 = \int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k,\mathbf{x})\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{T}^n} \mathbf{u}_*\,\mathrm{d}\mu = M_0\mathbf{u}_*,$$

Indeed, the assumed uniform upper-bound of the density implies

$$\begin{aligned}\int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k,\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{T}^n} \mathbf{u}_*\,\mathrm{d}\mu(\mathbf{x}) \\ = \int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k,\mathbf{x})\,\mathrm{d}\mathbf{x} - M_0\mathbf{u}_* = \int_{\mathbb{T}^n} (\mathbf{u}_k(\mathbf{x})-\mathbf{u}_*)\rho(t_k,\mathbf{x})\,\mathrm{d}\mathbf{x},\end{aligned}$$

and the latter is clearly bounded by $C|\mathbf{u}_k-\mathbf{u}_*|_2 \to 0$. We conclude that $\kappa_2(t) \geqslant c/t$, and the result follows from (3.10). □

Again the theorem does not make any use of topological alignment, and hence applies to a general class of alignment models based on local symmetric kernels with *geometric* singularity. Unconditional flocking is then achieved under a lower bound on the density, $\rho(t,\cdot) \gtrsim (1+t)^{-1/2}$. The difficulty is that this lower bound is too restrictive and is not given a priori for any strong solution. This brings us to the local topological models which yield unconditional flocking under more accessible lower-bounds on the density.

3.2. **Flocking for local models with topological kernels.** We now turn our attention to the (τ,α)-model which involves a communication kernel (2.13)

$$\phi(\mathbf{x},\mathbf{y}) = \frac{h(r(\mathbf{x},\mathbf{y}))}{(r(\mathbf{x},\mathbf{y}))^{(n+\alpha-\tau)}} \times \frac{1}{(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))^{\tau}}, \qquad \mathrm{d}_\rho(\mathbf{x},\mathbf{y}) = \Big(\int_{\mathbf{z}\in\Omega(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}\Big)^{1/n}. \tag{3.13}$$

This is a particular case within the general class of local symmetric kernels (3.1a) with singularity of order $n+\alpha$. Here, the topological distance d_ρ contributes a τ-singularity. The following theorem quantifies how this is used to tame the corresponding non-vacuous requirement on the decay of the density.

Theorem 3.2 (**Local topological kernels).** *Let $(\rho,\mathbf{u})$ be a global smooth solution of the (τ,α)-model (2.6),(3.13) based on the conical regions of topological communication $\Omega(\mathbf{x},\mathbf{y})$ in (2.3). Assume that the density $\rho(t,\cdot)$ satisfies, for all $t>0$,*

$$\rho(t,\mathbf{x}) \geqslant \frac{c}{(1+t)^{\beta}}, \quad 0\leqslant \beta \leqslant \beta_0 := \min\left\{1, \frac{n}{2n-\tau}\right\}, \tag{3.14}$$

and, in addition if $\tau > n+\alpha$, that

$$|\rho(t,\cdot)|_{\frac{\tau-n}{\alpha}} < C. \tag{3.15}$$

Then the solution aligns with at least algebraic rate given by

$$|\mathbf{u}(t)-\mathbf{u}_\infty|_\infty = \frac{o(1)}{t^\gamma} \quad \textit{where} \quad \gamma = \frac{1}{2}\left(1-\frac{\beta}{\beta_0}\right). \tag{3.16}$$

A few remarks are in order before we proceed to the proof.

Remark 3.3. (**Flocking with no rate**). Observe that in case of decay rate of order $\beta=\beta_0$, we still obtain unconditional flocking yet with no rate . When the decay rate is better than the end point β_0, the convergence *rate* of alignment is quantified in terms of the difference $\beta_0-\beta$. This version will be useful, in particular, when the density enjoys a maximum principle, as is the case in 1D and $e\equiv 0$, see Theorem 4.1. In this case $\beta=0$ and hence $\gamma=\frac{1}{2}$.

Remark 3.4. (**The one-dimensional case**). We highlight one notable application of this theorem to 1D settings. In this case the lower bound on the density (3.14) holds for any strong solutions with power 1, see Lemma 4.6. This fulfills (3.14) for any $\tau\geqslant 1$. For the topologically dominant case $\tau>1+\alpha$ the assumption (3.15) can be satisfied through a global control on $|\rho(t)|_\infty$ for a small initial data described in Lemma 4.7. We thus obtain an unconditional alignment in the range of $1\leqslant\tau\leqslant 1+\alpha$ which proves Theorem 1.8 stated in the Introduction. We note that the metric model, $\tau=0$, requires $\rho(t,x)\geqslant \frac{c}{\sqrt{1+t}}$, which does not seem to hold apriori.

Remark 3.5. (**About** $\tau=n$). We make another remark concerning the apparent threshold value of $\tau=n$. Clearly from (3.14), if $\tau\geqslant n$, then $\rho\geqslant\frac{1}{1+t}$ is the weakest assumption under which the theorem holds, while for $\tau<n$ a more stringent bound on ρ is required. This can be explained by the fact the the density on the bottom of ϕ needs to compensate the density on the top inside the diffusion term. Even more vividly the condition manifests itself after taking limit as $\alpha\to 2$. Such limits are standard in the elliptic theory and we will not provide many details here. One can verify the following:

$$\lim_{\alpha\to 2}(2-\alpha)\mathscr{L}_\phi f(x) = \nabla\cdot\left(\rho^{-\frac{\tau}{n}}\nabla f\right) := \mathscr{D}(f). \tag{3.17}$$

The commutator which would appear in the corresponding limit model reads

$$\mathscr{D}(\rho\mathbf{u}) - \mathbf{u}\mathscr{D}(\rho) = \frac{1}{\rho^{\gamma-1}}\Delta\mathbf{u} + \frac{2-\gamma}{\gamma}\nabla\mathbf{u}\nabla\rho, \qquad \gamma = \frac{\tau}{n}. \tag{3.18}$$

We can see that $\tau = n$ is the threshold that determines whether the density appears on the top or the bottom in front of the leading order term. For $\tau \geqslant n$ it amplifies dissipation in thinner regions as intended in the topological model.

Remark 3.6. (**A comparison with Motsch-Tadmor scaling**). It is instructive to compare the generic topological model with (n,α)-kernel

$$\phi(\mathbf{x},\mathbf{y}) = \psi_1(|\mathbf{x}-\mathbf{y}|) \times \frac{1}{m_t(\Omega(\mathbf{x},\mathbf{y}))}, \qquad m_t(\Omega) := \int_\Omega \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z},$$

with the Mostch-Tadmor scaling (1.14b) with local $\phi(r) = \chi_{R_0}$,

$$\phi(\mathbf{x},\mathbf{y}) = \psi_1(|\mathbf{x}-\mathbf{y}|) \times \frac{1}{m_t(B(\mathbf{x},R_0))}.$$

In the former, the pairwise interaction between two "agents" depends on the density in an intermediate region of communication; in the latter, the communication of each "agent" depends on how rarefied is the crowd in its own geometric neighborhood.

Proof. First we make one technical remark. With the assumptions at hand the following comparison holds between the topological metric and the kernel:

$$\frac{1}{\mathrm{d}_\rho^n(\mathbf{x},\mathbf{y})} \leqslant C(t)\phi(\mathbf{x},\mathbf{y}), \tag{3.19}$$

for some algebraic $C(t)$. Indeed, (3.19) is equivalent to a bound

$$\left[\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}\right]^{\frac{\tau}{n}-1} |\mathbf{x}-\mathbf{y}|^{n+\alpha-\tau} \leqslant C(t).$$

When $\tau < n$, the lower bound (3.14) applies to give $C(t) = Ct^{\beta(1-\frac{\tau}{n})}$. In case $n+\alpha \geqslant \tau \geqslant n$ the inequality holds automatically with uniform $C(t) = C$ since $\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z} \leqslant M_0$. Finally, in case $\tau > n+\alpha$, we use (3.15) and apply the Hölder inequality, again obtaining uniform C.

Now, let us fix a coordinate i and aim to prove (3.16) for u_i. We denote $u = u_i$ for notational simplicity. Using the Galilean invariance we can lift u if necessary and assume that $u(t) > 0$. Note that the extrema of $u(t)$, denoted $u_+(t)$ and $u_-(t)$, are monotonically decreasing and increasing, respectively.

We will make frequent use of the mass measure denoted

$$\mathrm{d}m_t = \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}.$$

Step 1: **flattening near extremes**. Let $\mathbf{x}_+(t)$ be a point of maximum for $u(t,\cdot)$ and $\mathbf{x}_-(t)$ a point of minimum. Let us fix a time-dependent $1 > \delta(t) > 0$ to be determined later, and consider the sets

$$G_\delta^+(t) = \{u < u_+(t)(1-\delta(t))\}, \quad G_\delta^-(t) = \{u > u_-(t)(1+\delta(t))\}.$$

The effect of flattening is expressed in terms of conditional expectations of the above sets in the balls $B(\mathbf{x}_\pm(t), R_0)$ with respect to the mass measure. Let us denote

$$\mathbb{E}_t[A|B] = \frac{m_t(A\cap B)}{m_t(B)}.$$

We show that

$$\int_0^\infty \frac{\delta(t)}{C(t)}\mathbb{E}_t[G^\pm_\delta(t)|B(\mathbf{x}_\pm(t), R_0)]\,\mathrm{d}t < \infty. \tag{3.20}$$

To this end, let us compute the equation pointwise at the critical point $(t, \mathbf{x}_+(t))$ utilizing the Rademacher Theorem: $(\partial_t u)(t, \mathbf{x}_+(t) = \partial_t u_+(t)$ a.e.,

$$\partial_t u_+(t) = \int \phi(\mathbf{x}_+(t), \mathbf{y})(u(\mathbf{y}) - u_+(t))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

At point $(\mathbf{x}_+(t), t)$ we estimate on the alignment term with the use of (3.19),

$$\begin{aligned}
-\partial_t u_+(t) &= \int \phi(\mathbf{x}_+, \mathbf{y})(u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\\
&\geqslant C^{-1}(t)\int_{B(\mathbf{x}_+,R_0)} \frac{1}{\mathrm{d}^n_\rho(\mathbf{x}_+, \mathbf{y})}(u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y},\\
&\geqslant \frac{C^{-1}(t)}{m_t(B(\mathbf{x}_+(t), R_0))}\int_{G^+_\delta(t)\cap B(\mathbf{x}_+(t),R_0)} (u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad (\text{since } \Omega(\mathbf{x}_+, \mathbf{y}) \subset B(\mathbf{x}_+, R_0))\\
&\geqslant \frac{C^{-1}(t)\delta(t)}{m_t(B(\mathbf{x}_+(t), R_0))}\int_{G^+_\delta(t)\cap B(\mathbf{x}_+(t),R_0)} \rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \times u_+(t)\\
&= \frac{\delta(t)}{C(t)}\mathbb{E}_t[G^+_\delta(t)|B(\mathbf{x}_+(t), R_0)] \times u_+(t).
\end{aligned}$$

The result follows by integration:

$$\int_0^\infty \frac{\delta(t)}{C(t)}\mathbb{E}_t[G^+_\delta(t)|B(\mathbf{x}_+(t), R_0)]\,\mathrm{d}t \leqslant \ln\frac{u_+(0)}{\lim_{t\to\infty} u_+(t)} \leqslant \ln\frac{u_+(0)}{u_-(0)}.$$

Step 2: **Campanato estimates**. On this next step we obtain proper Campanato estimates that measure deviation of u from its average values in terms of global enstrophy.

We denote the averages with respect to mass-measure by

$$u_{\mathbf{x},r} = \frac{1}{m_t(B(\mathbf{x}, r))}\int_{B(\mathbf{x},r)} u(t, \mathbf{z})\,\mathrm{d}m_t(\mathbf{z}).$$

Fix $\mathbf{x}_* \in \mathbb{T}^n$. By Hölder inequality, we have the following estimate:

$$\int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2\rho(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \int_{\substack{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}\\ |\mathbf{y}-\mathbf{x}_*|<r}} \frac{1}{m_t(B(\mathbf{x}_*, r))}|u(\mathbf{x}) - u(\mathbf{y})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$$

At this point we recall that the communication domain $\Omega(\mathbf{x}, \mathbf{y})$ in (2.3) has corner tips of opening $\frac{\pi}{2}$ degrees. Hence, we can make the following geometric observation.

Claim 3.7. If $|\mathbf{x} - \mathbf{x}_*| < \frac{1}{10}r$ and $|\mathbf{y} - \mathbf{x}_*| < r$, then $\Omega(\mathbf{x}, \mathbf{y}) \subset B(\mathbf{x}_*, r)$.

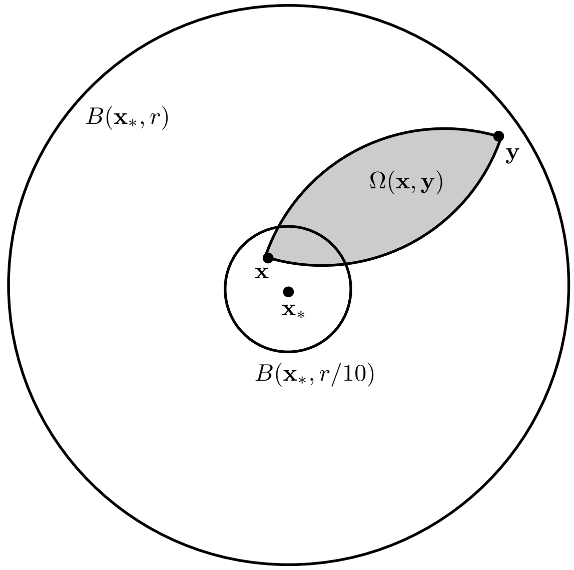

FIGURE 2. $\Omega(\mathbf{x}, \mathbf{y})$ is trapped in the outer ball if $\mathbf{x}$ is close to the center.

In other words if $\mathbf{y}$ is in a ball and $\mathbf{x}$ is close enough to the center of that ball then the domain $\Omega(\mathbf{x}, \mathbf{y})$ is entirely enclosed in the ball also, see Figure 2. This implies that $m_t(B(\mathbf{x}_*, r)) \geqslant m_t(\Omega(\mathbf{x}, \mathbf{y})) = \mathrm{d}_\rho^n(\mathbf{x}, \mathbf{y})$. We thus can further estimate, with the use of (3.19),

$$\int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2 \rho(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leqslant \int_{|\mathbf{x}-\mathbf{y}|<\frac{11}{10}r} \frac{1}{\mathrm{d}_\rho^n(\mathbf{x}, \mathbf{y})} |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y}) \,\mathrm{d}\mathbf{y} \,\mathrm{d}\mathbf{x}$$
$$= C(t) \int \phi(\mathbf{x}, \mathbf{y}) |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y}) \,\mathrm{d}\mathbf{y} \,\mathrm{d}\mathbf{x}.$$

The energy balance (3.3) (see also (2.8)) yields the space-time bound on the (components of) enstrophy on the right

$$\int_0^\infty \int_{\mathbb{T}^{2n}} \phi(\mathbf{x}, \mathbf{y}) |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{y} < \int_{\mathbb{T}^n} \rho_0 |\mathbf{u}_0|^2 \,\mathrm{d}\mathbf{x} < \infty,$$

hence we conclude with a time bound on the Campanato semi-norm,

$$\int_0^\infty \frac{1}{C(t)} [u]_\rho^2 \,\mathrm{d}t < \infty, \qquad [u]_\rho^2 := \sup_{\mathbf{x}_* \in \mathbb{T}^n, r < \frac{R_0}{2}} \int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2 \rho(\mathbf{x}) \,\mathrm{d}\mathbf{x}. \tag{3.21}$$

Combined with (3.20) we have obtained

$$I = \int_0^\infty \frac{1}{C(t)} \Big(\delta(t) \mathbb{E}_t[G_\delta^\pm(t) | B(\mathbf{x}_\pm(t), R_0)] + [u(t)]_\rho^2 \Big) \,\mathrm{d}t < \infty.$$

Clearly, there exists a constant C_0 and a vanishing quantity $o(1)$ such that

$$\int_T^{C_0 T} \frac{o(1)}{t} \,\mathrm{d}t > 2I \quad \text{for all} \quad T > 0.$$

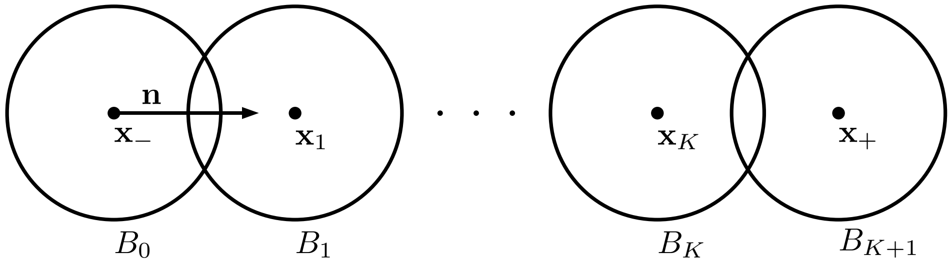

FIGURE 3.

Hence, for any $T > 0$ we can fix a $t \in [T, C_0 T]$ such that

$$
\begin{aligned}
&[u(t)]^2_\rho < \frac{o(1)C(t)}{t} \\
&\mathbb{E}_t[G^+_\delta(t)|B(\mathbf{x}_+(t), R_0)] + \mathbb{E}_t[G^-_\delta(t)|B(\mathbf{x}_-(t), R_0)] < \frac{o(1)C(t)}{t\delta(t)}
\end{aligned}
\tag{3.22}
$$

In view of the assumed lower bound on the density this implies in particular that

$$
\sup_{\mathbf{x}_*,\, r<\frac{R_0}{2}} \int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{o(1)C(t)}{t^{1-\beta}} = \frac{o(1)}{t^{2\gamma}}. \tag{3.23}
$$

STEP 3: **sliding averages**. We will now reconnect the two averages $u_{\mathbf{x}_+,r}$ and $u_{\mathbf{x}_-,r}$ sliding along the line connecting $\mathbf{x}_+$ and $\mathbf{x}_-$, and show that the variation of those averages on each step is decreasing over time.

Denote the direction vector $\mathbf{n} = \frac{\mathbf{x}_+ - \mathbf{x}_-}{|\mathbf{x}_+ - \mathbf{x}_-|}$ and define a sequence of *overlapping* balls, $B_k = B(\mathbf{x}_k, \frac{r}{10}),\ k = 0, \ldots, K$, with centers given by $\mathbf{x}_k = \mathbf{x}_- + \frac{19r}{100} k\mathbf{n}$, starting at $\mathbf{x}_-$ and ending, with $K = [\frac{|\mathbf{x}_+ - \mathbf{x}_-|}{19r/100}]$, at $\mathbf{x}_{K+1} = \mathbf{x}_+$, see Figure 3.

Chebychev inequality, followed by (3.23) applied to the ball centered at $\mathbf{x}_* = \mathbf{x}_0$, yields that for our fixed $t \in [T, C_0 T]$,

$$
|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| > \lambda\}| \leqslant \frac{1}{\lambda^2} \int_{B_0} |u(\mathbf{x}) - u_{\mathbf{x}_0,r}|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{o(1)}{\lambda^2 t^{2\gamma}}.
$$

We now fix scale $r := R_0/4$: noticing that $|B_k \cap B_{k+1}| = c_0 R_0^n$ for all $k \leqslant K$, we set $\lambda = 2\frac{\sqrt{o(1)}}{\sqrt{c_0} t^\gamma R_0^{n/2}}$ so that

$$
|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| > \lambda\}| \leqslant \frac{1}{4}|B_0 \cap B_1|.
$$

By shifting the same argument to the variation around the averaged value $u_{\mathbf{x}_1,r}$, centered at $\mathbf{x}_* = \mathbf{x}_1$, we obtain

$$
|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_1,r}| > \lambda\}| \leqslant \frac{1}{4}|B_0 \cap B_1|.
$$

Consequently the complements of the two sets must have a point in common in $B_0 \cap B_1$:

$$
\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| \leqslant \lambda\} \cap \{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_1,r}| \leqslant \lambda\} \neq \emptyset,
$$

which implies that

$$
|u_{\mathbf{x}_0,r} - u_{\mathbf{x}_1,r}| \leqslant 2\lambda.
$$

Continuing in the same manner we obtain the same bound for all consecutive averages:

$$|u_{\mathbf{x}_k,r} - u_{\mathbf{x}_{k+1},r}| \leqslant 2\lambda.$$

Hence,

$$|u_{\mathbf{x}_-,r} - u_{\mathbf{x}_+,r}| \leqslant 2(K+1)\lambda \lesssim o(1)t^{-\gamma}.$$

Note that $K \leqslant 400\pi/R_0$, so it is bounded by an absolute constant. Furthermore, in view of (3.22), we can estimate

$$\begin{aligned} u_{\mathbf{x}_+,r} &\geqslant \frac{1}{m_t(B(\mathbf{x}_+,r))} \int_{B(\mathbf{x}_+,r)\setminus G_\delta^+} u_+(t)(1-\delta(t))\,\mathrm{d}m_t \\ &\geqslant u_+(t)(1-\delta(t))(1-\mathbb{E}_t[G_\delta^+(t)|B(\mathbf{x}_+(t),R_0)]) \geqslant u_+(t)(1-\delta(t))\left(1-\frac{o(1)C(t)}{t\delta(t)}\right). \end{aligned}$$

Hence,

$$u_+(t) - u_{\mathbf{x}_+,r}(t) \lesssim \delta(t) + \frac{o(1)C(t)}{t\delta(t)}.$$

The optimal bound is then achieved when $\delta(t) = \sqrt{\frac{o(1)C(t)}{t}}$. Recall that $C(t)$ is either uniform when $\tau \geqslant n$, or $C(t) = t^{\beta(1-\tau/n)}$ if $\tau < n$. Consequently, either $\delta(t) = o(1)t^{-1/2}$ or $\delta(t) = o(1)t^{-\gamma}$, respectively. In either case, since $\gamma \leqslant \frac{1}{2}$, we obtain $\delta(t) \leqslant o(1)t^{-\gamma}$.

A similar argument goes through for the bottom average. Thus, we end up with th desired bound $u_+(t) - u_-(t) \lesssim o(1)t^{-\gamma}$. □

4. Global well-posedness in 1D

In this section we will construct a more complete theory of one-dimensional topological models:

$$\text{(4.1)}\quad \begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x = [\mathscr{L}_\phi, u](\rho), \quad \phi(x,y) = |x-y|^{-(1+\alpha-\tau)} \times \mathrm{d}_\rho^{-\tau}(x,y) \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}.$$

What distinguished the 1D case is that classical geometric-based models using radial kernels $\phi = |x-y|^{1+\alpha}$, satisfy an extra conservation law:

$$e_t + (ue)_x = 0, \qquad e := u_x + \mathscr{L}_\phi \rho. \tag{4.2}$$

The derivation of the conservative "e"-equation is straightforward with either smooth or singular *radial* kernels, [14, 53]. It plays a key role in the regularity and hence unconditional flocking of the 1D alignment with geometric-based communication, [14, 53, 55]. A priori, there is no reason for (4.2) to hold in our case: the derivation of such law stumbles upon the difficulty that the operator $\mathscr{L}_\phi$ does *not* commute with derivatives. Nevertheless, *it is remarkable that the law* (4.2) *still survives for anisotropic topological kernels.* To make our analysis rigorous we need to develop calculus of the operator $\mathscr{L}_\phi$ and collect several analytical facts before we can proceed. This will be done in Section 4.1.

Once we justify (4.2), we can proceed in section 4.2 to the regularity of the 1D solution along the lines of [53, 54]. Since the topological kernels lack translation invariance, we need to revisit the question of propagation of regularity, section 4.4 and Hölder regularization of the density on sections 4.5.1 and 4.5.3. Let us state the most complete global existence result in 1D settings, which covers Theorem 1.7 as a particular case.

Theorem 4.1. *Let $0 < \alpha < 2$. Consider the 1D (τ, α)-model (4.1) subject to given initial conditions $(\rho_0, u_0) \in H^{3+\alpha/2} \times H^4$, with non-vacuous density $0 < c_0 < \rho_0(x) < C_0$.*

(i) **Global existence**. *If either $\tau \leqslant \alpha$, or if $\tau > \alpha$ and in addition the following smallness condition holds,*[7]

$$M_0^\tau \left|\frac{e_0}{\rho_0}\right|_\infty < \frac{R_0^{\tau-\alpha}}{\tau - \alpha}, \qquad e_0 = u_0' + \mathscr{L}_\phi \rho_0,$$

then there exists a global in time smooth solution (ρ, u) in the class

$$\begin{aligned} &\rho \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^{3+\frac{\alpha}{2}}), \\ &u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^4) \cap L^2_{\text{loc}}(R^+; H^{4+\frac{\alpha}{2}}). \end{aligned} \tag{4.3}$$

(ii) **Alignment**. *If $\tau \geqslant 1$ or $e_0 = 0$ then* any *smooth solution aligns, $u(t) \to u_\infty$. In the case $e_0 = 0$, this alignment comes with the rate*

$$|\rho(t) - \bar{\rho}|_\infty + |u(t) - u_\infty|_\infty \leqslant \frac{o(1)}{\sqrt{t}}, \qquad \bar{\rho} = \frac{1}{2\pi} M_0, \;\; u_\infty = \frac{P_0}{M_0}. \tag{4.4}$$

Remark 4.2. (**Smooth solutions and alignment**). If $\tau > \alpha$, then the additional smallness assumption (i) is sufficient for a uniform upper bound, see Lemma 4.7, which fulfills the assumption (3.15) of Theorem 3.2 and existence of smooth solution follows.
We note that the alignment for $\tau \geqslant 1$ stated in case (ii) follows directly from theorem (3.2). Indeed, the lower bound on the density (3.14) in this case requires the rate of $1/(1+t)$ which will be established for any regular solutions in Lemma 4.6. If $e_0 = 0$ in case (ii) then $e(t, \cdot) \equiv 0$, and (4.1) is reduced to

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t = \displaystyle\int_{\mathbb{T}} \phi(x, y)(u(y) - u(x))\rho(y) dy. \end{cases}$$

Further, the structure of the density equation changes to a pure drift-diffusion, see (4.16), $\rho_t + u\rho_x = \rho \mathscr{L}_\phi(\rho)$, which enforces the maximum principle: $c_0 < \rho(t, x) < C_0$. Hence, Theorem 3.2 applies to give the claimed rate for u. We will postpone the discussion of flocking till Section 4.6.

Remark 4.3. (**Local existence**). The local existence of solutions in Sobolev classes stated in (4.3) follows along the lines of the result established in [53] based on the standard fixed point argument. Additional details pertaining to the topological component will already be a part of the main proof of Theorem 4.1 below. We therefore will omit the details.

The proof will be split into several stages. First, before we even embark into technicalities of the argument, we develop necessary tools to work with the operator $\mathscr{L}_\phi$ itself. It will be done in the next section. Second, we establish a priori estimates on the density the are necessary to sustain uniform parabolicity and conclude the alignment, see Section 4.3. Third, we prove a propagation of regularity result, Proposition 4.8, which states that if one can propagate some modulus of continuity of the density, then one can propagate any higher order regularity for both u and ρ. Fourth, we show how to gain a Hölder modulus of continuity from several sources. In the case $1 < \alpha < 2$ we reduce the problem to a known Schauder estimate for fractional singular operators. For the case $\alpha = 1$, we employ the

[7] This is a scaling invariant condition, see Section 4.5.3

DeGiorgi method along the lines of Caffarelli, Chan, and Vasseur work [9] with significant upgrades related to the presence of drift, source, and asymmetry of the kernel involved. We also treat the system as truly nonlinear, see also [27], and highlight scaling properties of the system which become very important, see (4.52)-(4.53). In the case $0 < \alpha < 1$ we adopt Silvestre's result [56] which essentially works in our settings due to gained $C^{1-\alpha}$ regularity of the drift.

4.1. **Leibnitz rules and regularization.** We start with basic product formulas for the derivative of $\mathscr{L}_\phi f$ provided f and ρ are smooth. First, let us observe that (2.16) in 1D case takes a simpler form:

$$\partial_x \mathrm{d}_\rho(x+z,x) = (\rho(x+z)-\rho(x))\operatorname{sgn}(z) = \delta_z\rho(x)\operatorname{sgn}(z). \tag{4.5}$$

A formal computation with the use of (4.5) yields

$$(\mathscr{L}_\phi f)'(x) = \mathscr{L}_\phi(f')(x) + \int \partial_d\phi(\mathrm{d}_\rho(x,y),x-y)(\rho(y)-\rho(x))\operatorname{sgn}(y-x)(f(y)-f(x))\,\mathrm{d}y.$$

The integral on the right hand side is again of the type $\mathscr{L}_{\phi'}(f)$, where with some abuse of notation we differentiated $\phi=\phi(d)$ at $d=\mathrm{d}_{\rho(t,x)}$,

$$\phi' = \partial_d\phi(\mathrm{d}_\rho(x,y),x-y)(\rho(y)-\rho(x))\operatorname{sgn}(y-x). \tag{4.6}$$

The symmetric kernel ϕ' is of the same order $1+\alpha$. So, we can make sense of the integral in the same way as we did for $\mathscr{L}_\phi$. Thus, the product formula we seek reads

$$(\mathscr{L}_\phi f)' = \mathscr{L}_\phi(f') + \mathscr{L}_{\phi'} f. \tag{4.7}$$

Justification is straightforward. For any $g\in C^\infty$, we have

$$\begin{aligned}
\langle(\mathscr{L}_\phi f)',g\rangle &= -\langle \mathscr{L}_\phi f, g'\rangle \\
&= \frac12\int \delta_z f(x)\delta_z g'(x)\phi(\mathrm{d}_\rho(x+z,x),z)\,\mathrm{d}x\,\mathrm{d}z \\
&= -\frac12\int \delta_z f'(x)\delta_z g(x)\phi(\mathrm{d}_\rho(x+z,x),z)\,\mathrm{d}x\,\mathrm{d}z - \frac12\int \delta_z f(x)\delta_z g(x)\psi(x,z)\,\mathrm{d}x\,\mathrm{d}z \\
&= \langle \mathscr{L}_\phi(f'),g\rangle + \langle \mathscr{L}_{\phi'} f, g\rangle.
\end{aligned}$$

Continuing in the same fashion we obtain

$$(\mathscr{L}_\phi f)'' = \mathscr{L}_\phi(f'') + 2\mathscr{L}_{\phi'} f' + \mathscr{L}_{\phi''} f, \tag{4.8}$$

where

$$\phi'' = \partial_{dd}\phi(\mathrm{d}_\rho(x,y),x-y)(\rho(y)-\rho(x))^2 + \partial_d\phi(\mathrm{d}_\rho(x,y),x-y)(\rho'(y)-\rho'(x))\operatorname{sgn}(y-x). \tag{4.9}$$

Clearly, one obtains higher order Leibnitz rules in similar fashion provided ρ is regular enough:

$$(\mathscr{L}_\phi f)^{(n)} = \sum_{k=0}^n \frac{n!}{k!(n-k)!}\mathscr{L}_{\phi^{(k)}} f^{(n-k)}. \tag{4.10}$$

We can now discuss a regularization property of the operator $\mathscr{L}_\phi$. In the classical case when $\alpha_1=0$, we would have the natural gain of α derivatives: if $\mathscr{L}_\phi f\in H^s$, then $f\in H^{s+\alpha}$. For the topological kernels this is not likely to be true. Instead, we can prove an $\frac{\alpha}{2}$-gain of derivatives.

Lemma 4.4. *Suppose $f,\rho\in H^n$, $n=0,1,2,\dots$, and suppose $\mathscr{L}_\phi f\in H^n$. Then $f\in H^{n+\frac{\alpha}{2}}$.*

Proof. Note that the case $n = 0$ is a simple consequene of Lemma 2.2. For $n = 1, 2, \dots,$ we have

$$|\langle (\mathscr{L}_\phi f)^{(n)}, f^{(n)} \rangle| \leqslant |\mathscr{L}_\phi f|_{H^n} |f|_{H^n}.$$

On the other hand, according to (4.10), when $k = 0$, the paring gives $H^{n+\alpha/2}$-norm of f:

$$|f|^2_{H^n} \lesssim \langle \mathscr{L}_\phi f^{(n)}, f^{(n)} \rangle \lesssim \sum_{k=1}^{n} |\langle \mathscr{L}_{\phi^{(k)}} f^{(n-k)}, f^{(n)} \rangle| + |\mathscr{L}_\phi f|_{H^n} |f|_{H^n}.$$

Thus, it remains to estimate all the terms in the sum. Note that the highest order of derivative of $\delta_z \rho(x)$ in the kernel $\phi^{(k)}$ is $k-1$. So, if $k \leqslant n-1$, then the highest order in the entire sum is $n-1$. Using that $\rho^{n-1} \in L^\infty$, we can simply use the bound $|\phi^{(k)}| \lesssim 1/|z|^{1+\alpha}$ and estimate

$$|\langle \mathscr{L}_{\phi^{(k)}} f^{(n-k)}, f^{(n)} \rangle| \lesssim |f^{(n-k)}|_{H^{\alpha/2}} |f^{(n)}|_{H^{\alpha/2}} \lesssim |f^{(n)}|_{H^{\alpha/2}}.$$

When $k = n$ the only term that remains to estimate is the one containing the highest derivative of the density:

$$I = \int \delta_z \rho^{(n-1)}(x) \delta_z f(x) \delta_z f^{(n)}(x) \frac{\operatorname{sgn} z}{(\mathrm{d}_\rho(x+z, x))^{\tau+1} |z|^{1+\alpha-\tau}} \, \mathrm{d}z \, \mathrm{d}x$$

For $n = 2, 3, \dots$ we simply replace $|\delta_z f(x)| \leqslant |f'|_\infty |z|$, and estimate the rest by Cauchy-Schwartz:

$$|I| \leqslant |f'|_\infty |\rho|_{H^{n-1+\alpha/2}} |f|_{H^{n+\alpha/2}} \leqslant |\rho|_{H^n} |f|_{H^n} |f|_{H^{n+\alpha/2}}$$

For $n = 1$, we obtain

$$\begin{aligned} |I| &\leqslant \int \frac{|\delta_z \rho(x)|}{|z|^2} \frac{|\delta_z f(x)|}{|z|^{\frac{\alpha-1}{2}}} \frac{|\delta_z f'(x)|}{|z|^{\frac{1+\alpha}{2}}} \, \mathrm{d}z \, \mathrm{d}x \leqslant |\rho|_{W^{4,\frac{3}{4}}} |f|_{W^{4,\frac{1}{4}}} |f|_{H^{1+\frac{\alpha}{2}}} \\ &\leqslant |\rho|_{H^1} |f|_{H^1} |f|_{H^{1+\frac{\alpha}{2}}}, \end{aligned}$$

where in the middle term we raised α to its highest value 2. This finishes the proof. ☐

4.2. **An additional conservation law.** The conservative "e"-equation (4.2) is a heart of matter for the 1D regularity theory, along the lines of [53, 54, 55, 25]. We derive it using use the product formula (4.7).

Lemma 4.5 (**The conservation law of** e)**.** *All topological* (τ, α)*-models obey the conservation law*

$$e_t + (ue)_x = 0, \qquad e = u_x + \mathscr{L}_\phi \rho, \quad \phi = \psi_1(r) \times \psi_2(\mathrm{d}_\rho)$$

Proof. Differentiating the velocity equation and using the product rule (4.7) we obtain

$$u_t' + u'u' + uu'' = \mathscr{L}_\phi((u\rho)') - u' \mathscr{L}_\phi(\rho) - u(\mathscr{L}_\phi(\rho))' + \mathscr{L}_{\phi'}(u\rho). \tag{4.11}$$

The finite difference in the integral representation of the last term is given by

$$u(y)\rho(y) - u(x)\rho(x) = \int_x^y (u\rho)'(\zeta) d\zeta = -\int_x^y \rho_t(\zeta) d\zeta = -\partial_t \mathrm{d}(x, y) \operatorname{sgn}(y - x).$$

Recall the formula for the distance $\mathrm{d}_\rho(x, y) = \left| \int_x^y \rho(t, z) \, \mathrm{d}z \right|$, we obtain

$$\int_x^y \rho_t(\zeta) d\zeta = \partial_t \mathrm{d}_\rho(x, y) \operatorname{sgn}(y - x).$$

Thus,

$$\mathscr{L}_{\phi'}(u\rho) = -\int \partial_t \mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x)\phi'(x,y)\,\mathrm{d}y.$$

Recalling the formula for ϕ' (4.6) we obtain the relationship:

$$\mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x)\phi'(x,y) = \partial_t\phi(x,y)(\rho(y)-\rho(x)).$$

So, $\mathscr{L}_{\phi'}(u\rho) = -\int \partial_t\phi(x,y)(\rho(y)-\rho(x))\,\mathrm{d}y$. Putting it together with the $\mathscr{L}_\phi((u\rho)')$ term we obtain

$$\mathscr{L}_\phi((u\rho)') + \mathscr{L}_{\phi'}(u\rho) = -\partial_t\mathscr{L}_\phi(\rho).$$

Grouping together terms in (4.11) we arrive at

$$(u' + \mathscr{L}_\phi(\rho))_t + u'(u' + \mathscr{L}_\phi(\rho)) + u(u' + \mathscr{L}_\phi(\rho))' = 0,$$

which is precisely the law (4.2). □

Paired with the mass equation we find that the ratio $q = e/\rho$ satisfies the transport equation

$$\frac{\mathrm{D}}{\mathrm{D}t} q = q_t + u q_x = 0.$$

Starting from sufficiently smooth initial condition with ρ_0 away from vacuum we can assume that $|q(t)|_\infty = |q_0|_\infty < \infty$. This gives a priori pointwise bound

$$|e(t,x)| \lesssim \rho(t,x). \tag{4.12}$$

The argument can be bootstrapped to higher order derivatives (see [53, Sec. 2]) as follows. The next order quantity $q_1 = q_x/\rho$ is again transported

$$\frac{\mathrm{D}}{\mathrm{D}t} q_1 = 0. \tag{4.13}$$

Solving for $e'(\cdot,t)$ we obtain another a priori pointwise bound

$$|e'(t,x)| \lesssim |\rho'(t,x)| + \rho(t,x). \tag{4.14}$$

Iterating we obtain

$$|e^{(k)}(t,x)| \lesssim |\rho^{(k)}(t,x)| + \ldots + \rho(t,x), \qquad k = 0,1,2\ldots \tag{4.15}$$

Using e allows one to rewrite the density equation in parabolic form:

$$\rho_t + u\rho_x + e\rho = \rho\mathscr{L}_\phi(\rho) \tag{4.16}$$

Similarly, one can write the equation for the momentum $m = \rho u$:

$$m_t + u m_x + e m = \rho\mathscr{L}_\phi(m). \tag{4.17}$$

With a priori bounds on the density we establish in the next section, this allows view equations (4.16) – (4.17) as a fractional parabolic system with rough drift and bounded force, which opens for possibility to apply recently developed tools of regularity theory for such equations. This will be the subject of all subsequent discussion.

4.3. **Bounds on the density.** Let us first make one trivial remark: if $e_0 = 0$, then the density equation becomes a pure drift-diffusion and hence by the maximum principle the density remains within the confines of its initial bounds:

$$\min \rho_0 \leqslant \rho(t,x) \leqslant \max \rho_0. \tag{4.18}$$

In general, however, the e-quantity introduces a Riccati term that needs to be controlled by the singularity of the kernel. First, we establish a bound from below.

Lemma 4.6. *Let (ρ, u) be a smooth solution to (4.1) subject to initial density ρ_0 away from vacuum. Then there is a positive constant $c = c(\rho_0, e_0) > 0$ such that*

$$\rho(t,x) \geqslant \frac{c}{1+t}, \qquad x \in \mathbb{T},\ t \geqslant 0. \tag{4.19}$$

Proof. Let us recall that the density equation can be rewritten as

$$\rho_t + u\rho_x = -q\rho^2 + \rho \mathscr{L}_\phi(\rho). \tag{4.20}$$

Let ρ_- and x_- be the minimum value of ρ and a point where such value is achieved. Invoking Lemma 5.1 to justify the pointwise evaluation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_- \geqslant -|q_0|_\infty \rho_-^2 + \rho_- \int_{\mathbb{T}} \phi(x_-,y)(\rho(y,t) - \rho_-)\,\mathrm{d}y \geqslant -|q_0|_\infty \rho_-^2.$$

The lower bound (4.19) follows. □

The next lemma gives a range of conditions implying boundedness from above.

Lemma 4.7. *Let (ρ, u) be a smooth solution of the (τ,α)-model (4.1), subject to initial density ρ_0 away from vacuum, $0 < c < \rho_0 < C < \infty$. Assume that either (i) $\tau \leqslant \alpha$, or else if $\tau > \alpha$, that (ii) the initial condition satisfies*

$$M_0^\tau |q_0|_\infty < \frac{R_0^{\tau-\alpha}}{\tau - \alpha}, \quad q_0 = \frac{e_0}{\rho_0}.$$

Then the density is uniformly bounded in time:

$$\rho(t,x) < C(M_0, |q_0|_\infty, \phi), \qquad x \in \mathbb{T},\ t \geqslant 0, \tag{4.21}$$

provided

Proof. Evaluating the mass equation at extreme maximum we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_+ \leqslant |q_0|_\infty \rho_+^2 + \rho_+ \int_{|z|<R_0} \frac{1}{M_0^\tau |z|^{1+\alpha-\tau}} (\rho(t, x_+ + z) - \rho_+)\,\mathrm{d}z.$$

Consider the case $\alpha \geqslant \tau$. Let us further reduce the region of integration to $\varepsilon < |z| < R_0$ for any fixed $\varepsilon > 0$. By choosing ε small enough we can ensure that

$$\int_{\varepsilon<|z|<R_0} \frac{1}{|z|^{1+\alpha-\tau}} > 2|q_0|_\infty M_0^\tau.$$

Then for that fixed ε we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_+ \leqslant -|q_0|_\infty \rho_+^2 + C\rho_+.$$

The result follows. Otherwise, for any $\varepsilon > 0$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_+ &\leqslant |q_0|_\infty \rho_+^2 + \rho_+ \frac{1}{M_0^\tau} \int_{\varepsilon<|z|<R_0} \frac{1}{|z|^{1+\alpha-\tau}} (\rho(x_+ + z, t) - \rho_+)\,\mathrm{d}z \\ &\leqslant |q_0|_\infty \rho_+^2 + \rho_+ \frac{1}{M_0^\tau}\left(M_0 \varepsilon^{-(1+\alpha-\tau)} - \rho_+ \frac{R_0^{\tau-\alpha} - \varepsilon^{\tau-\alpha}}{\tau - \alpha}\right)\end{aligned}$$

Clearly, under the smallness assumption of the lemma, for $\varepsilon > 0$ small enough the quadratic term gains a negative sign. The result follows. □

4.4. **Propagation of regularity.** Our goal in this section is to establish a general propagation result that relies on existence of a modulus of continuity for the density.

Proposition 4.8. *Consider a local solution to any (τ, α)-model, $0 < \alpha < 2$, $\tau > 0$:*

$$\begin{aligned} u &\in L^\infty_{\mathrm{loc}}([0,T); H^4) \cap L^2_{\mathrm{loc}}([0,T); H^{4+\frac{\alpha}{2}}) \\ e, \mathscr{L}_\phi \rho &\in L^\infty_{\mathrm{loc}}([0,T); H^3) \\ \rho &\in L^\infty_{\mathrm{loc}}([0,T); H^{3+\frac{\alpha}{2}}). \end{aligned}$$

Suppose there are constants $c, C > 0$ such that

$$c \leqslant \rho(t,x) \leqslant C, \quad (t,x) \in [0,T) \times \mathbb{T}. \tag{4.22}$$

Furthermore, suppose that ρ is uniformly continuous on $\mathbb{T} \times [0,T)$, i.e. there exits a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$, that is non-decreasing, bounded, and $\omega(0) = 0$, such that

$$|\rho(t, x+h) - \rho(t,x)| \leqslant \omega(|h|) \tag{4.23}$$

for any $x, h \in \mathbb{T}$, $t \in [0,T]$. Then the solution remains uniformly in the classes stated above on $[0,T]$ and, hence, can be extended beyond T.

Proof. We argue along the lines of our proof of propagation of regularity in the case of geometric singularity [53, 54]. The presence of topological kernel introduces additional computations that rely of the calculus of $\mathscr{L}_\phi$ developed in the previous parts of this section.

We have split the proof in seven steps. In steps 1–3 we establish control over first derivatives under the assumption (4.23). So, the goal is to show that $\sup_{t<T} (|\rho'(t,\cdot)|_\infty + |u'(t,\cdot)|_\infty) < \infty$. Higher derivatives are estimated in steps 4–7.

STEP 1: **Control over** ρ'. Let us differentiate (4.20):

$$\partial_t \rho' + u\rho'' + u'\rho' + e'\rho + e\rho' = \rho' \mathscr{L}_\phi \rho + \rho \mathscr{L}_\phi \rho' + \rho \mathscr{L}_{\phi'} \rho. \tag{4.24}$$

Using again $u' = e - \mathscr{L}_\phi \rho$ we rewrite

$$\partial_t \rho' + u\rho'' + e'\rho + 2e\rho' = 2\rho' \mathscr{L}_\phi \rho + \rho \mathscr{L}_\phi \rho' + \rho \mathscr{L}_{\phi'} \rho.$$

Evaluating at the maximum of ρ' and multiplying by ρ' we obtain

$$\partial_t |\rho'|^2 + e'\rho\rho' + 2e|\rho'|^2 = 2|\rho'|^2 \mathscr{L}_\phi \rho + \rho\rho' \mathscr{L}_\phi \rho' + \rho'\rho \mathscr{L}_{\phi'} \rho. \tag{4.25}$$

In view of (4.12) and (4.14) we can bound

$$|e'\rho\rho' + 2e|\rho'|^2| \leqslant C(|\rho'|^2 + |\rho'|).$$

Thus,

$$\partial_t |\rho'|^2 = C(|\rho'|^2 + |\rho'|) + 2|\rho'|^2 \mathscr{L}_\phi \rho + \rho\rho' \mathscr{L}_\phi \rho' + \rho'\rho \mathscr{L}_{\phi'} \rho. \tag{4.26}$$

Let us note that in view of Lemma 5.1 pointwise evaluation of all operators is justified. Due to the bound from below on ρ, we estimate

$$\rho\rho'\mathscr{L}_\phi\rho' \geqslant c_1 \int_{\mathbb{R}} \frac{(\rho'(x+z)-\rho'(x))\rho'(x+z)}{|z|^{1+\alpha}} h(z)\,\mathrm{d}z \geqslant c_2 \mathrm{D}_\alpha \rho'(x). \tag{4.27}$$

where

$$\mathrm{D}_\alpha\rho'(x) = \int_{\mathbb{R}} \frac{|\rho'(x)-\rho'(x+z)|^2}{|z|^{1+\alpha}} h(z)\,\mathrm{d}z.$$

According to [20], and complementing h to full unity, we obtain

$$\mathrm{D}_\alpha\rho'(x) \geqslant \frac{1}{2}\mathrm{D}_\alpha\rho'(x) + C\frac{|\rho'(x)|^{2+\alpha}}{|\rho|_\infty^\alpha} - c|\rho'|_2^2. \tag{4.28}$$

Because of the second term in (4.28), all the powers of ρ up to $2+\alpha$ are absorbed. The goal now is to find bounds on all the terms remaining in the energy budget that are ε-multiples of top power $|\rho'|_\infty^{2+\alpha}$. So, in particular at this stage we can rewrite (4.26) as

$$\partial_t|\rho'|^2 = C + 2|\rho'|^2\mathscr{L}_\phi\rho + \rho'\rho\mathscr{L}_{\phi'}\rho - \frac{1}{2}\mathrm{D}_\alpha\rho'(x) - c|\rho'(x)|^{2+\alpha}. \tag{4.29}$$

We now carry out a relatively simple weakly singular case.

Step 1.1: **The case** $0<\alpha<1$. Let us estimate $|\rho'|^2\mathscr{L}_\phi\rho$. To this end, we fix a small parameter $\varepsilon>0$ to be determined late. We then find a scale $\ell>0$ so that $\omega(\ell)<\varepsilon^{1+\alpha}$. Note that ℓ is independent of time. Next we consider another time-dependent scale $r=\frac{\varepsilon}{|\rho'|_\infty}$. If $\ell>r$, then we proceed as follows:

$$\begin{aligned} |\mathscr{L}_\phi\rho(x)| &\leqslant \int_{|z|<r} |\rho(x+z)-\rho(x)|\phi\,\mathrm{d}z + \int_{r<|z|<\ell} |\rho(x+z)-\rho(x)|\phi\,\mathrm{d}z \\ &\quad + \int_{|z|>\ell} |\rho(x+z)-\rho(x)|\phi\,\mathrm{d}z \\ &\leqslant |\rho'|_\infty r^{1-\alpha} + \omega(\ell)r^{-\alpha} + |\rho|_\infty \ell^{-\alpha}. \end{aligned}$$

Hence, given all the choices of constants we have made,

$$|\rho'|^2|\mathscr{L}_\phi\rho| \lesssim (\varepsilon^{1-\alpha}+\varepsilon)|\rho'|^{2+\alpha} + C(\ell).$$

For small ε the main term clearly gets absorbed into dissipation. If however $\ell<r$, then

$$|\rho'|_\infty \leqslant \varepsilon/\ell \tag{4.30}$$

In this case we simply split the integral between $|z|<1$ and $|z|>1$, and find a bound

$$|\rho'|^2|\mathscr{L}_\phi\rho| \leqslant C(\ell,\varepsilon),$$

which is uniform on $[0,T]$. In either case, we are left with a constant $C(\ell,\varepsilon)$.

It remains to estimate $\rho'\rho\mathscr{L}_{\phi'}\rho$. The nonlocal term takes form

$$\mathscr{L}_{\phi'}\rho = -\alpha_1 p.v. \int \frac{(\rho(x+z)-\rho(x))^2 \operatorname{sgn} z}{\mathrm{d}_\rho^{\tau+1}(x+z,x)|z|^{1+\alpha-\tau}} h(z)\,\mathrm{d}z.$$

We proceed similar to the above. If $r<\ell$, then

$$|\rho'\rho\mathscr{L}_{\phi'}\rho| \leqslant |\rho'|^3 r^{1-\alpha} + |\rho'|\omega^2(\ell)r^{-1-\alpha} + |\rho'|C(\ell) \leqslant (\varepsilon^{1-\alpha}+\varepsilon^{1+\alpha})|\rho'|^{2+\alpha} + |\rho'|C(\ell).$$

By Young, the last term is absorbed, as well as the first two for small ε. The case $\ell < r$ is handled as before with the advantage of time-independent bound (4.30). We arrive at

$$\partial_t |\rho'|^2 \leqslant c_1 - c_2 \mathrm{D}_\alpha \rho'. \tag{4.31}$$

This finished the proof of control over ρ'.

STEP 1.2: **The case** $1 \leqslant \alpha < 2$. Here our choice of r and ℓ will be the same as above. Moreover the case $\ell < r$ is straightforward due to (4.30). We proceed under the assumption that $r < \ell$. We use decomposition (5.9) with further breakdown of the integral:

$$\begin{aligned} \mathscr{L}_\phi \rho(x) = & \int_{|z|<r} (\rho(x+z) - \rho(x) - \rho'(x) z) \phi \, \mathrm{d}z + \rho'(x) b_r(x) \\ & + \int_{|z|>r} (\rho(x+z) - \rho(x)) \, \phi \, \mathrm{d}z = I + \rho'(x) b_r(x) + J. \end{aligned}$$

Using that

$$|\rho(x+z) - \rho(x) - \rho'(x) z| = \left| \int_0^z (\rho'(x+w) - \rho'(x)) \, \mathrm{d}w \right| \leqslant \sqrt{\mathrm{D}_\alpha \rho'(x)} |z|^{1+\frac{\alpha}{2}}, \tag{4.32}$$

we obtain $|I| \leqslant r^{1-\alpha/2} \sqrt{\mathrm{D}_\alpha \rho'(x)}$. Next, due to (5.10), $|b_r(x)| \leqslant c|\rho'|_\infty r^{2-\alpha}$. The J-term is similar to the previous case, resulting in the bound

$$|J| \leqslant \omega(\ell) r^{-\alpha} + |\rho|_\infty \ell^{-\alpha}.$$

Altogether we obtain

$$\begin{aligned} ||\rho'|^2 \mathscr{L}_\phi \rho| & \leqslant c_1 |\rho'|_\infty^2 r^{1-\alpha/2} \sqrt{\mathrm{D}_\alpha \rho'(x)} + c_2 |\rho'|_\infty^4 r^{2-\alpha} + c_3 |\rho'|_\infty^2 (\omega(\ell) r^{-\alpha} + \ell^{-\alpha}) \\ & \leqslant \frac{1}{4} \mathrm{D}_\alpha \rho'(x) + c_4 |\rho'|_\infty^4 r^{2-\alpha} + c_3 |\rho'|_\infty^2 (\omega(\ell) r^{-\alpha} + \ell^{-\alpha}) \\ & \leqslant \frac{1}{4} \mathrm{D}_\alpha \rho'(x) + c_4 |\rho'|_\infty^{2+\alpha} (\varepsilon^{2-\alpha} + \varepsilon^{1+\alpha}) + c_5 |\rho'|_\infty^2. \end{aligned}$$

Clearly all the terms get absorbed leaving a uniform constant out.

For the next term $\rho' \rho \mathscr{L}_{\phi'} \rho$ we have

$$\begin{aligned} -\mathscr{L}_{\phi'} \rho = & \int \frac{(\rho(x+z) - \rho(x))^2 \operatorname{sgn} z}{\mathrm{d}^{\tau+1}(x+z, x) |z|^{1+\alpha-\tau}} \, \mathrm{d}z \\ = & \frac{1}{2} \int \frac{(\rho(x+z) - \rho(x))^2 - (\rho(x-z) - \rho(x))^2}{(\mathrm{d}_\rho(x+z, x))^{\tau+1} |z|^{1+\alpha-\tau}} \operatorname{sgn} z \, \mathrm{d}z \\ & + \frac{1}{2} \int \frac{(\rho(x-z) - \rho(x))^2 \Big((\mathrm{d}_\rho(x+z, x))^{\tau+1} - (\mathrm{d}_\rho(x-z, x))^{\tau+1} \Big)}{(\mathrm{d}_\rho(x+z, x))^{\tau+1} (\mathrm{d}_\rho(x-z, x))^{\tau+1} |z|^{1+\alpha-\tau}} \, \mathrm{d}z = \frac{1}{2}(J_1 + J_2). \end{aligned}$$

To estimate the first integral we compute

$$\begin{aligned} |(\rho(x+z) - \rho(x))^2 - (\rho(x-z) - \rho(x))^2| & = |\rho(x+z) + \rho(x-z) - 2\rho(x)| |\rho(x+z) - \rho(x-z)| \\ & \leqslant \left| \int_0^z (\rho'(x+w) - \rho'(x) + \rho'(x) - \rho'(x-w)) \, \mathrm{d}w \right| |\rho'|_\infty |z| \leqslant \sqrt{D_\alpha \rho'(x)} |z|^{2+\alpha/2} |\rho'|_\infty. \end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
J_1 &\leqslant c_1 \int_{|z|<r} \sqrt{\mathrm{D}_\alpha \rho'(x)} |z|^{-\alpha/2} \, \mathrm{d}z + \int_{r<|z|<\ell} \omega^2(\ell) |z|^{-2-\alpha} \, \mathrm{d}z + C(\ell) \\
&\leqslant \sqrt{\mathrm{D}_\alpha \rho'(x)} r^{1-\alpha/2} + \omega^2(\ell) r^{-1-\alpha} + C(\ell).
\end{aligned} \tag{4.33}
$$

Hence,

$$
\begin{aligned}
|\rho'||J_1| &\leqslant |\rho'|^2 \sqrt{\mathrm{D}_\alpha \rho'} r^{1-\alpha/2} + |\rho'|(\omega^2(\ell) r^{-1-\alpha} + C(\ell)) \\
&\leqslant \frac{1}{4} \mathrm{D}_\alpha \rho'(x) + c|\rho'|_\infty^4 r^{2-\alpha} + |\rho'|(\omega^2(\ell) r^{-1-\alpha} + C(\ell)).
\end{aligned}
$$

This finishes the computation as before. Finally, as to the J_2-term, we utilize the same estimates as in the proof of Lemma 5.1 to obtain

$$
|(\mathrm{d}_\rho(x+z,x))^{\tau+1} - (\mathrm{d}_\rho(x-z,x))^{\tau+1}| \leqslant |\rho'|_\infty |z|^{\tau+2}.
$$

So, we proceed with the usual splitting:

$$
J_2 \leqslant \int_{|z|<r} |\rho'|^3 \frac{1}{|z|^{\alpha-1}} \, \mathrm{d}z + \int_{r<|z|<\ell} \omega(\ell)^2 \frac{1}{|z|^{2+\alpha}} \, \mathrm{d}z + C(\ell) \leqslant |\rho'|^3 r^{2-\alpha} + \omega(\ell)^2 r^{-1-\alpha} + C(\ell).
$$

$$
|\rho'||J_2| \leqslant |\rho'|^4 r^{2-\alpha} + |\rho'|(\omega(\ell)^2 r^{-1-\alpha} + C(\ell)).
$$

This finishes the bounds. Putting them together we obtain (4.31).

STEP 2: **Control over $\mathscr{L}_\phi \rho$.** Before we embark into the second part, it is essential to establish control over $|\mathscr{L}_\phi \rho|_\infty$. For the models with $0 < \alpha < 1$, this is straightforward from $|\mathscr{L}_\phi \rho|_\infty \lesssim |\rho'|_\infty$ and the established control over $|\rho'|_\infty$. For the case $\alpha \geqslant 1$, we resort to another energy-enstrophy estimate on ρ''. The overall goal of this section will be to prove

$$
\mathscr{L}_\phi \rho \in L^2([0,T); L^\infty).
$$

So, let us write the second derivative of density:

$$
\begin{aligned}
&\partial_t \rho'' + u\rho''' + u'\rho'' + e''\rho + 3e'\rho' + 2e\rho'' = \\
&\qquad 2\rho'' \mathscr{L}_\phi \rho + 3\rho' \mathscr{L}_{\phi'} \rho + 3\rho' \mathscr{L}_\phi \rho' + 2\rho \mathscr{L}_{\phi'} \rho' + \rho \mathscr{L}_{\phi''} \rho + \rho \mathscr{L}_\phi \rho''.
\end{aligned} \tag{4.34}
$$

Now, we use the test-function ρ''/ρ. Via routine computation with the use of the density equation, one can observe that

$$
\left\langle \partial_t \rho'' + u\rho''' + u'\rho'', \frac{\rho''}{\rho} \right\rangle = \frac{1}{2} \partial_t \int \frac{1}{\rho} |\rho''|^2 \, \mathrm{d}x.
$$

In view of the bounds on the density we note that $\int \frac{1}{\rho} |\rho''|^2 \, \mathrm{d}x \sim |\rho''|_2^2$. So, it is sufficient to bound the rest of the terms in terms of $|\rho''|_2^2$. Going back to the last three terms on the left hand side, we use a priori control (4.15) and the established bound on the ρ' to obtain

$$
\left\langle e''\rho + 3e'\rho' + 2e\rho'', \frac{\rho''}{\rho} \right\rangle \lesssim 1 + |\rho''|_2^2.
$$

Here we used the pointwise bound $|e''| \lesssim |\rho''|$. We will have to deal with the right hand side now. At this point we have (omitting all the terms that are already bounded)

$$
\begin{aligned}
\partial_t \int \frac{1}{\rho} |\rho''|^2 \,\mathrm{d}x &\lesssim 1 + |\rho''|_2^2 + \int |\rho''|^2 |\mathscr{L}_\phi \rho| \,\mathrm{d}x + \int |\rho''| |\mathscr{L}_{\phi'} \rho| \,\mathrm{d}x \\
&\quad + \int |\rho''| |\mathscr{L}_\phi \rho'| \,\mathrm{d}x + \int \rho'' \mathscr{L}_{\phi'} \rho' \,\mathrm{d}x + \int \rho'' \mathscr{L}_{\phi''} \rho \,\mathrm{d}x + \int \rho'' \mathscr{L}_\phi \rho'' \,\mathrm{d}x \\
&= 1 + |\rho''|_2^2 + I_1 + I_2 + I_3 + I_4 + I_5 + J.
\end{aligned} \tag{4.35}
$$

Clearly, the last term J is dissipative:

$$
J \lesssim - \int \mathrm{D}_\alpha \rho''(x) \,\mathrm{d}x - \int |\rho''|^{2+\alpha} \,\mathrm{d}x,
$$

where in the latter we dropped $\frac{1}{|\rho'|_\infty^\alpha}$ from inside the integral since this term is bounded from below.

We now estimate I_1. Let us fix an $\varepsilon > 0$ and use representation formula (5.9) with $r = \varepsilon$. The drift term is bounded by $\sim \varepsilon^{2-\alpha}$ while the $|z| > \varepsilon$ potion of the integral by $|\rho'|_\infty \varepsilon^{1-\alpha}$. Since ε is fixed this produces only a term of the form $C_\varepsilon |\rho''|_2^2$ out of I_1. For the remaining portion we have

$$
\begin{aligned}
&\left| \int_{|z| \leqslant \varepsilon} (\delta_z \rho(x) - \rho'(x) z) \phi \,\mathrm{d}z \right| = \left| \int_{|z| < \varepsilon} \int_0^z \rho''(x+w)(z-w) \,\mathrm{d}w \phi \,\mathrm{d}z \right| \\
&\leqslant \int_{|z| < \varepsilon} \frac{1}{|z|^\alpha} \int_0^z |\rho''(x+w)| \,\mathrm{d}w \,\mathrm{d}z = \int_{|w| < \varepsilon} |\rho''(x+w)| \int_{|w| < |z|} \frac{1}{|z|^\alpha} \,\mathrm{d}z \,\mathrm{d}w \\
&= \int_{|w| < \varepsilon} |\rho''(x+w)| |w|^{1-\alpha} \,\mathrm{d}w
\end{aligned} \tag{4.36}
$$

Note that the kernel $|w|^{1-\alpha}$ is integrable. Using Minkowskii inequality, we finally obtain

$$
\left| \int_{|z| \leqslant \varepsilon} (\delta_z \rho(\cdot) - \rho'(\cdot) z) \phi \,\mathrm{d}z \right|_{L^{\frac{2+\alpha}{\alpha}}} \leqslant \int_{|w| < \varepsilon} |\rho''|_{\frac{2+\alpha}{\alpha}} |w|^{1-\alpha} \,\mathrm{d}w = |\rho''|_{\frac{2+\alpha}{\alpha}} \varepsilon^{2-\alpha}.
$$

Continuing with the I_1-term we obtain

$$
|I_1| \leqslant C_\varepsilon |\rho''|_2^2 + \varepsilon^{2-\alpha} |\rho''|_{2+\alpha}^2 |\rho''|_{\frac{2+\alpha}{\alpha}} \leqslant C_\varepsilon |\rho''|_2^2 + \varepsilon^{2-\alpha} |\rho''|_{2+\alpha}^3 \leqslant C + C_\varepsilon |\rho''|_2^2 + \varepsilon^{2-\alpha} |\rho''|_{2+\alpha}^{2+\alpha},
$$

where in the last steps we used that $\alpha \geqslant 1$. This shows that the highest term is absorbed into dissipation.

Moving on to I_2, we reuse the previous estimates on $\mathscr{L}_{\phi'} \rho$ which after replacing ρ' with constants simply reads $|\mathscr{L}_{\phi'} \rho| \leqslant \sqrt{\mathrm{D}_\alpha \rho'}$. Thus, $|I_2| \leqslant |\rho''|_2^2 + |\rho'|_{H^{\alpha/2}}^2$, and both terms are absorbed. Next, in the I_3-term the computation in the previous subsection implies that $\mathscr{L}_\phi \rho'(x)$ is bounded by

$$
\mathscr{L}_\phi \rho'(x) = ((5.9), r = 1) \leqslant |\rho''(x)| + \sqrt{\mathrm{D}_\alpha \rho''(x)}.
$$

Thus,

$$
|I_3| \leqslant C_\varepsilon |\rho''|_2^2 + \varepsilon \int \mathrm{D}_\alpha \rho''(x) \,\mathrm{d}x,
$$

which is under control with the dissipative term. Note that I_4-term is similar since, once again, the order of singularity of the kernel ϕ' is the same due to obtained control over ρ'. Lastly, the term I_5 contains kernel ρ'' which according to (4.9) consists of two parts, $\phi_1 + \phi_2$

as listed in (4.9). The order of ϕ_1 is again $1+\alpha$, so this part it similar to I_1. And finally, let us observe that $\mathscr{L}_{\phi_2}\rho = \mathscr{L}_{\phi'}\rho'$. Hence this term is exactly equal to I_3.

We thus have obtained the estimate

$$\partial_t \int \frac{1}{\rho}|\rho''|^2 \, \mathrm{d}x \leqslant C_1 + C_2|\rho''|_2^2 - c_3|\rho''|_{H^{\alpha/2}}^2, \tag{4.37}$$

which implies that $\rho'' \in L^\infty L^2 \cap L^2 H^{\alpha/2}$ on the given time interval $[0,T]$. By imbedding, $\rho' \in C^{1/2}$ uniformly. Hence for $\alpha < \frac{3}{2}$, the term $\mathscr{L}_\phi \rho$ is bounded directly from (5.9). If, however, $\alpha \geqslant 3/2$, then of course $\rho'' \in L^2 L^\infty$. This shows that $\mathscr{L}_\phi\rho \in L^2L^\infty$ as well.

Step 3: **Control over** $|u'|_\infty$. Again the case $0<\alpha<1$ is straightforward from $|\mathscr{L}_\phi\rho|_\infty \lesssim |\rho'|_\infty$, and uniform bound on e, (4.12). For $\alpha \geqslant 1$ we set out to make another round of estimates. It is more economical to deal with the momentum equation (4.17) for this purpose. Note that bounds on m' and u' are equivalent at this point.

So, we write

$$\partial_t m' + um'' + u'm' + e'm + em' = -\rho'\mathscr{L}_\phi m - \rho\mathscr{L}_\phi m' - \rho \mathscr{L}_{\phi'} m.$$

Evaluating at the maximum, replacing $u' = e - \mathscr{L}_\phi \rho$, and using the already established control over ρ', we obtain, up to a constant

$$\partial_t |m'|^2 \leqslant C + |m'|^2 + |m'|^2|\mathscr{L}_\phi\rho|_\infty + |m'||\mathscr{L}_\phi m| + |m'||\mathscr{L}_{\phi'} m| - \mathrm{D}_\alpha m'.$$

Absorbing $|m'|^2$ into the nonlinear lower bound on $\mathrm{D}_\alpha m'$ we further obtain

$$\partial_t |m'|^2 \leqslant C + |m'|^2|\mathscr{L}_\phi\rho|_\infty + |m'||\mathscr{L}_\phi m| + |m'||\mathscr{L}_{\phi'} m| - \frac{1}{2}\mathrm{D}_\alpha m'.$$

From the previous subsection, we know that $|\mathscr{L}_\phi\rho|_\infty$ is an integrable multiplier. So, it presents no problems in application of Grönwall's lemma. It remains to consider the remaining two terms, which are similar due to the same singularity in the kernels ϕ and ϕ'. But as is done several times previously, splitting the integral, this time with $r=1$, we immediately obtain

$$|m'||\mathscr{L}_\phi m| \leqslant |m'|\sqrt{\mathrm{D}_\alpha m'} + |m'| \leqslant \varepsilon \mathrm{D}_\alpha m' + |m'|^2 + |m'|.$$

which is readily absorbed. We arrive at

$$\partial_t|m'|^2 \leqslant C + |m'|^2 f(t), \quad f \in L^2(0,T),$$

and the desired result follows.

Step 4: **Control over** $|u|_{H^2}$ **and** $|u|_{H^3}$. Let us note that at this stage we established control over slopes and

$$e \in L^\infty([0,T];H^2), \quad \rho \in L^\infty([0,T];H^2) \cap L^2([0,T];H^{2+\frac{\alpha}{2}}).$$

following from (4.37), and pointwise $|e''| \lesssim |\rho''|$. It is more than sufficient to establish control over $|u|_{H^2}$. It is also sufficient to establish control in $u \in L^\infty H^3 \cap L^2 H^{3+\alpha/2}$. We will not show details of computations for this stage since those details are entirely similar to (and a subcase of) what we will perform in the top regularity spaces. We thus assume that

$$e \in L^\infty H^2, \quad \rho \in L^\infty H^2 \cap L^2 H^{2+\frac{\alpha}{2}}, \quad u \in L^\infty H^3 \cap L^2 H^{3+\alpha/2}$$

and move on to the next stage.

Step 5: **Control over** $|\rho''|_\infty$. We note that this is an intermediate step necessary to conclude the pointwise non-linear lower bound

$$\mathrm{D}_\alpha \rho'''(x) \geqslant c\frac{|\rho'''(x)|^{2+\alpha}}{|\rho''|_\infty} \gtrsim |\rho'''(x)|^{2+\alpha} \tag{4.38}$$

which will be used on the next stage. So, let us test (4.34) with ρ'' evaluated at a point of maximum. Given the quoted bounds available at this stage all the terms on the left are bounded by $C_1 + C_2|\rho''|_\infty^2$. Replacing $\mathscr{L}_\phi \rho$ on the right hand side in the first term with $e - u'$ we also find it bounded. So, given that ρ and ρ' are also bounded it remains to estimate

$$J_1 = \rho'' \mathscr{L}_{\phi'} \rho; \quad J_2 = \rho'' \mathscr{L}_\phi \rho'; \quad J_3 = \rho'' \mathscr{L}_{\phi'} \rho'; \quad J_4 = \rho'' \mathscr{L}_{\phi''} \rho$$

with the help of dissipation term

$$\rho'' \mathscr{L}_\phi \rho'' \lesssim -\mathrm{D}_\alpha \rho''(x) - |\rho''(x)|^{2+\alpha}.$$

For J_1 we recall the estimate from (4.33) and below with $r = 1$ so that $|J_1| \leqslant |\rho''|\sqrt{\mathrm{D}_\alpha \rho'(x)} + |\rho''| + C$. However, trivially, $|\mathrm{D}_\alpha \rho'(x)| \leqslant C|\rho''|^2 + C$. Thus, $|J_1| \leqslant c_1|\rho''|^2 + c_2$. As to J_2 we first invoke Lemma 5.1 to bound

$$|\mathscr{L}_\phi \rho'(x)| \leqslant C|\rho''(x)| + \left| \int \phi(x+z,x)(\rho'(x+z) - \rho'(x) - \rho''(x)z)\,\mathrm{d}z \right|.$$

As before, $|\rho'(x+z) - \rho'(x) - \rho''(x)z| \leqslant |z|^{1+\alpha/2}\sqrt{\mathrm{D}_\alpha \rho''(x)}$, hence, continuing,

$$\leqslant C|\rho''(x)| + \sqrt{\mathrm{D}_\alpha \rho''(x)} \int |z|^{-\alpha/2}\,\mathrm{d}z \lesssim |\rho''(x)| + \sqrt{\mathrm{D}_\alpha \rho''(x)}.$$

Thus, $|J_2| \leqslant C_\varepsilon |\rho''(x)|^2 + \varepsilon \mathrm{D}_\alpha \rho''(x)$, which is under control with dissipation.

Moving to J_3, first clearly for $0 < \alpha < 1$, $|\mathscr{L}_{\phi'} \rho'| \leqslant C|\rho''||\rho'|$ and we are done. For $\alpha \geqslant 1$ we first estimate :

$$\begin{aligned}\mathscr{L}_{\phi'} \rho' &= \int (\delta_z \rho'(x) - \rho''(x)z)\delta_z \rho(x) \frac{\operatorname{sgn}(z)}{\mathrm{d}_\rho^{\tau+1}(x+z,x)|z|^{1+\alpha-\tau}}\,\mathrm{d}z \\ &\quad + \rho''(x) \int \frac{\delta_z \rho(x)}{\mathrm{d}_\rho^{\tau+1}(x+z,x)|z|^{\alpha-\tau}}\,\mathrm{d}z \leqslant |\rho'|\sqrt{\mathrm{D}_\alpha \rho''(x)} + |\rho''(x)||\mathscr{L}_{\phi_1}\rho|,\end{aligned}$$

where ϕ_1 is exactly the $(\tau+1,\alpha)$-kernel. Lemma 5.1 applies to yield $|\mathscr{L}_{\phi_1}\rho| \leqslant |\rho''| + |\rho'|^2$. This finishes estimate for J_3. Lastly, J_4 splits into further two terms according to (4.9). The second part is exactly equal to $\mathscr{L}_{\phi'}\rho'$, so it has been estimated already. And the first part gives rise to the integral

$$\begin{aligned}J_5 &= \int \frac{(\delta_z \rho(x))^3}{\mathrm{d}_\rho^{\tau+2}(x+z,x)|z|^{1+\alpha-\tau}}\,\mathrm{d}z = \int \frac{(\delta_z \rho(x))^2(\delta_z \rho(x) - \rho'(x)z)}{\mathrm{d}_\rho^{\tau+2}(x+z,x)|z|^{1+\alpha-\tau}}\,\mathrm{d}z \\ &\quad + \rho'(x) \int \frac{(\delta_z \rho(x))^2 \operatorname{sgn} z}{\mathrm{d}_\rho^{\tau+2}(x+z,x)|z|^{\alpha-\tau}}\,\mathrm{d}z\end{aligned}$$

the first being bounded as before by $\sqrt{\mathrm{D}_\alpha \rho'(x)} \leqslant |\rho''|_\infty$, while for the second the estimate of (4.33) applies with τ replaced by $\tau+1$. This finishes all estimates.

Step 6: **Control over** $|\rho|_{H^3}, |e|_{H^3}$. The goal at this stage will be to upgrade the above memberships to

$$e \in L^\infty H^3, \quad \rho \in L^\infty H^3 \cap L^2 H^{3+\frac{\alpha}{2}}. \tag{4.39}$$

The computation here will be similar to that done for ρ'', however different at various places. First, in the top class we cannot use the point-wise bound $|e'''| \lesssim |\rho'''|$ because initially e''' is no longer bounded. Second, we pick up many more terms from dissipation that require more careful control.

Let us start with the following a priori bound

$$|e'''|_2 \leqslant C_1|\rho'''|_2 + C_2. \tag{4.40}$$

It goes by observing that the quantity

$$Q = \frac{1}{\rho}\left(\frac{1}{\rho}\left(\frac{1}{\rho}\left(\frac{e}{\rho}\right)'\right)'\right)'$$

satisfies the basic transport equation in weak form:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q + uQ_x = 0.$$

Since the drift u is smooth at this stage, we conclude that

$$|Q(t)|_2 \leqslant |Q_0|_2 \exp\left\{\int_0^t |u'|_\infty \,\mathrm{d}s\right\} \leqslant C, \quad t < T.$$

Unwrapping the derivatives in Q and using the already known bounds on lower order terms we readily obtain (4.40).

Let us now focus on ρ''':

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho''' + u\rho^{(4)} + 3u'\rho''' + 3u''\rho'' + u'''\rho' + e'''\rho + 3e''\rho' + 3e'\rho'' + e\rho''' \\ = \rho'''\mathscr{L}_\phi\rho + 3\rho''(\mathscr{L}_\phi\rho)' + 3\rho'(\mathscr{L}_\phi\rho)'' + \rho(\mathscr{L}_\phi\rho)'''. \end{aligned} \tag{4.41}$$

Testing with $\frac{1}{\rho}\rho'''$ we obtain

$$\left\langle \partial_t\rho''' + u\rho^{(4)} + u'\rho''', \frac{\rho''}{\rho}\right\rangle = \frac{1}{2}\partial_t \int \frac{1}{\rho}|\rho'''|^2 \,\mathrm{d}x,$$

and in view of (4.39) and (4.40),

$$\left\langle 2u'\rho''' + 3u''\rho'' + u'''\rho' + e'''\rho + 3e''\rho' + 3e'\rho'' + e\rho''', \frac{\rho''}{\rho}\right\rangle \leqslant C_1 + C_2|\rho'''|_2^2.$$

Replacing $\mathscr{L}_\phi$ with $e - u'$ in the first three terms on the right hand side of (4.41) we can again bound those similarly by $C_1 + C_2|\rho'''|_2^2$. We thus arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\partial_t \int \frac{1}{\rho}|\rho'''|^2 \,\mathrm{d}x \leqslant C_1 + C_2|\rho'''|_2^2 + \int \rho'''(\mathscr{L}_\phi\rho)''' \,\mathrm{d}x. \tag{4.42}$$

Let us expand according to (4.10):

$$(\mathscr{L}_\phi\rho)''' = \mathscr{L}_\phi\rho''' + 3\mathscr{L}_{\phi'}\rho'' + 3\mathscr{L}_{\phi''}\rho' + \mathscr{L}_{\phi'''}\rho.$$

Clearly, due to (4.38),

$$\int \rho'''\mathscr{L}_\phi\rho''' \,\mathrm{d}x \lesssim -\int \mathrm{D}_\alpha\rho'''(x)\,\mathrm{d}x \leqslant -\frac{1}{2}\int \mathrm{D}_\alpha\rho'''(x)\,\mathrm{d}x - c\int |\rho'''(x)|^{2+\alpha}\,\mathrm{d}x.$$

The analysis of terms $\langle \rho''', \mathscr{L}_{\phi'}\rho''\rangle$ and $\langle \rho''', \mathscr{L}_{\phi''}\rho'\rangle$ is entirely the same as that of I_4 and I_5 of (4.35), respectively, with replacement of ρ with ρ'. It remains to analyze $I_6 = \langle \rho''', \mathscr{L}_{\phi'''}\rho\rangle$. Since the kernel ϕ''' is symmetric, we obtain

$$I_6 = \int \delta_z\rho'''(x)\delta_z\rho(x)\phi'''(x+z,x)\,\mathrm{d}z\,\mathrm{d}x.$$

Given the known bounds $|\rho^{(j)}|_\infty < C$, $j = 0,1,2$, all of the terms involved in representation of $\phi'''(x+z,x)$ are of the order $\frac{1}{|z|^{1+\alpha}}$, except for $\dfrac{\delta_z\rho''(x)\,\mathrm{sgn}\, z}{(\mathrm{d}_\rho(x+z,x))^{\tau+1}|z|^{1+\alpha-\tau}}$. However, since $|\delta_z\rho(x)| \lesssim |z|$, by Cauchy-Schwartz,

$$\left|\int \delta_z\rho'''(x)\delta_z\rho(x)\frac{\delta_z\rho''(x)\,\mathrm{sgn}\, z}{(\mathrm{d}_\rho(x+z,x))^{\tau+1}|z|^{1+\alpha-\tau}}\,\mathrm{d}z\,\mathrm{d}x\right|$$
$$\leqslant \sqrt{\int \mathrm{D}_\alpha\rho'''\,\mathrm{d}x}\sqrt{\int \mathrm{D}_\alpha\rho''\,\mathrm{d}x} \leqslant \varepsilon\int \mathrm{D}_\alpha\rho'''\,\mathrm{d}x + C_\varepsilon f(t),$$

where $f(t)$ is an integrable function on $[0,T]$. We arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\partial_t\int \frac{1}{\rho}|\rho'''|^2\,\mathrm{d}x \leqslant C_1 f(t) + C_2|\rho'''|_2^2 - |\rho'''|^2_{H^{\alpha/2}}.$$

This finishes the desired result at this stage.

Step 7: **Control over $|u|_{H^4}$ and $|\rho|_{H^{3+\alpha/2}}$.** We now get the final estimate in top regularity class for u, $u \in L^\infty H^4 \cap L^2 H^{4+\alpha/2}$. Let us note that since $e \in H^3$ this would automatically imply that $\mathscr{L}_\phi\rho \in H^3$, and by Lemma 4.4 that $\rho \in H^{3+\alpha/2}$. We will use the u-equation directly, as opposed to m-equation since the latter would inevitably require a bound on $e^{(4)}$ which is not available. We thus differentiate the u-equation, and test with $u^{(4)}$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^{(4)}|_2^2 + \int (uu')^{(4)}u^{(4)}\,\mathrm{d}x = \int \mathscr{C}_\phi^{(4)}(\rho,u)u^{(4)}\,\mathrm{d}x. \tag{4.43}$$

For the term on the left hand side have, using the classical commutator estimate,

$$\int (uu')^{(4)}u^{(4)}\,\mathrm{d}x = \int [(uu')^{(4)} - uu^{(5)}]u^{(4)}\,\mathrm{d}x + \int uu^{(5)}u^{(4)}\,\mathrm{d}x$$
$$\leqslant |(uu')^{(4)} - uu^{(5)}|_2|u^{(4)}|_2 - \frac{1}{2}\int u'|u^{(4)}|^2\,\mathrm{d}x$$
$$\leqslant C|u'|_\infty|u^{(4)}|_2^2.$$

The main bulk of the estimates will be performed on the right hand side. We expand using the product rule:

$$\mathscr{C}_\phi^{(4)}(\rho,u) = \sum_{k_1+k_2+k_3=4}\frac{4!}{k_1!k_2!k_3!}\mathscr{C}_{\phi^{(k_1)}}(\rho^{(k_2)},u^{(k_3)}).$$

We will use a short notation for triple products:

$$T_\phi(f,g,h) = \int f(x,z)g(x,z)h(x,z)\phi(x+z,x)\,\mathrm{d}z\,\mathrm{d}x.$$

Also, denote

$$I_{k_1,k_2,k_3} := \left\langle \mathscr{C}_{\phi^{(k_1)}}(\rho^{(k_2)},u^{(k_3)}),u^{(4)}\right\rangle.$$

Let us first consider the case $k_1 = 0$. We thus have five terms at hand:

$$I_{0,0,4}, I_{0,1,3}, I_{0,2,2}, I_{0,3,1}, I_{0,4,0}.$$

Clearly, $I_{0,0,4}$ is dissipative. We have by symmetrization

$$I_{0,0,4} = \frac{1}{2}T_\phi(\rho, \delta_z u^{(4)}, \delta_z u^{(4)}) + \frac{1}{2}T_\phi(\delta_z\rho, \delta_z u^{(4)}, u^{(4)}) \leqslant -c_0 \int \mathrm{D}_\alpha u^{(4)}\,\mathrm{d}x + \frac{1}{2}T_\phi(\delta_z\rho, \delta_z u^{(4)}, u^{(4)}).$$

For the case $0 < \alpha < 1$ the second term is easy: the bound $|\delta_z\rho| \leqslant c|z|$ desingularizes the kernel, and hence, $|T_\phi(\delta_z\rho, \delta_z u^{(4)}, u^{(4)})| \leqslant |u^{(4)}|_2^2$. In the sequel, we will not make references to the case $0 < \alpha < 1$ again and focus on more challenging range $1 \leqslant \alpha < 2$. Thus, we have

$$T_\phi(\delta_z\rho, \delta_z u^{(4)}, u^{(4)}) = T_\phi(\delta_z\rho - \rho'(x)z, \delta_z u^{(4)}, u^{(4)}) + T_\phi(\rho'(x)z, \delta_z u^{(4)}, u^{(4)}).$$

By (4.32), and using that $|\mathrm{D}_\alpha\rho'| \leqslant |\rho''|_\infty < C$,

$$|T_\phi(\delta_z\rho - \rho'(x)z, \delta_z u^{(4)}, u^{(4)})| \leqslant C|u^{(4)}|_2^2.$$

In the second, we distribute power of z in the z-integral:

$$\left|\int z\delta_z u^{(4)}\phi\,\mathrm{d}z\right| \leqslant \int \frac{1}{|z|^{\frac{\alpha-1}{2}}}\frac{|\delta_z u^{(4)}(x)|}{|z|^{\frac{\alpha+1}{2}}}\,\mathrm{d}z \leqslant C\sqrt{\mathrm{D}_\alpha u^{(4)}}. \tag{4.44}$$

Thus,

$$T_\phi(\rho'(x)z, \delta_z u^{(4)}, u^{(4)}) \leqslant \varepsilon \int \mathrm{D}_\alpha u^{(4)}\,\mathrm{d}x + C_\varepsilon|u^{(4)}|_2^2.$$

We thus obtain

$$I_{0,0,4} \leqslant -c_1 \int \mathrm{D}_\alpha u^{(4)}\,\mathrm{d}x + C|u^{(4)}|_2^2.$$

To streamline our subsequent work, in the course of estimating the terms we note a few recurring themes. Once used they will be reused subsequently without commenting. Any quantity that is known to be bounded at this stage will be replaced by a constant C also without commenting. So, let us consider the next term

$$\begin{aligned}
I_{0,1,3} &= T_\phi(\delta_z\rho', \delta_z u''', u^{(4)}) + T_\phi(\rho', \delta_z u''', \delta_z u^{(4)}) \\
&\leqslant |\rho''|_\infty \left\langle \sqrt{\mathrm{D}_\alpha u'''}, u^{(4)} \right\rangle + |\rho'|_\infty \left\langle \sqrt{\mathrm{D}_\alpha u'''}, \sqrt{\mathrm{D}_\alpha u^{(4)}} \right\rangle \\
&\leqslant C_\varepsilon |u|_{H^{3+\alpha/2}}^2 + |u^{(4)}|_2^2 + \varepsilon|\mathrm{D}_\alpha u^{(4)}|_1.
\end{aligned}$$

Since $|u|_{H^{3+\alpha/2}}^2 \in L^1$, the above estimate is sufficient for application of the Grönwall inequality. Next, for $I_{0,2,2}$ we will not do symmetrization, instead, just add and subtract $u^{(4)}(y)$:

$$I_{0,2,2} = \int \rho''(y)u^{(4)}(y)\mathscr{L}_\phi u''(y)\,\mathrm{d}y + T_\phi(\rho'', \delta_z u'', \delta_z u^{(4)}).$$

We note that in view of (5.11) and (4.32) we obtain a bound

$$|\mathscr{L}_\phi u''(y)| \leqslant \sqrt{\mathrm{D}_\alpha u'''(y)} + |u'''(y)|.$$

Hence,

$$I_{0,2,2} \leqslant |u^{(4)}|_2^2 + |u|_{H^{3+\alpha/2}}^2 + |u'''|_2^2 + \varepsilon|\mathrm{D}_\alpha u^{(4)}|_1.$$

Next,

$$I_{0,3,1} = \int \rho'''(x)\mathscr{L}_\phi u'(x)u^{(4)}(x)\,\mathrm{d}x + T_\phi(\delta_z\rho''', \delta_z u', \delta_z u^{(4)}),$$

noting that $\mathscr{L}_\phi u' \in L^\infty$, and $|\delta_z u'| \lesssim |z|$ with (4.44) in mind,

$$I_{0,3,1} \leqslant |\rho'''|_2^2 + \int \mathrm{D}_\alpha \rho'''(x)\,\mathrm{d}x + |u^{(4)}|_2^2.$$

Finally,

$$I_{0,4,0} = \int \rho^{(4)}(x+z)\delta_z u(x) u^{(4)}(x)\phi\,\mathrm{d}z\,\mathrm{d}x.$$

Writing $\rho^{(4)}(x+z) = (\delta_z\rho'''(x))'_z$ and integrating by parts, we obtain

$$I_{0,4,0} = -\int \delta_z\rho'''(x)u'(x+z)u^{(4)}(x)\phi\,\mathrm{d}z\,\mathrm{d}x - \int \delta_z\rho'''(x)\delta_z u(x)u^{(4)}(x)[\phi_d\rho(x+z)\operatorname{sgn} z + \phi_z]\,\mathrm{d}z\,\mathrm{d}x.$$

In the first integral after symmetrization we obtain

$$T_\phi(\delta_z\rho''', \delta_z u', u^{(4)}) + T_\phi(\delta_z\rho''', u', \delta_z u^{(4)}) \leqslant |u''|_\infty \left\langle \sqrt{\mathrm{D}_\alpha\rho'''}, u^{(4)}\right\rangle + |u'|_\infty \left\langle \sqrt{\mathrm{D}_\alpha\rho'''}, \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle.$$

This leads to the desired bound. In the second integral, also symmetrizing we obtain

$$\begin{aligned}
&\int \delta_z\rho'''(x)\delta_z u(x)\delta_z u^{(4)}(x)[\phi_d\rho(x+z)\operatorname{sgn} z + \phi_z]\,\mathrm{d}z\,\mathrm{d}x \\
&+ \int \delta_z\rho'''(x)\delta_z u(x)u^{(4)}(x)\phi_d\delta_z\rho(x)\operatorname{sgn} z\,\mathrm{d}z\,\mathrm{d}x \\
&\leqslant |u'|_\infty \left\langle \sqrt{\mathrm{D}_\alpha\rho'''}, \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle + |u'|_\infty|\rho'|_\infty \left\langle \sqrt{\mathrm{D}_\alpha\rho'''}, u^{(4)}\right\rangle.
\end{aligned}$$

This finishes our first installment of estimates.

Next we focus on terms $I_{1,0,3}, I_{1,1,2}, I_{1,2,1}, I_{1,3,0}$. Note that the kernel ϕ_r is of the same order $1/|z|^{1+\alpha}$. So, performing similar manipulations as before and using uniform bounds on $|\rho', \rho'', u', u''|_\infty$ throughout we obtain

$$\begin{aligned}
2I_{1,0,3} &= T_{\phi'}(\delta_z\rho, \delta_z u''', u^{(4)}) + T_{\phi'}(\rho, \delta_z u''', \delta_z u^{(4)}) \lesssim \left\langle \sqrt{\mathrm{D}_\alpha u'''}, u^{(4)}\right\rangle + \left\langle \sqrt{\mathrm{D}_\alpha u'''}, \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle \\
2I_{1,1,2} &= T_{\phi'}(\delta_z\rho', \delta_z u'', u^{(4)}) + T_{\phi'}(\rho', \delta_z u'', \delta_z u^{(4)}) \lesssim \left\langle \sqrt{\mathrm{D}_\alpha u''}, u^{(4)}\right\rangle + \left\langle \sqrt{\mathrm{D}_\alpha u''}, \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle \\
2I_{1,2,1} &= T_{\phi'}(\delta_z\rho'', \delta_z u', u^{(4)}) + T_{\phi'}(\rho'', \delta_z u', \delta_z u^{(4)}) \lesssim \left\langle \sqrt{\mathrm{D}_\alpha \rho''}, u^{(4)}\right\rangle + \left\langle \sqrt{\mathrm{D}_\alpha u'}, \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle \\
2I_{1,3,0} &= T_{\phi'}(\delta_z\rho''', \delta_z u, u^{(4)}) + T_{\phi'}(\rho''', \delta_z u, \delta_z u^{(4)}) \lesssim \left\langle \sqrt{\mathrm{D}_\alpha \rho'''}, u^{(4)}\right\rangle + \left\langle \rho''', \sqrt{\mathrm{D}_\alpha u^{(4)}}\right\rangle.
\end{aligned}$$

Note that all the terms on the right hand side are bounded by

$$\varepsilon|\mathrm{D}_\alpha u^{(4)}|_1 + |u^{(4)}|_2^2 + f(t), \quad f \in L^1([0,T)).$$

Next, the kernel ρ'' is still bounded by $1/|z|^{1+\alpha}$ due to $|\rho''|_\infty < C$. Yet, fewer derivatives fall onto ρ and u inside $I_{2,k,p}$, $k+p=2$. We therefore skip these estimates as they repeat the previous. As to $I_{3,k,p}$, $k+p=1$, let us expand the kernel:

$$\phi''' = c_1\frac{(\delta_z\rho(x))^3\operatorname{sgn} z}{\mathrm{d}_\rho^{\tau+3}(x+z,x)|z|^{1+\alpha-\tau}} + c_2\frac{\delta_z\rho(x)\delta_z\rho'(x)}{\mathrm{d}_\rho^{\tau+2}(x+z,x)|z|^{1+\alpha-\tau}} + c_3\frac{\delta_z\rho''(x)\operatorname{sgn} z}{\mathrm{d}_\rho^{\tau+1}(x+z,x)|z|^{1+\alpha-\tau}}.$$

It is clear that the first two parts are bounded by $1/|z|^{1+\alpha}$, and hence, the estimates for those terms follow as before. For the remaining part, after symmetrization we obtain

$$\int \frac{\delta_z\rho''(x)\delta_z\rho^{(k)}(x)\delta_z u^{(p)}u^{(4)}\operatorname{sgn} z}{(\mathrm{d}_\rho(x+z,x))^{\tau+1}|z|^{1+\alpha-\tau}}\,\mathrm{d}z\,\mathrm{d}x + \int \frac{\delta_z\rho''(x)\rho^{(k)}(x)\delta_z u^{(p)}\delta_z u^{(4)}\operatorname{sgn} z}{\mathrm{d}_\rho^{\tau+1}(x+z,x)|z|^{1+\alpha-\tau}}\,\mathrm{d}z\,\mathrm{d}x$$

Using that $|\delta_z \rho^{(k)}(x) \delta_z u^{(p)}| \leqslant C|z|^2$ in the first term, we obtain the bound by $\langle \sqrt{\mathrm{D}_\alpha \rho''}, u^{(4)} \rangle$. In the second we use $|\rho^{(k)}(x) \delta_z u^{(p)}| \leqslant C|z|$ and hence bound by $\langle \sqrt{\mathrm{D}_\alpha \rho''}, \sqrt{\mathrm{D}_\alpha u^{(4)}} \rangle$.

Lastly, in the term $I_{4,0,0}$ the kernel reads

$$\begin{aligned}\phi^{(4)} &= c_1 \frac{(\delta_z \rho(x))^4}{(\mathrm{d}_\rho(x+z,x))^{\tau+4}|z|^{1+\alpha-\tau}} + c_2 \frac{(\delta_z \rho(x))^2 \delta_z \rho'(x) \operatorname{sgn} z}{(\mathrm{d}_\rho(x+z,x))^{\tau+3}|z|^{1+\alpha-\tau}} + c_3 \frac{\delta_z \rho''(x) \delta_z \rho(x) + (\delta_z \rho'(x))^2}{(\mathrm{d}_\rho(x+z,x))^{\tau+2}|z|^{1+\alpha-\tau}} \\ &+ c_4 \frac{\delta_z \rho'''(x) \operatorname{sgn} z}{(\mathrm{d}_\rho(x+z,x))^{\tau+1}|z|^{1+\alpha-\tau}}.\end{aligned}$$

For the first three terms we argue exactly as before. For the last we have after symmetrization

$$\begin{aligned}&\int \frac{\delta_z \rho'''(x) \delta_z \rho(x) \delta_z u(x) u^{(4)}(x) \operatorname{sgn} z}{\mathrm{d}^{\tau+1}(x+z,x)|z|^{1+\alpha-\tau}} \,\mathrm{d}z \,\mathrm{d}x + \int \frac{\delta_z \rho'''(x) \rho(x) \delta_z u(x) \delta_z u^{(4)}(x) \operatorname{sgn} z}{(\mathrm{d}_\rho(x+z,x))^{\tau+1}|z|^{1+\alpha-\tau}} \,\mathrm{d}z \,\mathrm{d}x \\ &\leqslant \left\langle \sqrt{\mathrm{D}_\alpha \rho'''}, u^{(4)} \right\rangle + \left\langle \sqrt{\mathrm{D}_\alpha \rho'''}, \sqrt{\mathrm{D}_\alpha u^{(4)}} \right\rangle.\end{aligned}$$

This finishes the proof. □

4.5. **Hölder regularization of the density.** In this section we focus on obtaining Hölder regularity of the density by a various fractional techniques depending on the range of α. Combined with Proposition 4.8 we immediately obtain global existence and conclude Theorem 4.1.

4.5.1. **Case $1 < \alpha < 2$ via Schauder**. In this particular case the regularization will follow from a kinematic argument based on the Schauder estimates as in [10, 36]. So, we start by rewriting the relation between ρ, u, and e as follows

$$\partial_x^{-1} \mathscr{L}_\phi \rho = \partial_x^{-1} e - u \in L^\infty. \tag{4.45}$$

In the purely metric case, $\tau = 0$, this of course implies $\rho \in C^{1-\alpha}$ immediately. For the topological models the conclusion is not so straightforward, and in fact may not even be true up to regularity $1 - \alpha$.

First let us make an observation that $\mathscr{L}_\phi \rho = \partial_x(\mathscr{F}\rho)$, where

$$\mathscr{F}\rho(x) = -\frac{1}{\tau - 1} \int \frac{\operatorname{sgn}(z)}{(\mathrm{d}_\rho(x+z,x))^{\tau-1}|z|^{\alpha-\tau+1}} h(z) \,\mathrm{d}z$$

if $\tau \neq 1$, and

$$\mathscr{F}\rho(x) = \int \frac{\operatorname{sgn}(z) \ln \mathrm{d}_\rho(x+z,x)}{|z|^\alpha} h(z) \,\mathrm{d}z,$$

if $\tau = 1$. Next, by symmetrization if $\tau \neq 1$,

$$\mathscr{F}\rho(x) = -\frac{1}{2(\tau - 1)} \int \frac{(\mathrm{d}_\rho(x+z,x))^{\tau-1} - (\mathrm{d}_\rho(x-z,x))^{\tau-1}}{(\mathrm{d}_\rho(x+z,x))^{\tau-1}(\mathrm{d}_\rho(x-z,x))^{\tau-1}|z|^{1+\alpha-\tau}} \operatorname{sgn}(z) h(z) \,\mathrm{d}z,$$

and correspondingly for $\tau = 1$,

$$\mathscr{F}\rho(x) = \frac{1}{2} \int \frac{\ln \mathrm{d}_\rho(x+z,x) - \ln \mathrm{d}_\rho(x-z,x)}{|z|^\alpha} \operatorname{sgn}(z) h(z) \,\mathrm{d}z,$$

Now we use the expansion

$$\begin{aligned}&(\mathrm{d}(x+z))^{\tau-1} - (\mathrm{d}_\rho(x-z))^{\tau-1} \\ &= (\tau - 1)[(\mathrm{d}_\rho(x+z)) - \mathrm{d}_\rho(x-z)] \int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{\tau-2} \,\mathrm{d}\theta,\end{aligned} \tag{4.46}$$

and correspondingly, for $\tau = 1$,

$$\ln \mathrm{d}_\rho(x+z,x) - \ln \mathrm{d}_\rho(x-z,x) \\ = [\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)] \int_0^1 \frac{\mathrm{d}\theta}{\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)}.$$

Next,

$$[\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)]\,\mathrm{sgn}(z) = \int_x^{x+z} \rho(y)\,\mathrm{d}y + \int_x^{x-z} \rho(y)\,\mathrm{d}y = \int_{-z}^{z} \rho(x+w)\,\mathrm{sgn}\, w\,\mathrm{d}w.$$

We can now subtract the total mass from the density without changing the result. However, the function $\rho - M_0$ is a mean-zero function. Hence, $\rho - M_0 = f'$, for some f. Continuing we obtain

$$[\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)]\,\mathrm{sgn}(z) = \int_{-z}^{z} f'(x+w)\,\mathrm{sgn}(w)\,\mathrm{d}w = f(x+z) + f(x-z) - 2f(x),$$

which is the second order finite difference of f. We thus obtain

$$\mathscr{F}\rho(x) = \int [f(x+z) + f(x-z) - 2f(x)] K(x,z,t)\,\mathrm{d}z,$$

where the kernel $K(x,z,t)$ is given by

$$K(x,z,t) = (\tau-1)\frac{h(z)\int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{\tau-2}\,\mathrm{d}\theta}{(\mathrm{d}(x+z,x))^{\tau-1}(\mathrm{d}(x-z,x))^{\tau-1}|z|^{1+\alpha-\tau}}$$

when $\tau \neq 1$, and

$$K(x,z,t) = \frac{h(z)}{|z|^\alpha} \int_0^1 \frac{\mathrm{d}\theta}{\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)},$$

when $\tau = 1$. For all $\tau > 0$ the kernel satisfies the following four conditions:

(i) $\dfrac{\Lambda^{-1}}{|z|^{1+\alpha}}\chi_{[0,R_0]}(z) \leqslant K(x,z,t) \leqslant \dfrac{\Lambda}{|z|^{1+\alpha}}\chi_{[0,2R_0]}(z)$ for some $\Lambda > 0$;
(ii) $K(x,-z,t) = K(x,z,t)$;
(iii) $|z|^{2+\alpha}|K(x+h,z,t) - K(x,z,t)| \leqslant C|h|$;
(iv) $|\partial_z(|z|^{1+\alpha}K(x,z,t))| \leqslant C|z|^{-1}$.

Here the constants $\Lambda, C > 0$ depend only on the density itself and not its derivatives. Indeed, (i) and (ii) are trivial. As to (iv), (we focus on $\tau \neq 1$ case only, and we omit adimensional constants) we have

$$|z|^{1+\alpha}K(x,z,t) = h(z)\frac{|z|^\tau \int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{\tau-2}\,\mathrm{d}\theta}{(\mathrm{d}_\rho(x+z,x))^{\tau-1}\mathrm{d}_\rho(x-z,x))^{\tau-1}} \tag{4.47}$$

Given that $\mathrm{d}_\rho(x+z,x) \sim |z|$, it is clear that this expression along is uniformly bounded by a constant. Hence, so it will remain if ∂_z falls on h. The bound gains $|z|^{-1}$ order when ∂_z falls on $|z|^\tau$. Next, observe that

$$\partial_z \mathrm{d}_\rho(x \pm z, x) = \rho(x \pm z)\,\mathrm{sgn}(z),$$

which is a uniformly bounded quantity. So, any derivative that falls on the distance inside the expression (4.47) reduces the power of that term by 1, while the rest remains uniformly bounded.

To verify (iii) we can even prove a stronger inequality

$$|z|^{2+\alpha}|\partial_x K(x,z,t)| \leqslant C.$$

Indeed, in this case we recall (4.5) which implies that $\partial_x \mathrm{d}_\rho(x \pm z, x)$ remains uniformly bounded. So, we have

$$|z|^{2+\alpha}\partial_x K(x,z,t) = h(z)|z|^{1+\tau}\partial_x \left(\frac{\int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{\tau-2} \, \mathrm{d}\theta}{(\mathrm{d}_\rho(x+z,x))^{\tau-1}(\mathrm{d}_\rho(x-z,x))^{\tau-1}} \right).$$

In view of the above observation, the order of the partial of the entire expression in parenthesis is $|z|^{-1-\tau}$. This finishes the verification.

So, the initial relation (4.45) can be stated now as a fractional elliptic problem:

$$\int [f(x+z) + f(x-z) - 2f(x)]K(x,z,t)\,\mathrm{d}z = g(x) \in L^\infty. \tag{4.48}$$

Under the assumptions (i) – (iv), it is known, see for example [10, 36], that any bounded solution f to (4.48) satisfies $f \in C^{1+\gamma}$ for some positive $\gamma > 0$. This readily implies $\rho \in C^\gamma$ and concludes the argument.

4.5.2. **Case $0 < \alpha < 1$ via Silvestre**. The Hölder regularization result obtained in [56] for forced drift-diffusion equations with pure fractional Laplacian, as note by the author, applied to more general kernels, even in z: $K(x,z,t) = K(x,-z,t)$. This condition, however is necessary only for the range $\alpha \geqslant 1$ to justify pointwise evaluation of the integral on smooth function. For $0 < \alpha < 1$, such condition is not required and the proof goes through as is, except for one point to be elaborated below. Another necessary condition is regularity of the drift $u \in C^{1-\alpha}$.

Let us start with one specific point at the proof where more regularity of the kernel is required. In the proof of the diminish of oscillation lemma, Lemma 3.1, namely in the construction of the barrier, the author makes use of the fact that the application $(-\Delta)^{\alpha/2}\eta$ is a continuous function for smooth cut-off function η. More specifically, it is needed that for values $0 \leqslant \eta(x) \leqslant \beta$ small enough, $(-\Delta)^{\alpha/2}\eta(x) \leqslant 0$. This certainly follows from the fact that if $\eta(x) = 0$, then $(-\Delta)^{\alpha/2}\eta(x) < 0$. The value of β enters into the size of diminishing amplitude of the solution, propagates through the proof, and enters in the penultimate Hölder exponent. Hence, it must not depend on any parameter that deteriorates in time. In general, β depends on some modulus of continuity of the kernel away from the singularity. In our case, the kernel in the density equation is given by

$$K(x,z,t) = \rho(x)\phi(x+z,x),$$

and certainly such modulus depends on one of ρ, the very quantity we are trying to control. However, since $\rho(x)$ appears on the outside, we have $\mathscr{L}_K\rho = \rho\mathscr{L}_\phi\rho$, and consequently, the sign of $\mathscr{L}_K\rho$ is controlled only by the operator $\mathscr{L}_\phi\rho$. The kernel $\phi(x+z,x)$ *does* possess a Lipschitz modulus, clearly, since $d(x+z,x)$ is Lipschitz in x uniformly in time (with constant depending only on $|\rho|_\infty$, which we control uniformly on a given time interval).

Second, we obtain regularity $u \in C^{1-\alpha}$, necessary to apply [56]. For this we use the representation (4.45): $u = \partial_x^{-1}e - \mathscr{F}\rho$. Since $\partial_x^{-1}e \in W^{1,\infty}$, it remains to check that $\mathscr{F}\rho \in C^{1-\alpha}$. The verification again goes via an optimization over cut-off scale argument.

Let $\tau \neq 1$. Then, omitting constants,

$$\begin{aligned}\mathscr{F}\rho(x+h) - \mathscr{F}\rho(x) = \int_{|z|\geqslant h} \frac{\mathrm{d}_\rho(x+h+z,x+h))^{\tau-1} - \mathrm{d}_\rho(x+z,x))^{\tau-1}}{\mathrm{d}_\rho(x+h+z,x+h))^{\tau-1}\mathrm{d}_\rho(x+z,x))^{\tau-1}} \frac{\operatorname{sgn}(z)h(z)}{|z|^{\alpha-\tau+1}}\,\mathrm{d}z \\ + \int_{|z|\leqslant h}\left(\frac{1}{(\mathrm{d}_\rho(x+h+z,x+h))^{\tau-1}} - \frac{1}{(\mathrm{d}_\rho(x+z,x))^{\tau-1}}\right)\frac{\operatorname{sgn}(z)}{|z|^{\alpha-\tau+1}} h(z)\,\mathrm{d}z\end{aligned}$$

In the latter integral we simply use the order of singularity $|z|^{-\alpha}$, which implies bound by $|h|^{1-\alpha}$, as needed. In the first, we use Taylor formula (4.46) which yields a bound by $|h|/|z|^{1+\alpha}$, with a uniform constant depending only on (4.22). This results again in $|h|^{1-\alpha}$, as needed. This finishes the proof.

4.5.3. **Case $\alpha = 1$ via De Giorgi**. In this section we present a regularization result for the case $\alpha = 1$. We state our result more precisely in the following proposition.

Proposition 4.9. *Consider the case $\alpha = 1$. Assume the density is uniformly bounded (4.22). Then there exists a $\gamma > 0$ such that $[\rho]_\gamma \leqslant \dfrac{C}{t^\gamma}$ for all $t \in (0, T]$. Here C depends on the bounds on the density on $[0, T]$.*

Let us make some preliminary remarks. Our proof is based on blending our model into the settings of Caffarelli, Chan, Vasseur work [9] which adopts the method of De Giorgi to non-local equation with symmetric kernels. We note however that the result of [9] is not directly applicable to our model due to the presence of drift and force in the density equation, and in addition we lack symmetry of the kernel. The forced case was considered in a similar situation in Golse et al [27], where the control over the force is achieved via pre-scaling of the equation. We will use a similar argumentation here as well. We proceed in five steps.

Step 1: **Symmetric form of the density equation**. Let us recall the density equation in parabolic form:

$$\rho_t + u\rho_x = \rho\mathscr{L}_\phi\rho - e\rho. \tag{4.49}$$

To get rid of the ρ prefactor we will perform the following procedure: divide (4.49) by ρ and write evolution equation for the new variable $w = \ln\rho$,

$$w_t + uw_x = \mathscr{L}_\phi e^w - e.$$

Using that

$$e^{w(y)} - e^{w(x)} = (w(y) - w(x))\int_0^1 \rho(y)^\theta\rho(x)^{1-\theta}\,\mathrm{d}\theta,$$

we further rewrite the equation as

$$w_t + uw_x = \mathscr{L}_K w - e. \tag{4.50}$$

where

$$K(x,y,t) = \phi(x,y)\int_0^1 \rho(y)^\theta\rho(x)^{1-\theta}\,\mathrm{d}\theta$$

In view of the bounds on the density, the new kernel satisfies

$$\frac{1}{\Lambda|x-y|^{1+\alpha}}\chi_{|x-y|<R_0} \leqslant K(x,y) \leqslant \frac{\Lambda}{|x-y|^{1+\alpha}}\chi_{|x-y|<2R_0}, \tag{4.51}$$

and now is fully symmetric

$$K(x,y,t) = K(y,x,t).$$

Clearly, Hölder continuity of w is equivalent to that of ρ, so we will work with (4.50) instead.

In what follows we treat the term $-e$ as a passive source. However we cannot treat u similarly since the derivative u_x that will come up in the truncated energy inequality will have to be recycled back through its connection with e. We therefore first discuss scaling properties of the system.

STEP 2: **Rescaling**. Let us adopt the point of view that our solution (u, ρ) is defined periodically on the real line $\mathbb{R}$. Elementary computation shows that if (u, ρ) is a solution and $R > 0$, then the new pair

$$u_R = u\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right), \quad \rho_R = \rho\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right) \tag{4.52}$$

satisfies the rescaled system

$$\begin{cases} \partial_t \rho_R + R^{1-\alpha}(\rho_R u_R)_x = 0, \\ \quad \partial_t u_R + R^{1-\alpha} u_R u_R' = \displaystyle\int_{\mathbb{R}} \rho_R(y)(u_R(y) - u_R(x))\phi_R(x,y)\,\mathrm{d}y, \end{cases} \tag{4.53}$$

where the new kernel is given by

$$\phi_R(x,y,t) = \frac{1}{R^{1+\alpha}}\phi\left(x_0 + \frac{x}{R}, x_0 + \frac{y}{R}, t_0 + \frac{t}{R^\alpha}\right).$$

Note that for a given bound on the density $c < \rho < C$ on a given time interval I, the new kernel still satisfies

$$\frac{1}{\Lambda|x-y|^{1+\alpha}}\chi_{|x-y|\leqslant R_0 R} \leqslant \phi_R(x,y) \leqslant \frac{\Lambda}{|x-y|^{1+\alpha}}\chi_{|x-y|<2R_0 R},$$

on time interval $R^\alpha(I - t_0)$, and the constant Λ is independent of R. Thus, if $R > 1$, the bound from below holds on a wider space and time intervals. The corresponding e-quantity rescales to

$$e_R = R^{1-\alpha}u_R' + \mathscr{L}_{\phi_R}\rho_R = \frac{1}{R^\alpha}e\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right),$$

and satisfies

$$\partial_t e_R + R^{1-\alpha}(u_R e_R)_x = 0.$$

Hence, e_R/ρ_R is transported and as a consequence we obtain an priori bound

$$|e_R| \lesssim \frac{1}{R^\alpha}\rho_R \lesssim \frac{1}{R^\alpha}. \tag{4.54}$$

The rescaled density equation becomes

$$\partial_t \rho_R + R^{1-\alpha}u_R\rho_R' + e_R\rho_R = \rho_R\mathscr{L}_{\phi_R}\rho_R.$$

The corresponding w-equation reads

$$\partial_t w_R + R^{1-\alpha}u_R w_R' = \mathscr{L}_{K_R} w - e_R,$$

where the kernel K_R satisfies the same bound (4.51) for all $R \geqslant 1$.

So, it is clear that the drift remains under control for $\alpha \geqslant 1$, and is scaling invariant in the case $\alpha = 1$.

STEP 3: **First De Giorgi lemma**. We return to the symmetrized version of the density equation (4.50), where the only extra term the prevents us to directly apply [9] is the drift.

Since, in addition the drift is not div-free and non-linearly depends upon ρ we will take extra care of keeping protocol of relation between w and u after re-scalings.

First, we start by noting that it suffices to work on time interval $[-3,0]$ and prove uniform Hölder continuity on $[-1,0]$. Second, in view of (4.54) if necessary we can rescale the equation by a large $R>1$ and assume without loss of generality that $|e|_{L^\infty(\mathbb{R}\times[-3,0))} = \varepsilon_0 < 1$, where ε_0 will be determined at a later stage and will in fact depend only on Λ.

The argument of [9] uses rescaling of the form $\omega = \frac{w_R}{C_1} + C_2$, where $R \geqslant 1$, and $|C_1| \leqslant C_0 = \max\{1, |w|_\infty\}$, and w is the original solution. Let us note that the new quantity ω satisfies

$$
\begin{aligned}
&\omega_t + u_R \omega_x = \mathscr{L}_{K_R}\omega + f_{R,C_1}, \\
&|f_{R,C_1}|_\infty \leqslant \frac{\varepsilon_0}{RC_1}.
\end{aligned} \tag{4.55}
$$

To keep control over the source we therefore impose the following assumption on all rescalings

$$
RC_1 > 1. \tag{4.56}
$$

We will now derive a truncated energy inequality for ω.

Let ψ be a Lipschitz function on $\mathbb{R}$. We always assume that our Lipschitz functions have slopes bounded by a universal constant. Testing (4.55) with $(\omega-\psi)_+$ we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\psi)_+^2 \,\mathrm{d}x - \frac{1}{2}\int (u_R)_x (\omega-\psi)_+^2 \,\mathrm{d}x - \frac{1}{2}\int u_R \psi_x (\omega-\psi)_+ \,\mathrm{d}x \\
= -B_R(\omega, (\omega-\psi)_+) + \int f_{R,C_1}(\omega-\psi)_+ \,\mathrm{d}x,
\end{aligned}
$$

where

$$
B_R(h,g) = \frac{1}{2}\int K_R(x,y)(h(y)-h(x))(g(y)-g(x))\,\mathrm{d}y\,\mathrm{d}x.
$$

Continuing we obtain

$$
(u_R)_x = e_R - \mathscr{L}_{\phi_R}\rho_R = e_R - \mathscr{L}_{K_R} w_R = e_R - C_1 \mathscr{L}_{K_R}\omega.
$$

We also note that in view of our assumptions and the maximum principle we have a scaling invariant bound $|u_R\psi_x| \leqslant C$. So, as long as in addition $RC_1 > 1$, we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\psi)_+^2 \,\mathrm{d}x + B_R(\omega,(\omega-\psi)_+) \leqslant \frac{C_1}{2} B_R(\omega, (\omega-\psi)_+^2) + C(|(\omega-\psi)_+|_1 + |(\omega-\psi)_+|_2^2).
$$

Note that the B-term on the right hand side is cubic, while on the left hand side it is quadratic. This will help hide the cubic term with the help of the following smallness assumption:

$$
|(\omega-\psi)_+|_\infty \leqslant \frac{1}{2C_0}. \tag{4.57}
$$

Under this assumption we have

$$
B_R(\omega,(\omega-\psi)_+) - \frac{C_1}{2} B_R(\omega,(\omega-\psi)_+^2) = B_{R,\omega}(\omega,(\omega-\psi)_+),
$$

where $B_{R,\omega}$ is the bilinear form associated with the kernel

$$
K_{R,\omega}(x,y) = K_R(x,y)\left[1 - \frac{C_1}{2}((\omega-\psi)_+(x) + (\omega-\psi)_+(y))\right],
$$

which under (4.57) satisfies similar bounds as the original kernel and is symmetric. Continuing with the energy inequality, we write $\omega - \psi = (\omega-\psi)_+ - (\omega-\psi)_-$ and obtain

$$B_{R,\omega}(\omega,(\omega-\psi)_+) = B_{R,\omega}((\omega-\psi)_+,(\omega-\psi)_+) - B_{R,\omega}((\omega-\psi)_-,(\omega-\psi)_+) + B_{R,\omega}(\psi,(\omega-\psi)_+).$$

The first is the main dissipative term for which we have a coercive bound

$$B_{R,\omega}((\omega-\psi)_+,(\omega-\psi)_+) \geqslant c_{\Lambda,C_0}|(\omega-\psi)_+|^2_{H^{1/2}} - |(\omega-\psi)_+|_2^2.$$

For the second we have after cancellations

$$-B_{R,\omega}((\omega-\psi)_-,(\omega-\psi)_+) = 2\int K_{R,\omega}(x,y)(\omega-\psi)_-(y)(\omega-\psi)_+(z)\,\mathrm{d}y\,\mathrm{d}z := P$$

which is positive and can be dismissed for the application of the First DeGiorgi Lemma. Finally, as in [9] we obtain

$$|B_{R,\omega}(\psi,(\omega-\psi)_+)| \leqslant \frac{1}{2}B_R((\omega-\psi)_+,(\omega-\psi)_+) + |(\omega-\psi)_+|_1 + |\{\omega-\psi>0\}|.$$

We thus have proved the following energy bound under (4.57) and for any rescaled solution with $RC_1 > 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\omega-\psi)_+^2\,\mathrm{d}x + |(\omega-\psi)_+|^2_{H^{1/2}} \lesssim |(\omega-\psi)_+|_2^2 + |(\omega-\psi)_+|_1 + |\{\omega-\psi>0\}|.$$

We now recap the First DeGiorgi Lemma: there exists $\delta > 0$ and $\theta \in (0,1)$ such that any solution ω to (4.55) satisfying

$$\omega(t,x) \leqslant 1 + (|x|^{1/4}-1)_+ \quad \text{on } \mathbb{R}\times[-2,0],$$

and

$$|\{\omega>0\}\cap(B_2\times[-2,0])| \leqslant \delta,$$

must have a bound

$$\omega(t,x) \leqslant 1-\theta.$$

The proof proceeds as in [9] with extra care given for (4.57). We consider Lipschitz function

$$\psi_{L_k}(x) = 1-\theta-\frac{\theta}{2^k} + (|x|^{1/2}-1)_+.$$

For θ small enough it is clear that $(\omega-\psi_{L_k})_+$ can be made as small as we wish for all $k\in\mathbb{N}$, in particular satisfying (4.57). With θ fixed we can then apply the energy inequality for all terms $(\omega-\psi_{L_k})_+$, and the argument of [9] proceeds.

STEP 4: **The second De Giorgi lemma**. In the Second DeGiorgi Lemma the energy bound is used in a somewhat different way. Here the presence of the drift term requires extra attention as well as condition (4.57). We recall the lemma first. For a $\lambda < 1/3$ we define $\psi_\lambda(x) = ((|x|-\frac{1}{\lambda^4})_+^{1/4}-1)_+$. Let also F be non-increasing with $F=1$ on B_1 and $F=0$ outside B_2. Define

$$\phi_j = 1+\psi_\lambda - \lambda^j F, \quad j=0,1,2.$$

The lemma claims that there exist $\mu,\lambda,\gamma>0$ depending only on Λ such that if

$$\omega(t,x) < 1+\psi_\lambda(x) \text{ on } \mathbb{R}\times[-3,0],$$

and

$$|\{\omega < \phi_0\} \cap B_1 \times (-3,-2)| \geqslant \mu,$$
$$|\{\omega > \phi_2\} \cap \mathbb{R} \times (-2,0)| \geqslant \delta,$$

then necessarily

$$|\{\phi_0 < \omega < \phi_2\} \cap \mathbb{R} \times (-3,0)| \geqslant \gamma.$$

So, if the function has substantial subzero presence and later over $1-\lambda^2$ presence then it has to leave some appreciable mass in between. The proof goes by application of the energy inequality to $(\omega - \phi_1)_+$. However, $(\omega-\phi_1)_+ \leqslant \lambda$ pointwise. Hence, to satisfy (4.57) it is it sufficient to pick $\lambda < 1/2C_0$, among further restrictions which come subsequently in the course of the proof. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\phi_1)_+^2 \,\mathrm{d}x + B_{R,\omega}((\omega-\phi_1)_+,(\omega-\phi_1)_+) + P = -B_{R,\omega}(\phi_1,(\omega-\phi_1)_+)$$
$$+\int \left(\frac{1}{2}u_R(\phi_1)_x + f_{R,C_1}\right)(\omega-\phi_1)_+ \,\mathrm{d}x.$$

All the terms are exactly the same as in [9] except the last one. To bound the last term we note that $(\omega-\phi_1)_+$ is supported on B_2, where $\phi_1 = 1+\lambda F$, hence $|(\phi_1)_x|_{L^\infty(B_2)} \leqslant C\lambda$. Furthermore, as noted above, $(\omega-\phi_1)_+ \leqslant \lambda$. Hence,

$$\left|\frac{1}{2}\int u_R(\phi_1)_x(\omega-\phi_1)_+ \,\mathrm{d}x\right| \leqslant C\lambda^2.$$

As to the source term, we obtain the same bound provided $\varepsilon_0 < \lambda$. The resulting bound repeats another estimate on the term $B_{R,\omega}(\phi_1,(\omega-\phi_1)_+)$, and hence, blends with the rest of Section 4 in [9].

The rest of the proof makes no further direct use of the energy inequality and thus proceeds ad verbatim. The penultimate constant λ ends up being dependent only on Λ and C_0 which are scaling invariant.

STEP 5: **Diminishing oscillation and C^γ regularity**. The first and second lemmas are not being used to prove that any solution with controlled tails on $[-3,0]\times\mathbb{R}$,

$$-1-\psi_{\varepsilon,\lambda} \leqslant w \leqslant 1+\psi_{\varepsilon,\lambda},$$

where

$$\psi_{\varepsilon,\lambda}(x) = \begin{cases} 0 & , \quad \text{if } |x| < \lambda^{-4} \\ [(|x|-\lambda^{-4})^\varepsilon - 1]_+ & , \quad \text{if } |x| \geqslant \lambda^{-4} \end{cases}$$

satisfies

$$\sup_{[-1,0]\times B_1} w - \inf_{[-1,0]\times B_1} w < 2-\lambda^*,$$

for some $\lambda^* > 0$. The proof goes by application of shift-amplitude rescalings of the form

$$w_{k+1} = \frac{1}{\lambda^2}(w_k - (1-\lambda^2)) = \frac{1}{\lambda^{2k}}w + C_k.$$

For our sourced equation this is the worst kind of rescaling since it doesn't come with a compensated space-time stretching. However, in the argument the number of iterations is limited to $k_0 = |[-3,0]\times B_3|/\gamma$, and hence depends only on Λ. We can pre-scale the equation in the beginning using $R_0 > 0$ so large that $\varepsilon_0 = |f_{R_0}|_\infty < \lambda^{2k_0}C_0 \leqslant \lambda^{2k_0}$. Hence, on each step of the iteration we have $|f_k| < \lambda$, fulfilling the requirement of the previous Lemma automatically.

The final iteration consists on zooming and shifting process:

$$w_1 = w/|w|_\infty,$$
$$w_{k+1} = \frac{1}{1-\lambda^*/4}((w_k)_R - \bar{w_k}),$$

where $\bar{w_k}$ is the average over $[-1,0]\times B_1$. On the first step we still have the bound $|f_1| < \lambda^{2k_0}$. Subsequently, among other restrictions put on R in [9] we set in addition $R(1-\lambda^*/4) > 1$, which preserves the bound $|f| < \varepsilon_0$ for all steps. This finishes the proof.

4.6. **Flocking to a uniform state when** $e = 0$**.** In the case if $e = 0$ we once again take advantage of the density equation (4.49). Note that the equation has a structure similar to the u-equation while the density remains uniformly bounded from above and below, see (4.18). Moreover, testing with ρ and using that $u_x = -\mathscr{L}_\phi \rho$, we obtain the energy equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho|_2^2 = \int |\rho|^2 \mathscr{L}_\phi \rho \,\mathrm{d}x.$$

Symmetrizing we obtain

$$\int |\rho|^2 \mathscr{L}_\phi \rho \,\mathrm{d}x = -\frac{1}{2}\int \phi(x,y)(\rho(x)+\rho(y))(\rho(x)-\rho(y))^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Since the pre-factor $(\rho(x)+\rho(y))$ is uniformly bounded from above and below this supplies the energy inequality analogous to (2.8a). We now have all ingredients for a direct application of Theorem 3.2 (with $\beta = 0$) to the density equation. This finishes the argument.

5. Appendix on topological alignment

5.1. **Mean-field limit, kinetic description.** We now give a formal derivation of a kinetic model (1.13) that describes evolution the large crowd dynamics (1.2) with geometric-topological communication kernel, $\phi(\mathbf{x},\mathbf{y}) = \psi_1(r(\mathbf{x},\mathbf{y})) \times \psi_2(\mathrm{d}_\rho(\mathbf{x},\mathbf{y}))$, in terms of a mass probability distribution of agents $f(t,\mathbf{x},\mathbf{v})$ in phase space $(\mathbf{x},\mathbf{v})$. Our argument is very similar to that given in Ha, Tadmor [31] in the homogeneous case with a few notable distinctions pertaining to modeling of the collective power terms m_{ij}. At this point we make no specific assumptions on the geometric term ψ.

We consider a probability density $P^N = P^N(\mathbf{x}_1,\mathbf{v}_1,...,\mathbf{x}_N,\mathbf{v}_N,t)$ of a system of N agents in the configuration space $(\mathbf{x}_1,\mathbf{v}_1,...,\mathbf{x}_N,\mathbf{v}_N) \in \mathbb{R}^{2nN}$. The conservation of mass in the Gibbs ensemble propagated according to the given system (2.2) leads to the classical Liouville equation:

$$P_t^N + \sum_{i=1}^N \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} P^N + \sum_{i=1}^N \nabla_{\mathbf{v}_i} \cdot (\dot{\mathbf{v}}_i P^N) = 0. \tag{5.1}$$

In the limit as $N \to \infty$ we assume that the total mass $M = \sum m_i$ remains constant while $\max_i m_i \to 0$. As a result, the agents become more and more indistinguishable, which we reflect in the symmetry condition

$$P^N(t,...,\mathbf{x}_i,\mathbf{v}_i,...,\mathbf{x}_j,\mathbf{v}_j,...) = P^N(t,...,\mathbf{x}_j,\mathbf{v}_j,...,\mathbf{x}_i,\mathbf{v}_i,...).$$

We seek to derive an equation for the first marginal

$$P^{1,N}(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^{2n(N-1)}} P^N(\mathbf{x},\mathbf{v},\bar{\mathbf{x}},\bar{\mathbf{v}},t)\mathrm{d}\bar{\mathbf{x}}\mathrm{d}\bar{\mathbf{v}},$$

where $\bar{\mathbf{x}} = (\mathbf{x}_2,...,\mathbf{x}_N)$ and $\bar{\mathbf{v}} = (\mathbf{v}_2,...,\mathbf{v}_N)$. Thus, integrating in $\bar{\mathbf{x}},\bar{\mathbf{v}}$ in (5.1) we obtain

$$P_t^{1,N} + \mathbf{v}\cdot\nabla_{\mathbf{x}}P^{1,N} + \lambda\nabla_{\mathbf{v}}\cdot\int_{\mathbb{R}^{2n(N-1)}}\sum_{j=2}^N m_j\psi_1(|\mathbf{x}-\mathbf{x}_j|)\times\psi_2(m_{1j})(\mathbf{v}_j-\mathbf{v})P^N\mathrm{d}\bar{\mathbf{x}}\mathrm{d}\bar{\mathbf{v}} = 0.$$

Note that the mass coefficients m_{1j} encode a rather complex dependence on all the spacial positions involved. However, as the number of agents increases it is reasonable to expect that m_{1j}'s are well approximated by the macroscopic mass of the communication domain:

$$m_{1j} \sim M\int_{\Omega(\mathbf{x},\mathbf{x}_j)\times\mathbb{R}^n} P^{1,N}(t,\mathbf{z},\mathbf{v})\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{v} := \mathcal{m}_N(\Omega(\mathbf{x},\mathbf{x}_j)). \tag{5.2}$$

By replacing m_{1j}'s in the equation with its approximation (5.2), and in view of the symmetry of P^N, we achieve equality of the integrals in the sum above, and hence,

$$P_t^{1,N} + \mathbf{v}\cdot\nabla_{\mathbf{x}}P^{1,N}$$
$$+\lambda(M-m_1)\nabla_{\mathbf{v}}\cdot\int_{\mathbb{R}^{2n(N-1)}}\psi_1(|\mathbf{x}-\mathbf{y}|)\times\psi_2(\mathcal{m}_N(\Omega(\mathbf{x},\mathbf{y})))(\mathbf{w}-\mathbf{v})P^{2,N}(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{w} = 0.$$

Denoting the limiting densities by $P = \lim_{N\to\infty} P^{1,N}$, $Q = \lim_{N\to\infty} P^{2,N}$, and accordingly,

$$\mathcal{m}_N(\Omega(\mathbf{x},\mathbf{y})) \to \mathcal{m}_t(\Omega(\mathbf{x},\mathbf{y})) = M\int_{\Omega(\mathbf{x},\mathbf{y})\times\mathbb{R}^n} P(t,\mathbf{z},\mathbf{w})\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{w} \tag{5.3}$$

we obtain

$$P_t + \mathbf{v}\cdot\nabla_{\mathbf{x}}P + \lambda M\nabla_{\mathbf{v}}\cdot\int_{\mathbb{R}^{2n}}\psi_1(|\mathbf{x}-\mathbf{y}|)\psi_2(\mathcal{m}_t(\Omega(\mathbf{x},\mathbf{y})))(\mathbf{w}-\mathbf{v})Q(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{w} = 0.$$

We close by making the molecular chaos assumption

$$Q(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w}) = P(t,\mathbf{x},\mathbf{v})P(t,\mathbf{y},\mathbf{w}),$$

which results in the following Vlasov-type equation for the mass density $f = MP$

$$f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}}f + \lambda\nabla_{\mathbf{v}}\cdot Q(f,f) = 0, \tag{5.4}$$

where

$$Q(f,f)(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^{2n}}\psi_1(|\mathbf{x}-\mathbf{y}|)\times\psi_2(\mathcal{m}_t(\Omega(\mathbf{x},\mathbf{y})))(\mathbf{w}-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{w}.$$

Note. A more straightforward way to see a connection between discrete and kinetic models is to consider (5.4) for measure-valued solutions and to compute it for the empirical measure given by

$$\mu_t^N = \sum_{i=1}^N m_i\delta_{\mathbf{x}_i(t)}\otimes\delta_{\mathbf{v}_i(t)} \quad \text{subject to} \quad \sum m_j = \rho_0, \ \sum_j m_j\mathbf{v}_j = \rho_0\mathbf{u}_0. \tag{5.5}$$

In fact this approach is more practical for rigorous justification purposes, [30, 45]. We say that $\{\mu_t\}_{0\leqslant t<T}$ is a measure-valued solution to (5.4) with initial condition μ_0 if for any test-function $g \in C_0^\infty([0,T)\times\mathbb{R}^{2n})$ one has

$$\langle \mu_0, g(0,\cdot,\cdot)\rangle + \int_0^T \langle \mu_t, \partial_t g + \mathbf{v}\cdot\nabla_{\mathbf{x}} g\rangle \,\mathrm{d}t + \int_0^T \langle \mu_t, F(\mu_t)\cdot\nabla_{\mathbf{v}} g\rangle \,\mathrm{d}t = 0. \tag{5.6}$$

Here, by $F(\mu_t)$ we understand a function of $(t,\mathbf{x},\mathbf{v})$ given by

$$F(\mu_t)(t,\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^{2n}} \psi_1(|\mathbf{x}-\mathbf{y}|)\times\psi_2(\mathscr{m}_t(\Omega(\mathbf{x},\mathbf{y})))(\mathbf{w}-\mathbf{v})\,\mathrm{d}\mu_t(\mathbf{y},\mathbf{w}),$$
$$\mathscr{m}_t(\Omega(\mathbf{x},\mathbf{y})) = \int_{\mathbb{R}^n}\int_{\Omega(\mathbf{x},\mathbf{y})} 1\ \mathrm{d}\mu_t(\mathbf{z},\,\mathrm{d}\mathbf{v}).$$

Thus, for the empirical measures μ_t^N, we recover the mass coefficients $m_{ij} = \mathscr{m}_t(\Omega(\mathbf{x}_i,\mathbf{x}_j))$.

The above formulation undoubtedly presents several technical difficulties which we plan to address in the future. We only remark that at least for the weakly singular kernels when the total order of the singularity is less than n, and the spatial marginals of μ_t satisfy the no-vacuum condition:

$$\mathscr{m}_t(\Omega(\mathbf{x},\mathbf{y})) \geqslant c_T|\mathbf{x}-\mathbf{y}|,\ \text{for all } \mathbf{x},\mathbf{y}\in \operatorname{supp}\int_{\mathbb{R}^n}\mu_t(\cdot,\,\mathrm{d}\mathbf{v}), \tag{5.7}$$

one can verify directly that if $(\mathbf{x}_i(t),\mathbf{v}_i(t))$ is a solution to (2.2), then the empirical measure (5.5) satisfies weak kinetic formulation (5.6), and clearly it satisfies (5.7). We refer to [45, page 281] for a similar computation in CS case. Rigorous justification of the limit, as $N\to\infty$, requires a separate study and we will not pursue it at the moment.

5.2. **Pointwise evaluation of topological alignment.** Here we collect necessary formalities related to pointwise evaluations of the operator $\mathscr{L}_\phi$ and the commutator $\mathscr{C}_\phi$. The statements come with corresponding estimates we used throughout the text.

Lemma 5.1. *For any $0<\alpha<2$ one has the natural pointwise representation formula*

$$\mathscr{L}_\phi f(\mathbf{x}) = p.v.\int_{\mathbb{T}^n}(f(\mathbf{x}+\mathbf{z})-f(\mathbf{x}))\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\,d\mathbf{z}. \tag{5.8}$$

Moreover, for any $r>0$,

$$\mathscr{L}_\phi f(\mathbf{x}) = \int_{\mathbb{T}^n}(f(\mathbf{x}+\mathbf{z})-f(\mathbf{x})-\mathbf{z}\cdot\nabla f(\mathbf{x})\chi_{|\mathbf{z}|<r}(\mathbf{z}))\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\,d\mathbf{z} + b_r(\mathbf{x})\cdot\nabla f(\mathbf{x}), \tag{5.9}$$

where

$$|b_r|_\infty \leqslant C|\nabla\rho|_\infty r^{2-\alpha}. \tag{5.10}$$

Proof. At the core of the proof is a bound on the operator given by

$$B_r\zeta(\mathbf{x}) = p.v.\int_{|\mathbf{z}|<r}\zeta(\mathbf{x}+\mathbf{z})\,\mathbf{z}\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\,\mathrm{d}\mathbf{z}.$$

Clearly, $B_r 1 = b_r$. We address it more generally as was used in preceding sections. By symmetrization,

$$B_r\zeta(\mathbf{x}) = \frac{1}{2}\int_{|\mathbf{z}|<r} \frac{\mathrm{d}_\rho^\tau(\mathbf{x}-\mathbf{z},\mathbf{x}) - \mathrm{d}_\rho^\tau(\mathbf{x}+\mathbf{z},\mathbf{x})}{\mathrm{d}_\rho^\tau(\mathbf{x}+\mathbf{z},\mathbf{x})\mathrm{d}_\rho^\tau(\mathbf{x}-\mathbf{z},\mathbf{x})|\mathbf{z}|^{n+\alpha-\tau}}\zeta(\mathbf{x}+\mathbf{z})\mathbf{z}h(\mathbf{z})\,\mathrm{d}\mathbf{z}$$
$$+\frac{1}{2}\int_{|\mathbf{z}|<r} \frac{\zeta(\mathbf{x}+\mathbf{z})-\zeta(\mathbf{x}-\mathbf{z})}{\mathrm{d}_\rho^\tau(\mathbf{x}-\mathbf{z},\mathbf{x})|\mathbf{z}|^{n+\alpha-\tau}}\mathbf{z}h(\mathbf{z})\,\mathrm{d}\mathbf{z} = I(\mathbf{x}) + J(\mathbf{x}).$$

In what follows the constant C will change line to line and may depend on the underlying bounds on the density at hand, (2.11). As to J, we directly obtain

$$|J(\mathbf{x})| \leqslant C|\nabla\zeta|_\infty r^{2-\alpha}.$$

For $I(\mathbf{x})$ we first observe

$$\mathrm{d}_\rho^\tau(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}_\rho^\tau(\mathbf{x}-\mathbf{z},\mathbf{x}) = \frac{\tau}{n}[\mathrm{d}_\rho^n(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}_\rho^n(\mathbf{x}-\mathbf{z},\mathbf{x})]\times$$
$$\times\int_0^1 \left[\theta\mathrm{d}_\rho^n(\mathbf{x}+\mathbf{z},\mathbf{x}) + (1-\theta)\mathrm{d}_\rho^n(\mathbf{x}-\mathbf{z},\mathbf{x})\right]^{\frac{\tau}{n}-1}\,\mathrm{d}\theta.$$

Note that

$$|\mathrm{d}_\rho^n(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}_\rho^n(\mathbf{x}-\mathbf{z},\mathbf{x})| = \left|\int_{\Omega(\mathbf{z},0)}(\rho(\mathbf{x}+\mathbf{w})-\rho(\mathbf{x}-\mathbf{w}))\,\mathrm{d}w\right| \leqslant |\nabla\rho|_\infty|\mathbf{z}|^{n+1},$$

and clearly,

$$\int_0^1 \left[\theta\mathrm{d}_\rho^n(\mathbf{x}+\mathbf{z},\mathbf{x}) + (1-\theta)\mathrm{d}_\rho^n(\mathbf{x}-\mathbf{z},\mathbf{x})\right]^{\frac{\tau}{n}-1}\,\mathrm{d}\theta \leqslant C|\mathbf{z}|^{\tau-n}.$$

Consequently,

$$|I(\mathbf{x})| \leqslant C|\nabla\rho|_\infty|\zeta|_\infty \int_{|\mathbf{z}|<r}\frac{1}{|\mathbf{z}|^{n+\alpha-2}}\,\mathrm{d}\mathbf{z} \sim |\nabla\rho|_\infty|\zeta|_\infty r^{2-\alpha}.$$

In conclusion we obtain the bound

$$|B_r\zeta|_\infty \leqslant C\left(|\nabla\rho|_\infty|\zeta|_\infty + |\nabla\zeta|_\infty\right)r^{2-\alpha}. \tag{5.11}$$

Note that the bounds above provide a common integrable dominant for the integrands parametrized by $\mathbf{x}$. So, in addition $B_r\zeta \in C(\mathbb{T}^n)$.

The bound (5.10) now follows directly from (5.11), and we also have $b_r \in C(\mathbb{T}^n)$. With the knowledge that the drift is finite, clearly, the right hand sides of (5.8) and (5.9) coincide. Denote them $L_\phi f(\mathbf{x})$. We now have a task to pass to the limit

$$\langle \mathscr{L}_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0),$$

for *every* $\mathbf{x}_0 \in \mathbb{T}^n$. Splitting the integral we obtain

$$\langle \mathscr{L}_\phi f, g_\varepsilon\rangle = \frac{1}{2}\int_{\mathbb{R}} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\chi_{|\mathbf{x}-\mathbf{y}|<r})(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_{\mathbb{R}} \phi(\mathbf{x},\mathbf{y})\nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\chi_{|\mathbf{x}-\mathbf{y}|<r})(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = I + J.$$

Note that $J = \frac{1}{2}\langle b_r\cdot\nabla f, g_\varepsilon\rangle + \frac{1}{2}\langle B_r\nabla f, g_\varepsilon\rangle$. By continuity of B_r proved above,

$$J \to \frac{1}{2}b_r(\mathbf{x}_0)\cdot\nabla f(\mathbf{x}_0) + \frac{1}{2}(B_r\nabla f)(\mathbf{x}_0). \tag{5.12}$$

As to I we can unwind the symmetrization since each part of the integral is not singular any more:

$$I = \frac{1}{2}\int_{\mathbb{R}} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\chi_{|\mathbf{x}-\mathbf{y}|<r})g_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$$
$$- \frac{1}{2}\int_{\mathbb{R}} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\chi_{|\mathbf{x}-\mathbf{y}|<r})g_\varepsilon(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.$$

Passing to the limit in each integral we obtain

$$\begin{aligned}
I &\to \frac{1}{2}\int_{\mathbb{T}}(f(\mathbf{y}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0))\phi(\mathbf{x}_0,\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&- \frac{1}{2}\int_{\mathbb{T}^2}(f(\mathbf{x}_0) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{x}_0-\mathbf{x}))\phi(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}} \phi(\mathbf{x}_0,\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}_0) - \frac{1}{2}(\nabla f(\mathbf{x}_0) + \nabla f(\mathbf{y}))(\mathbf{y}-\mathbf{x}_0)\chi_{|\mathbf{x}_0-\mathbf{y}|<r})\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}} \phi(\mathbf{x}_0,\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0)\chi_{|\mathbf{x}_0-\mathbf{y}|<r})\,\mathrm{d}\mathbf{y} \\
&+ \frac{1}{2}\int_{\mathbb{R}} \phi(\mathbf{x}_0,\mathbf{y})(\nabla f(\mathbf{x}_0) - \nabla f(\mathbf{y}))(\mathbf{y}-\mathbf{x}_0)\chi_{|\mathbf{x}_0-\mathbf{y}|<r}\,\mathrm{d}\mathbf{y} \\
&= L_\phi f(\mathbf{x}_0) - \frac{1}{2}b_r(\mathbf{x}_0)\cdot\nabla f(\mathbf{x}_0) - \frac{1}{2}(B_r\nabla f)(\mathbf{x}_0).
\end{aligned}$$

Thus, combining with (5.12) we obtain

$$I + J \to L_\phi f(\mathbf{x}_0).$$

This finishes the proof. □

As a corollary we obtain analogous representation formula for the commutator.

Lemma 5.2. *For any $0 < \alpha < 2$ one has the following pointwise representation*

$$\mathscr{C}_\phi(f,\zeta)(\mathbf{x}) = p.v.\int_{\mathbb{T}^n} \phi(\mathbf{x}+\mathbf{z},\mathbf{x})\zeta(\mathbf{x}+\mathbf{z})(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}))\,d\mathbf{z}. \tag{5.13}$$

Moreover, the following representation holds for any $r > 0$:

$$\begin{aligned}
\mathscr{C}_\phi(f,\zeta)(\mathbf{x}) = \int_{\mathbb{R}} &\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\zeta(\mathbf{x}+\mathbf{z})(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) - \mathbf{z}\cdot\nabla f(\mathbf{x})\chi_{|\mathbf{z}|<r})\,d\mathbf{z} \\
&+ (\zeta(\mathbf{x})b_r(\mathbf{x}) + a_r(\mathbf{x}))\cdot\nabla f(\mathbf{x}),
\end{aligned} \tag{5.14}$$

where b_r is as before, and

$$|a_r|_\infty \leqslant C|\nabla\zeta|_\infty r^{2-\alpha}. \tag{5.15}$$

The proof goes by a direct application of Lemma 5.1. For the residual drift we obtain

$$a_r(\mathbf{x}) = \int_{|\mathbf{z}|<r} \phi(\mathbf{x}+\mathbf{z},\mathbf{x})(\zeta(\mathbf{x}+\mathbf{z}) - \zeta(\mathbf{x}))\mathbf{z}\,\mathrm{d}\mathbf{z}.$$

The bound (5.15) follows immediately.

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