The Vlasov-Poisson-Fokker-Planck System with Uncertainty and a One-dimensional Asymptotic Preserving Method *

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Abstract

We develop a stochastic Asymptotic Preserving (s-AP) scheme for the Vlasov-Poisson-Fokker-Planck (VPFP) system in the high field regime with uncertainty based on the generalized Polynomial Chaos Stochastic Galerkin framework (gPC-SG). We first prove that, for a given electric field with uncertainty, the regularity of initial data in the random space is preserved by the analytical solution at later time, which allows us to establish the spectral convergence of the gPC-SG method. We follow the framework developed in [15] to numerically solve the resulting system in one space dimension, and show formally that the fully discretized scheme is s-AP in the high field regime. Numerical examples are given to validate the accuracy and s-AP properties of the proposed method.

Key words. Vlasov-Poisson-Fokker-Planck system, Uncertainty Quantification, Asymptotic Preserving, Polynomial Chaos, Stochastic Galerkin

AMS subject classifications.

1 Introduction

In this paper we are interested in developing a stochastic Asymptotic-preserving scheme for the Vlasov-Poisson-Fokker-Planck (VPFP) system with random inputs, which arises in the kinetic modeling of the Brownian motion of a large system of particles in a surrounding bath [2]. One application of such system is in electrostatic plasma, in which one considers the interactions between the electrons and a surrounding bath via the Coulomb force. The equation takes the form of a Liouville equation with a Fokker-Planck operator in the velocity space, coupled with a Poisson equation for the electric field. See Section 2 for details of the equations. The unknown in the system is $f(t, \mathbf{x}, \mathbf{v})$, the particle density distribution of particles at time $t > 0$, position $\boldsymbol{x} \in \mathbb{R}^N$ with velocity $\boldsymbol{v} \in \mathbb{R}^N$. In addition to the classical difficulty of high dimensionality

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to solve equations in the phase space, the problem under study has two more computational challenges: *Multi-scale and uncertainty.*

In this paper the high field regime, in which the strong forcing term balances the Fokker-Planck diffusion term [1], will be considered. In this problem, numerical stiffness arises due to the strong field and diffusion term. On the other hand, in this regime one can approximate the VPFP system by its high field limit, which has the form of a transport-Poisson system for the density and electric potential [10, 20]. One successful numerical strategy to efficiently compute into such asymptotic regimes is to develop *Asymptotic-Preserving (AP) schemes*, which preserves the continuous asymptotic limit in the discrete space in a numerically uniformly stable way [12]. This strategy has been widely used in kinetic and hyperbolic equations with multiple time and space scales (see [13] for a general review and [4] for applications in plasma). For its development for the high-field limit, see [15, 16, 8, 3].

Another difficulty here is to treat the uncertainty. Due to modeling and measurement errors, uncertainties in kinetic modeling could arise from initial and boundary data, and the forcing term. In this paper we will consider the cases in which the electric potential and initial data contain random inputs, modeled by random variables with given probability density functions. In recent years, the generalized polynomial chaos approximation based stochastic Galerkin (gPC-SG) methods have found many applications in a wide range of physical and engineering problems, see [6, 23, 21], although its applications in kinetic problems are scarce, see recent efforts in [17, 11, 14]. It is the goal of this paper to develop a gPC-SG method for the VPFP system with random inputs that are *stochastic Asymptotic-Preserving* (s-AP). As defined in [17], for the s-AP scheme, a stochastic Galerkin method for the VPFP system, in the high field limit, becomes a stochastic Galerkin method for the limiting transport-Poisson system, when all the numerical parameters are held fixed. For this scheme, one can use a fixed mesh size, time step, and the number of gPC modes, in different asymptotic regimes. In particular, one does not need to numerically resolve the physically small scale and still capture the correct solutions of the high field limit.

For a given electric potential that contains uncertainty (thus the underlying problem becomes linear), we first prove, in section 3, that the system preserves the regularity of the initial data in the random space. In section 4 we introduce the gPC-SG method for the VPFP system, and the regularity result in section 3 naturally leads to the proof of the spectral accuracy of the method in section 5. Since the gPC-SG system is a vector version of the deterministic VPFP system, in section 6, in the one dimensional case, we will use the AP scheme developed for its deterministic counterpart in [15] for time, spatial and velocity discretizations, and the method is shown formally to be s-AP, namely, in the high field limit, it gives the gPC-SG method–actually a kinetic scheme– for the limiting system. Numerical experiments are conducted to demonstrate asymptotic property, accuracy and other properties of the method in section 7.

In the near future we will also develop multi-dimensional s-AP schemes for the VPFP system.

2 The Background and Model

2.1 The VPFP System with Uncertainty

In the VPFP system with uncertainty, the time evolution equations of particle density distribution function $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z})$ under the action of an electrical potential $\phi(t, \boldsymbol{x}, \boldsymbol{z})$ is

$$\begin{cases} \partial_t f + \boldsymbol{v} \cdot \bigtriangledown_{\boldsymbol{x}} f - \frac{1}{\epsilon} \bigtriangledown_{\boldsymbol{x}} \phi \cdot \bigtriangledown_{\boldsymbol{v}} f = \frac{1}{\epsilon} \bigtriangledown_{\boldsymbol{v}} \cdot [\boldsymbol{v} f + \bigtriangledown_{\boldsymbol{v}} f], \\ - \triangle_{\boldsymbol{x}} \phi = \rho - h, \ t > 0, \quad \boldsymbol{x} \in \mathbb{R}^N, \ \boldsymbol{v} \in \mathbb{R}^N, \ \boldsymbol{z} \in I_z, \end{cases} \tag{2.1}$$

with the following initial condition:

$$f(0, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}) = f^0(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}), \quad \boldsymbol{x} \in \mathbb{R}^N, \ \boldsymbol{v} \in \mathbb{R}^N, \ \boldsymbol{z} \in I_z. \tag{2.2}$$

Here the distribution function $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z})$ depends on time t, position $\boldsymbol{x}$, velocity $\boldsymbol{v}$ and random variable $\boldsymbol{z} \in I_{\boldsymbol{z}} \subseteq \mathbb{R}^d$. $\boldsymbol{z}$ is in a properly defined probability space $(\Sigma, \mathbb{A}, \mathbb{P})$, whose event space is Σ and is equipped with σ-algebra $\mathcal{A}$ and probability measure $\mathbb{P}$. $\phi(t, \boldsymbol{x}, \boldsymbol{z})$ is the self-consistent electrical potential, and $h(\boldsymbol{x}, \boldsymbol{z})$ is a given positive background charge with global neutrality relation

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f^0(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}) d\boldsymbol{x} d\boldsymbol{v} = \int_{\mathbb{R}^N} h(\boldsymbol{x}, \boldsymbol{z}) d\boldsymbol{x}, \tag{2.3}$$

and the density function $\rho(t, \boldsymbol{x}, \boldsymbol{z})$ is defined as

$$\rho(t, \boldsymbol{x}, \boldsymbol{z}) = \int_{\mathbb{R}^N} f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}) d\boldsymbol{v}. \tag{2.4}$$

Besides, we define operators $\mathcal{L}$, $\mathcal{L}_\phi$ as,

$$\mathcal{L}(f, \phi) = \partial_t f + \boldsymbol{v} \cdot \bigtriangledown_{\boldsymbol{x}} f - \frac{1}{\epsilon} \bigtriangledown_{\boldsymbol{x}} \phi \cdot \bigtriangledown_{\boldsymbol{v}} f - \frac{1}{\epsilon} \bigtriangledown_{\boldsymbol{v}} \cdot [\boldsymbol{v} f + \bigtriangledown_{\boldsymbol{v}} f], \tag{2.5}$$

$$\mathcal{L}_\phi(f, \phi) = - \triangle_{\boldsymbol{x}} \phi - (\rho - h). \tag{2.6}$$

2.2 The High Field Limit

Here we will show the formal limit of (2.1) when $\epsilon \to 0$.

First, integrate (2.1) over $\boldsymbol{v}$,

$$\partial_t \int_{\mathbb{R}^N} f d\boldsymbol{v} + \bigtriangledown_{\boldsymbol{x}} \cdot \int_{\mathbb{R}^N} \boldsymbol{v} f d\boldsymbol{v} - \frac{1}{\epsilon} \int_{\mathbb{R}^N} \bigtriangledown_{\boldsymbol{v}} \cdot (\bigtriangledown_{\boldsymbol{x}} \phi f d\boldsymbol{v}) = \frac{1}{\epsilon} \int_{\mathbb{R}^N} \bigtriangledown_{\boldsymbol{v}} \cdot (\boldsymbol{v} f + \bigtriangledown_{\boldsymbol{v}} f) d\boldsymbol{v}. \tag{2.7}$$

Define the flux

$$j = \int_{\mathbb{R}^N} \boldsymbol{v} f d\boldsymbol{v}. \tag{2.8}$$

After integrating by parts, one has

$$\partial_t \rho + \bigtriangledown_{\boldsymbol{x}} \cdot j = 0. \tag{2.9}$$

Then multiply $\boldsymbol{v}$ to both sides of (2.1) and integrate over $\boldsymbol{v}$,

$$\epsilon \partial_t \int_{\mathbb{R}^N} \boldsymbol{v} f + \epsilon \int_{\mathbb{R}^N} \boldsymbol{v} \boldsymbol{v} \cdot \bigtriangledown_{\boldsymbol{x}} f = \int_{\mathbb{R}^N} \boldsymbol{v} \bigtriangledown_{\boldsymbol{v}} \cdot (f \bigtriangledown_{\boldsymbol{x}} \phi) + \boldsymbol{v} \bigtriangledown_{\boldsymbol{v}} \cdot [\boldsymbol{v} f + \bigtriangledown_{\boldsymbol{v}} f]. \tag{2.10}$$

Let $\epsilon \to 0$ it becomes

$$0 = \int_{\mathbb{R}^N} \boldsymbol{v} \left[\nabla_{\boldsymbol{v}} \cdot (f \nabla_{\boldsymbol{x}} \phi + \boldsymbol{v} f + \nabla_{\boldsymbol{v}} f) \right] d\boldsymbol{v}, \tag{2.11}$$

which implies,

$$0 = \int_{\mathbb{R}^N} f \nabla_{\boldsymbol{x}} \phi + \boldsymbol{v} f + \nabla_{\boldsymbol{v}} f d\boldsymbol{v}. \tag{2.12}$$

Therefore, one has,

$$j = -\rho(\nabla_{\boldsymbol{x}} \phi). \tag{2.13}$$

Finally plugging (2.13) into (2.9), one gets the high field limit of system (2.1),

$$\begin{cases} \partial_t \rho - \nabla_{\boldsymbol{x}} \cdot (\rho \nabla_{\boldsymbol{x}} \phi) = 0, \\ -\triangle_{\boldsymbol{x}} \phi = \rho - h. \end{cases} \tag{2.14}$$

For each fixed $\boldsymbol{z}$, the rigorous proof for the high field limit of VPFP system in one dimension can be found in [10, 20].

3 Regularity of the Solution in the Random Space

In this section, we study the regularity of $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z})$ for a given potential function $\phi(t, \boldsymbol{x}, \boldsymbol{z})$. In this setting, the equation is linear. This regularity will be needed to prove the spectral convergence of the gPC approximation in Section 5.3. To simplify the notation we also assume $z \in I_z \subset \mathbb{R}$. All the theory can be extended to $\boldsymbol{z} \in \mathbb{R}^d$ easily.

Before we start, let us first define $\pi(\boldsymbol{z}) : I_{\boldsymbol{z}} \longrightarrow \mathbb{R}^+$ as the probability density function of the random variable $\boldsymbol{z}(\omega)$, $\omega \in \Sigma$. So one can define a corresponding L^2_π space with inner product,

$$< f, g >_\pi := \int_{I_z} f g \pi(\boldsymbol{z}) \, d\boldsymbol{z}, \tag{3.1}$$

and weighted norm in $\boldsymbol{x}$, $\boldsymbol{v}$, $\boldsymbol{z}$ space

$$||f||_\pi = \left(\int_{I_z} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f|^2 \, \pi(\boldsymbol{z}) d\boldsymbol{x} d\boldsymbol{v} d\boldsymbol{z} \right)^{\frac{1}{2}}. \tag{3.2}$$

3.1 Regularity of Solution in the Random Space

Theorem 3.1. *Given $\phi(t, \boldsymbol{x}, z)$, if there exists some integer $m > 0$, and positive constants C_f, C_ϕ, such that $||\partial_z^l f^0||_\pi \le C_f$, $||\partial_z^l \nabla_{\boldsymbol{x}} \phi||_{L^\infty} \le C_\phi$, for $l = 0, \cdots, m$, then*

$$||\partial_z^l f(t)||_\pi \le D_l e^{\frac{G_l t}{\epsilon}}, \quad \text{for } l = 0, \cdots, m, \tag{3.3}$$

where $D_l = 2a^l C_f \, l!$, $a = \max\{C_\phi, 1\}$, $G_l = \frac{1}{2}(l+1)$,

Proof. For notation simplicity, we take $N = 1$. However, the proof can be easily extended to multi-dimensional $\boldsymbol{x}$ and $\boldsymbol{v}$.

First, multiply $2f\pi(z)$ to (2.1) and integrate it over x, v and z, after integration by parts, one gets,

$$\epsilon \partial_t ||f||^2_\pi = ||f||^2_\pi - 2||\partial_v f||^2_\pi. \tag{3.4}$$

For $l = 1, \cdots, m$, take l-th derivative in z to (2.1), one gets,

$$\epsilon\partial_t\partial_z^l f + \epsilon v\partial_x\partial_z^l f - \partial_x\phi\partial_v\partial_z^l f - \sum_{i=0}^{l-1}\binom{l}{i}(\partial_z^{l-i}\partial_x\phi)(\partial_v\partial_z^i f) = \partial_v(v\partial_z^l f + \partial_v\partial_z^l f). \tag{3.5}$$

Multiplying $2\pi(z)\partial_z^l f$ and integrating over x, v and z, then the second and third terms vanish, so one has, for $l = 1, \cdots, m$

$$\epsilon\partial_t||\partial_z^l f(t)||_\pi^2 = \int_{I_z}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\sum_{i=0}^{l-1} 2\binom{l}{i}(\partial_z^{l-i}\partial_x\phi)(\partial_v\partial_z^i f)\partial_z^l f\pi(z)\ dxdvdz + ||\partial_z^l f||_\pi^2 - 2||\partial_v\partial_z^l f||_\pi^2. \tag{3.6}$$

Using Young's Inequality and the boundedness of $\|\partial_z^l\nabla_{\boldsymbol{x}}\phi\|_\infty$, one gets,

$$\epsilon\partial_t||\partial_z^l f(t)||_\pi^2 \leq C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_v\partial_z^i f||_\pi^2 + (l+1)\,||\partial_z^l f(t)||_\pi^2 - 2||\partial_v\partial_z^l f||_\pi^2\ . \tag{3.7}$$

Multiplying a constant A_l^m to (3.7) and summing l from 1 to m, then adding $A_0^m\times$(3.4) gives,

$$\begin{aligned}
&\epsilon\partial_t\left(\sum_{l=0}^{m} A_l^m||\partial_z^l f(t)||_\pi^2\right)\\
\leq& C_\phi^2\sum_{l=1}^{m}\sum_{i=0}^{l-1} A_l^m\binom{l}{i}^2||\partial_v\partial_z^i f||_\pi^2 + \sum_{l=0}^{m}(l+1)\,A_l^m||\partial_z^l f(t)||_\pi^2 - 2\sum_{l=0}^{m} A_l^m||\partial_v\partial_z^l f||_\pi^2\\
=&\sum_{i=0}^{m-1}\left(\sum_{l=i+1}^{m} C_\phi^2\binom{l}{i}^2 A_l^m - 2A_i^m\right)||\partial_v\partial_z^i f||_\pi^2 - 2A_m^m||\partial_v\partial_z^i f||_\pi^2 + \sum_{l=0}^{m}(l+1)\,A_l^m||\partial_z^l f(t)||_\pi^2.
\end{aligned} \tag{3.8}$$

Let $A_m^m = 1$ and $\sum_{l=i+1}^m C_\phi^2\binom{l}{i}^2 A_l^m - 2A_i^m = 0$, for $i = 0, \cdots, m-1$, (3.8) becomes

$$\epsilon\partial_t\left(\sum_{l=0}^{m} A_l^m||\partial_z^l f(t)||_\pi^2\right) \leq \sum_{l=0}^{m}(l+1)\,A_l^m||\partial_z^l f(t)||_\pi^2. \tag{3.9}$$

and one has a linear system for $A_i^m, i = 0, \cdots, m-1$:

$$\begin{pmatrix}
-\frac{2}{C_\phi^2} & \binom{1}{0}^2 & \binom{2}{0}^2 & \cdots & \binom{m-1}{0}^2\\
 & -\frac{2}{C_\phi^2} & \binom{2}{1}^2 & \cdots & \binom{m-1}{1}^2\\
 & & \ddots & \vdots & \vdots\\
 & & & -\frac{2}{C_\phi^2} & \binom{m-1}{m-2}^2\\
 & & & & -\frac{2}{C_\phi^2}
\end{pmatrix}
\begin{pmatrix} A_0^m\\ A_1^m\\ \vdots\\ A_{m-2}^m\\ A_{m-1}^m\end{pmatrix}
= -\begin{pmatrix}\binom{m}{0}^2\\ \binom{m}{1}^2\\ \vdots\\ \binom{m}{m-2}^2\\ \binom{m}{m-1}^2\end{pmatrix}. \tag{3.10}$$

Lemma 3.2. *Solving the linear system (3.10), one has,*

$$0 < A_l^m \leq b^{m-l}\left(\frac{m!}{l!}\right)^2, \quad \text{where } b = \max\{1, C_\phi^2\} \tag{3.11}$$

Proof. See Appendix A.1 ■

Therefore, by Lemma 3.2, and apply Gronwall's Inequality to (3.9), one obtains,

$$\begin{aligned}\sum_{l=0}^{m} A_l^m||\partial_z^l f(t)||_\pi^2 &\le e^{\frac{(m+1)t}{\epsilon}}\left(\sum_{l=0}^{m} A_l^m||\partial_z^l f(0)||_\pi^2\right) \le e^{\frac{(m+1)t}{\epsilon}} C_f^2 \sum_{l=0}^{m} b^{m-l}\left(\frac{m!}{l!}\right)^2 \\ &\le e^{\frac{(m+1)t}{\epsilon}} C_f^2 (m!)^2 b^m \left[\left(\frac{1}{0!}\right)^2 + \sum_{l=1}^{m}\frac{1}{b^l 4^{(l-1)}}\right] \\ &\le \frac{7}{3} b^m (m!)^2 e^{\frac{(m+1)t}{\epsilon}} C_f^2, \end{aligned} \tag{3.12}$$

which implies,

$$||\partial_z^m f(t)||_\pi \le (2a^m m!) e^{\frac{(m+1)t}{2\epsilon}} C_f, \tag{3.13}$$

where $a = \max\{C_\phi, 1\}$ ■

3.2 Regularity of $\nabla_{\boldsymbol{v}} f$ in the Random Space

Theorem 3.3. *Given $\phi(t, \boldsymbol{x}, z)$, if there exists some integer $m > 0$, and positive constants C_f, C_ϕ, such that $||\partial_z^l \nabla_{\boldsymbol{v}} f(0)||_\pi \le C_f$, $||\partial_z^l \nabla_{\boldsymbol{x}} f(0)||_\pi \le C_f$, $||\partial_z^l \nabla_{\boldsymbol{x}}\phi||_{L^\infty} \le C_\phi$, $||\partial_z^l \nabla_{\boldsymbol{x}}^2\phi||_{L^\infty} \le C_\phi$, for $l = 0, \cdots, m$, then,*

$$||\partial_z^l \nabla_{\boldsymbol{v}} f(t)||_\pi \le C_l e^{\frac{L_l}{\epsilon} t}, \quad \textit{for } l = 0, \cdots, m, \tag{3.14}$$

where $C_l = 3a^l C_f\, l!$, $a = \max\{C_\phi, 1\}$, $L_l = \frac{1}{2}(\epsilon + C_\phi + 5 + 2l)$.

Proof. Applying $\partial_z^l \partial_v$ and $\partial_z^l \partial_x$ to (2.1), $l = 1, \cdots, m$, gives

$$\begin{aligned} &\epsilon\partial_t\partial_z^l\partial_v f + \epsilon v\partial_x\partial_z^l\partial_v f + \epsilon\partial_z^l\partial_x f - \sum_{i=0}^{l-1}\tbinom{l}{i}\partial_z^{l-i}\partial_x\phi\partial_z^i\partial_v^2 f - \partial_x\phi\partial_v\partial_z^l\partial_v f \\ =&\partial_v(\partial_z^l f + v\partial_z^l\partial_v f + \partial_z^l\partial_v^2 f)\ ; \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\epsilon\partial_t\partial_z^l\partial_x f + \epsilon v\partial_x\partial_z^l\partial_x f - \partial_x^2\phi\partial_v\partial_z^l f - \partial_x\phi\partial_v\partial_x\partial_z^l f - \sum_{i=0}^{l-1}\tbinom{l}{i}\partial_x^2\partial_z^{l-i}\phi\partial_v\partial_z^i f \\ -&\sum_{i=0}^{l-1}\tbinom{l}{i}\partial_x\partial_z^{l-i}\phi\partial_v\partial_z^i\partial_x f = \partial_v(v\partial_z^l\partial_x f + \partial_v\partial_z^l\partial_x f)\ . \end{aligned} \tag{3.16}$$

Multiplying $2\pi(z)\partial_z^l\partial_v f$ to (3.15) and $2\pi(z)\partial_z^l\partial_x f$ to (3.16), and integrating over x, v and z, one has respectively,

$$\begin{aligned} &\epsilon\partial_t||\partial_z^l\partial_v f||_\pi^2 + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 2\epsilon < \partial_z^l\partial_x f,\ \partial_z^l\partial_v f >_\pi -2\sum_{i=0}^{l-1} < \tbinom{l}{i}\partial_z^{l-i}\partial_x\phi\partial_z^i\partial_v^2 f,\ \partial_z^l\partial_v f >_\pi dxdv \\ =&3||\partial_z^l\partial_v f||_\pi^2 - 2||\partial_z^l\partial_v^2 f||_\pi^2\ ; \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\epsilon\partial_t||\partial_z^l\partial_x f||_\pi^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 2\sum_{i=0}^{l-1} < \tbinom{l}{i}\partial_x^2\partial_z^{l-i}\phi\partial_z^i\partial_v f,\ \partial_z^l\partial_x f >_\pi dxdv \\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 2\sum_{i=0}^{l-1} < \tbinom{l}{i}\partial_x\partial_z^{l-i}\phi\partial_z^i\partial_v\partial_x f,\ \partial_z^l\partial_x f >_\pi -2 < \partial_x^2\phi\partial_z^l\partial_v f,\ \partial_z^l\partial_x f >_\pi dxdv \\ =&||\partial_z^l\partial_x f||_\pi^2 - 2||\partial_z^l\partial_v\partial_x f||_\pi^2\ . \end{aligned} \tag{3.18}$$

By Young's Inequality, one gets,

$$\epsilon\partial_t||\partial_z^l\partial_v f||_\pi^2 \leq \epsilon||\partial_z^l\partial_x f||_\pi^2 + (\epsilon+3+l)\,||\partial_z^l\partial_v f||_\pi^2 + C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v^2 f||_\pi^2 - 2||\partial_z^l\partial_v^2 f||_\pi^2\ ; \quad (3.19)$$

$$\epsilon\partial_t||\partial_z^l\partial_x f||_\pi^2 \leq (C_\phi+1+2l)\,||\partial_z^l\partial_x f||_\pi^2 + C_\phi||\partial_z^l\partial_v f||_\pi^2 + C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v f||_\pi^2$$
$$+ C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v\partial_x f||_\pi^2 - 2||\partial_z^l\partial_v\partial_x f||_\pi^2. \quad (3.20)$$

Summing the two inequalities yields,

$$\epsilon\partial_t\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right)$$
$$\leq (\epsilon+C_\phi+3+2l)\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right) + C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v f||_\pi^2$$
$$+ C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v^2 f||_\pi^2 - 2||\partial_z^l\partial_v^2 f||_\pi^2 + C_\phi^2\sum_{i=0}^{l-1}\binom{l}{i}^2||\partial_z^i\partial_v\partial_x f||_\pi^2 - 2||\partial_z^l\partial_v\partial_x f||_\pi^2\ . \quad (3.21)$$

Similarly, for $l=0$, one has,

$$\epsilon\partial_t(||\partial_x f||_\pi^2 + ||\partial_v f||_\pi^2) \leq (C_\phi+3+\epsilon)(||\partial_x f||_\pi^2 + ||\partial_v f||_\pi^2) - 2||\partial_x\partial_v f||_\pi^2 - 2||\partial_v\partial_v f||_\pi^2\ . \quad (3.22)$$

Multiplying A_l^m to (3.20) and Summing it from 1 to m over l, then adding $A_0^m\times$(3.22), gives,

$$\epsilon\partial_t\sum_{l=0}^{m}A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right)$$
$$\leq \sum_{l=0}^{m}(\epsilon+C_\phi+3+2l)\,A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right) + \sum_{i=0}^{m-1}\left(C_\phi^2\sum_{l=i+1}^{m}\binom{l}{i}^2A_l^m||\partial_z^i\partial_v f||_\pi^2\right)$$
$$+ \sum_{i=0}^{m-1}\left(C_\phi^2\sum_{l=i+1}^{m}\binom{l}{i}^2A_l^m - 2A_i^m\right)\left(||\partial_z^i\partial_v^2 f||_\pi^2 + ||\partial_z^i\partial_v\partial_x f||_\pi^2\right). \quad (3.23)$$

Let $A_m^m=1$ and A_i^m solves (3.10), for $i=0,\cdots,m-1$, one has,

$$\epsilon\partial_t\sum_{l=0}^{m}A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right)$$
$$\leq \sum_{l=0}^{m}(\epsilon+C_\phi+3+2l)\,A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right) + \sum_{i=0}^{m-1}2A_i^m||\partial_z^i\partial_v f||_\pi^2$$
$$\leq \sum_{l=0}^{m}(\epsilon+C_\phi+5+2l)\,A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right), \quad (3.24)$$

then by Lemma 3.2 and Gronwall's Inequality, one obtains,

$$\sum_{l=0}^{m}A_l^m\left(||\partial_z^l\partial_v f||_\pi^2 + ||\partial_z^l\partial_x f||_\pi^2\right) \leq \frac{7}{3}b^m(m!)^2e^{\frac{\epsilon+C_\phi+5+2m}{\epsilon}t}2C_f^2. \quad (3.25)$$

Therefore, one can get,

$$||\partial_z^m\partial_v f||_\pi \leq 3a^m m!e^{\frac{\epsilon+C_\phi+5+2m}{2\epsilon}t}C_f. \quad (3.26)$$

which completes the proof. ■

Remark 3.4. *Theorems 3.1 and 3.3 imply that if f and $\partial_{\boldsymbol{v}} f$ are in $H^m = \{f \mid \|\partial_z^l f\|_\pi < \infty, 0 \leq l \leq m\}$ initially, then under suitable assumption on the regularity of ϕ as given in Theorems 3.1 and 3.3, f and $\nabla_{\boldsymbol{v}} f$ remain in H^m at later time. Thus the regularity in z of the initial data is preserved in time.*

4 The gPC Method for the VPFP System

4.1 The Method of gPC

Let W_π^K be the orthogonal polynomial space corresponding to the random space $(\Sigma, \mathbb{A}, \mathbb{P})$,

$$W_\pi^K = \{g : I_{\boldsymbol{z}} \longrightarrow \mathbb{R} : g \in \text{span}\{\Phi_k(\boldsymbol{z})\}_{k=0}^K\}, \tag{4.1}$$

where Φ_k, $k = 0, \cdots, K$ is a set of d-variate orthonormal polynomials of degree k satisfying,

$$< \Phi_k, \Phi_l >_\pi = \mathbb{E}(\Phi_k \Phi_l) = \int_{I_{\boldsymbol{z}}} \Phi_k(\boldsymbol{z}) \Phi_l(\boldsymbol{z}) \pi(\boldsymbol{z}) d\boldsymbol{z} = \delta_{kl}. \tag{4.2}$$

Here $\mathbb{E}$ means the expected value, and δ_{kl} is the Kronecker delta function. By the classical approximation theory, W_π^∞ is a Hilbert space with inner product $< \cdot, \cdot >_\pi$. Thus the solution $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z})$, $\phi(t, \boldsymbol{x}, \boldsymbol{z})$ to (2.1) can be represented as

$$f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}) = \sum_{k=0}^\infty \bar{f}_k(t, \boldsymbol{x}, \boldsymbol{v}) \Phi_k(\boldsymbol{z}), \quad \phi(t, \boldsymbol{x}, \boldsymbol{z}) = \sum_{k=0}^\infty \bar{\phi}_k(t, \boldsymbol{x}) \Phi_k(\boldsymbol{z}), \quad \text{in } L_\pi^2 \ . \tag{4.3}$$

In the gPC stochastic Galerkin (gPC-SG) method, one seeks an approximation to the exact solution f and ϕ in the subspace W_π^K, i.e. the approximation solution $\hat{f}^K$, $\hat{\phi}^K$ are in the form of,

$$\hat{f}^K(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{z}) = \sum_{k=0}^K \hat{f}_k(t, \boldsymbol{x}, \boldsymbol{v}) \Phi_k(\boldsymbol{z}) \triangleq \hat{\boldsymbol{f}}^K \cdot \boldsymbol{\Phi}^K, \quad \hat{\phi}^K(t, \boldsymbol{x}, \boldsymbol{z}) = \sum_{k=0}^K \hat{\phi}_k(t, \boldsymbol{x}) \Phi_k(\boldsymbol{z}) \triangleq \hat{\boldsymbol{\phi}}^K \cdot \boldsymbol{\Phi}^K \ , \tag{4.4}$$

where $\boldsymbol{\Phi}^K = (\Phi_0, \cdots, \Phi_K)$, and $\hat{f}_k =< \hat{f}^K, \Phi_k >_\pi$, $\hat{\phi}_k =< \hat{\phi}^K, \Phi_k >_\pi$, which are independent of $\boldsymbol{z}$, are the components of vector $\hat{\boldsymbol{f}}^K, \hat{\boldsymbol{\phi}}^K$ satisfying, for $0 \leq j \leq K$,

$$\begin{aligned} &< \mathcal{L}(\hat{f}^K, \hat{\phi}^K), \Phi_j >_\pi = 0, \\ &< \mathcal{L}_\phi(\hat{f}^K, \hat{\phi}^K), \Phi_j >_\pi = 0. \end{aligned} \tag{4.5}$$

We also approximate the given charge h by

$$\hat{h}^K(\boldsymbol{x}, \boldsymbol{z}) = \sum_{k=0}^K \hat{h}_k \Phi_k \triangleq \hat{\boldsymbol{h}}^K \cdot \boldsymbol{\Phi}^K, \tag{4.6}$$

where $\hat{h}_k(\boldsymbol{x}) =< h, \Phi_k >_\pi$, for $k = 0, \cdots, K$.
By the definition of ρ in (2.4), the numerical approximation of ρ is,

$$\hat{\rho}^K(t, \boldsymbol{x}, \boldsymbol{z}) = \sum_{k=0}^K \hat{\rho}_k \Phi_k \triangleq \hat{\boldsymbol{\rho}}^K \cdot \boldsymbol{\Phi}^K, \tag{4.7}$$

where $\hat{\rho}_k(t,\boldsymbol{x}) = \int_{\mathbb{R}^N} \hat{f}_k(t,\boldsymbol{x},\boldsymbol{v})d\boldsymbol{v}$, for $k = 0, \cdots, K$.

By equation (4.5), we have for each $j = 0, \cdots, K$,

$$\begin{cases} \partial_t f_j + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_j - \frac{1}{\epsilon}\sum_{k,l=0}^K \nabla_{\boldsymbol{x}}\phi_k \cdot \nabla_{\boldsymbol{v}} f_l (E_j)_{kl} = \frac{1}{\epsilon}\nabla_{\boldsymbol{v}}\cdot[\boldsymbol{v} f_j + \nabla_{\boldsymbol{v}} f_j], \\ -\triangle_{\boldsymbol{x}}\, \phi_j = \rho_j - h_j, \quad \text{for } 1 \le j \le K, \end{cases} \tag{4.8}$$

where E_j, $(0 \le j \le K)$, is a $(K+1)$-dimensional matrix, and $(E_k)_{jl} = \mathbb{E}\Phi_j\Phi_l\Phi_k$.

In order to express the system in a simple form, also for the sake of combining the stiff terms and forming an AP scheme as in [15], we give the following Lemma.

Lemma 4.1. *For matrix E_i, $0 \le i \le K$, defined above, one has*

$$\sum_{k,l=0}^K \nabla_{\boldsymbol{x}}\phi_k \cdot \nabla_{\boldsymbol{v}} f_l (E_j)_{kl} = \nabla_{\boldsymbol{v}} \cdot \left[\sum_{k=0}^K (E_k \hat{\boldsymbol{f}}^K)_j (\nabla_{\boldsymbol{x}}\phi_k)^\top\right]. \tag{4.9}$$

Proof.

$$\begin{aligned} &\sum_{k,l=0}^K \nabla_{\boldsymbol{x}}\phi_k \cdot \nabla_{\boldsymbol{v}} f_l (E_j)_{kl} \\ =&\sum_{k,l=0}^K\sum_{i=0}^N \partial_{x_i}\phi_k \partial_{v_i} f_l (E_j)_{kl} = \sum_{k=0}^K\sum_{i=0}^N \partial_{x_i}\phi_k \sum_{l=1}^K (E_k)_{jl}\partial_{v_i} f_l = \sum_{k=0}^K\sum_{i=0}^N \partial_{x_i}\phi_k \partial_{v_i}\left[\sum_{l=0}^K (E_k)_{jl} f_l\right] \\ =&\sum_{k=0}^K\sum_{i=0}^N \partial_{x_i}\phi_k\partial_{v_i}(E_k\hat{\boldsymbol{f}}^K)_j = \sum_{k=0}^K \nabla_{\boldsymbol{v}}\cdot[\partial_{x_1}\phi_1(E_k\hat{\boldsymbol{f}}^K)_j, ..., \partial_{x_N}\phi_k(E_k\hat{\boldsymbol{f}}^K)_j] \\ =&\nabla_{\boldsymbol{v}}\cdot\left[\sum_{k=1}^K (E_k\hat{\boldsymbol{f}}^K)_j \nabla_{\boldsymbol{x}}\phi_k\right]. \end{aligned} \tag{4.10}$$

∎

Now by Lemma 4.1, (4.8) can be written in a vector form as

$$\begin{cases} \partial_t \hat{\boldsymbol{f}}^K + (\nabla_{\boldsymbol{x}}\hat{\boldsymbol{f}}^K)\boldsymbol{v} - \dfrac{1}{\epsilon}\nabla_{\boldsymbol{v}}\cdot\left[\displaystyle\sum_{k=0}^K E_k\hat{\boldsymbol{f}}^K\nabla_{\boldsymbol{x}}\phi_k\right] = \dfrac{1}{\epsilon}\nabla_{\boldsymbol{v}}\cdot\left[\hat{\boldsymbol{f}}^K\boldsymbol{v}^\top + \nabla_{\boldsymbol{v}}\hat{\boldsymbol{f}}^K\right], \\ -\triangle_{\boldsymbol{x}}\hat{\boldsymbol{\phi}}^K = \hat{\boldsymbol{\rho}}^K - \hat{\boldsymbol{h}}^K. \end{cases} \tag{4.11}$$

5 The Spectral Convergence of the gPC-SG Method

In this section, we establish the spectral convergence of the gPC-SG method for a given potential $\phi(t,\boldsymbol{x},\boldsymbol{z})$.

5.1 Stability

We first prove a stability result, estimating the evolution of $||\hat{f}^K(t)||_\pi$

Theorem 5.1. *For $\forall\ t > 0$,*

$$||\hat{f}^K(t)||_\pi \le e^{\frac{3Nt}{\epsilon}}||\hat{f}^K(0)||_\pi \tag{5.1}$$

Proof. Due to the orthogonality of $\phi_k(z)$, one has $||\hat{f}^K||_\pi = ||\hat{\boldsymbol{f}}^K||_{L^2}$, with $||\cdot||_{L^2}$ defined as,

$$||\cdot||_{L^2} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^N} ||\cdot||_2^2 \; dxdv \right)^{\frac{1}{2}}, \tag{5.2}$$

where $||\cdot||_2$ is the regular Euclidean norm for vectors. Therefore one only needs to prove the theorem for $||\hat{\boldsymbol{f}}^K(t)||_{L^2}$.

Multiplying $\hat{f}_j$ to (4.8) and integrating over $\boldsymbol{x}$ and $\boldsymbol{v}$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[\partial_t \left(\frac{1}{2} \hat{f}_j^2 \right) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \left(\frac{1}{2} \hat{f}_j^2 \right) - \frac{1}{\epsilon} \sum_{k,l,i=0}^{K} \partial_{x_i} \phi_k \hat{f}_j \partial_{v_i} \hat{f}_l (E_j)_{kl} \right] \boldsymbol{dxdv} \\ =& - \frac{1}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \boldsymbol{v} \cdot \nabla_{\boldsymbol{v}} \left(\frac{1}{2} \hat{f}_j^2 \right) \; \boldsymbol{dxdv} + \frac{N}{\epsilon} ||\hat{f}_j^2||_{L^2}^2 - \frac{1}{\epsilon} ||\nabla_{\boldsymbol{v}} \hat{f}_j||_{L^2}^2, \end{aligned} \tag{5.3}$$

After integration by parts the second term on the LHS vanishes, and the first term of the RHS becomes $\frac{N}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{2} \hat{f}_j^2 \; \boldsymbol{dxdv}$. Sum j from 1 to K, one gets,

$$\frac{1}{2} \partial_t ||\hat{\boldsymbol{f}}^K||_{L^2}^2 - \frac{1}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \sum_{k,l,i,j=0}^{K} \partial_{x_i} \phi_k \hat{f}_j \partial_{v_i} \hat{f}_l (E_j)_{kl} \boldsymbol{dxdv} \leq \left(\frac{N}{2\epsilon} + \frac{N}{\epsilon} \right) ||\hat{\boldsymbol{f}}^K||_{L^2}^2. \tag{5.4}$$

Note the second term on the LHS also vanishes, since

$$\begin{aligned} &\frac{1}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \sum_{k,l,i,j=0}^{K} \partial_{x_i} \phi_k \hat{f}_j \partial_{v_i} \hat{f}_l (E_j)_{kl} \; \boldsymbol{dxdv} \\ =& \frac{1}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[\sum_{k,i=0}^{K} \sum_{j=0}^{K} \partial_{x_i} \phi_k (E_j)_{kj} \partial_{v_i} \left(\frac{1}{2} \hat{f}_j^2 \right) + \sum_{k,i=0}^{K} \sum_{j \neq l}^{K} \partial_{x_i} \phi_k \hat{f}_j \partial_{v_i} \hat{f}_l (E_k)_{jl} \right] \; \boldsymbol{dxdv}, \\ =& \frac{1}{\epsilon} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \sum_{k,i=0}^{K} \sum_{j=0}^{K} \partial_{x_i} \phi_k (E_j)_{kj} \partial_{v_i} \left(\frac{1}{2} \hat{f}_j^2 \right) + \sum_{k,i=0}^{K} \sum_{j>l}^{K} \partial_{x_i} \phi_k (E_k)_{jl} \partial_{v_i} (\hat{f}_j \hat{f}_l) \; \boldsymbol{dxdv}. \end{aligned} \tag{5.5}$$

By the symmetric of E_k, where the last inequality uses the symmetry of E_k. Both terms in (5.5) vanish after integration by parts, so (5.4) implies,

$$\frac{1}{2} \partial_t ||\hat{\boldsymbol{f}}^K||_{L^2}^2 \leq \left(\frac{N}{2\epsilon} + \frac{N}{\epsilon} \right) ||\hat{\boldsymbol{f}}^K||_{L^2}^2, \tag{5.6}$$

By Gronwall's Inequality,

$$||\hat{\boldsymbol{f}}^K(t)||_{L^2} \leq e^{\frac{3Nt}{\epsilon}} ||\hat{\boldsymbol{f}}^K(0)||_{L^2}, \tag{5.7}$$

which completes the proof. ■

5.2 The Spectral Convergence

Before we start to prove the convergence of the numerical approximation $\hat{f}$, for the sake of convenience, we assume $z \in \mathbb{R}$, and all the proof can be easily extended to multi-dimensional $\boldsymbol{z}$. We define operators $\mathcal{L}_f$, $\mathcal{K}$ as,

$$\mathcal{L}_f := \epsilon \partial_t + \epsilon \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} - \boldsymbol{v} \cdot \nabla_{\boldsymbol{v}} - N - \triangle_{\boldsymbol{v}}, \quad \mathcal{K} := \nabla_{\boldsymbol{x}} \phi \cdot \nabla_{\boldsymbol{v}}, \quad \text{then} \quad \mathcal{L} = \mathcal{L}_f - \mathcal{K}. \tag{5.8}$$

Let the projection of the exact solution $f(t,\boldsymbol{x},\boldsymbol{v},z)$ to the subspace W_π^K be $\mathcal{P}_K f$,

$$\mathcal{P}_K f := \sum_{k=0}^{K} < f, \Phi_k >_\pi \Phi_k(z) := \sum_{k=0}^{K} \bar{f}_k(t,x,v)\Phi_k(z) := \bar{\boldsymbol{f}}^K \cdot \boldsymbol{\Phi}^K, \tag{5.9}$$

where $\bar{\boldsymbol{f}} = (\bar{f}_0, \cdots, \bar{f}_K)^\top$. As defined in (4.4), the numerical approximation $\hat{f}^K = \hat{\boldsymbol{f}}^K \cdot \boldsymbol{\Phi}^K$, then the error can be split into two parts,

$$f - \hat{f}^K = (f - \mathcal{P}^K f) + (\mathcal{P}^K f - \hat{f}^K) := R^K + \mu^K, \tag{5.10}$$

Where

$$R^K = \sum_{k=K+1}^{\infty} \bar{f}_k(t,\boldsymbol{x},\boldsymbol{v})\Phi_k(z), \tag{5.11}$$

is the projection error. Define vector

$$\boldsymbol{\mu}^K = (\mu_0, \cdots, \mu_K) \quad \text{with} \quad \mu_i = \bar{f}_i - \hat{f}_i, \quad i = 0, \cdots, K. \tag{5.12}$$

So $\mu^K = \boldsymbol{\mu}^K \cdot \boldsymbol{\Phi}^K$ is the error of the gPC-SG approximation.

Theorem 5.2. *Given $\phi(t,\boldsymbol{x},z)$, if for some integer $m>0$, and positive constants C_f, C_ϕ, such that $||\partial_z^l \nabla_{\boldsymbol{v}} f(0)||_\pi \le C_f$, $||\partial_z^l \nabla_{\boldsymbol{x}}\phi||_{L^\infty} \le C_\phi$, $||\partial_z^l \nabla_{\boldsymbol{x}}^2\phi||_{L^\infty} \le C_\phi$, for $l = 0, \cdots, m$, then for $0 < t < T$,*

$$||\mu^K(t)||_\pi^2 \le \frac{H_m e^{\frac{2L_m+3N}{2\epsilon}t}}{K^m}, \tag{5.13}$$

where $H_m = \frac{C_A C_m C_\phi}{\sqrt{2L_m}}$, with C_A a constant depending on polynomials $\{\Phi_k(z) \mid 0 \le k \le m\}$.

Proof. Subtracting $< \mathcal{L}f, \boldsymbol{\Phi}^K >_\pi = 0$ by $< \mathcal{L}\hat{f}^K, \boldsymbol{\Phi}^K >_\pi = 0$, one has

$$< \mathcal{L}_f(f - \hat{f}^K), \boldsymbol{\Phi}^K) >_\pi - < \mathcal{K}(f - \hat{f}^K), \boldsymbol{\Phi}^K >_\pi = 0. \tag{5.14}$$

Since $\mathcal{L}_f$ is independent of z,

$$< \mathcal{L}_f(f - \hat{f}^K), \boldsymbol{\Phi}^K) >_\pi = \mathcal{L}_f < f - \hat{f}^K, \boldsymbol{\Phi}^K >_\pi = \mathcal{L}_f(\boldsymbol{\mu}^K). \tag{5.15}$$

Plugging (5.15) into (5.14) gives,

$$\mathcal{L}_f(\boldsymbol{\mu}^K) - < \mathcal{K}(\mu^K + R^K), \boldsymbol{\Phi}^K >_\pi = 0. \tag{5.16}$$

Taking dot product of $2\boldsymbol{\mu}^K$ to (5.16), then integrating over $\boldsymbol{x}$, $\boldsymbol{v}$, yields,

$$\begin{aligned}
0 =& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[2\mathcal{L}_f(\boldsymbol{\mu}^K)\cdot\boldsymbol{\mu}^K - 2 < \mathcal{K}(\mu^K + R^K), \boldsymbol{\Phi}^K >_\pi \cdot \boldsymbol{\mu}^K\right] \, d\boldsymbol{x}d\boldsymbol{v} \\
=& \epsilon\partial_t ||\mu^K||_\pi^2 - 2N||\mu^K||_\pi^2 + 2||\nabla_{\boldsymbol{v}}\mu^K||_\pi^2 - 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} < \mathcal{K}(R^K), \mu^K >_\pi \, d\boldsymbol{x}d\boldsymbol{v} \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{I_z} \partial_x\phi\nabla_{\boldsymbol{v}}(\mu^K)^2\pi(z)\, dz d\boldsymbol{x}d\boldsymbol{v} \\
=& \epsilon\partial_t ||\mu^K||_\pi^2 - 2N||\mu^K||_\pi^2 + 2||\nabla_{\boldsymbol{v}}\mu^K||_\pi^2 - 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} < \mathcal{K}(R^K), \mu^K >_\pi \, d\boldsymbol{x}d\boldsymbol{v}.
\end{aligned} \tag{5.17}$$

This gives,

$$\epsilon \partial_t ||\mu^K||_\pi^2 \leq (2N+N)||\mu^K||_\pi^2 + C_\phi^2 ||\nabla_{\boldsymbol{v}} R^K||_\pi^2. \tag{5.18}$$

Since $||\mu^K(0)||_\pi = \int\int ||\boldsymbol{u}^K(0)||_2 \; d\boldsymbol{x} d\boldsymbol{v} = 0$, and by Grownwall's inequality implies,

$$||\mu^K(t)||_\pi^2 \leq \frac{1}{\epsilon}\left(C_\phi^2 \int_0^t ||\nabla_{\boldsymbol{v}} R^K(s)||_\pi^2 ds \right) e^{\frac{3N}{\epsilon}t}. \tag{5.19}$$

By classical approximation theory and Theorem 3.3,

$$||\nabla_{\boldsymbol{v}} R^K||_\pi \leq \frac{C_A ||\partial_z^m \nabla_{\boldsymbol{v}} f||_\pi}{K^m} \leq \frac{C_A C_m e^{\frac{L_m}{\epsilon}t}}{K^m}, \tag{5.20}$$

where C_A is a constant depending on polynomials $\{\Phi_k(z) \mid 0 \leq k \leq m\}$. Plugging (5.20) into (5.19) yields,

$$||\mu^K(t)||_\pi^2 \leq \frac{H_m^2 (e^{\frac{2L_m}{\epsilon}t} - 1)}{K^{2m}} e^{\frac{3N}{\epsilon}t}, \tag{5.21}$$

where $H_m = \frac{C_A C_m C_\phi}{\sqrt{2L_m}}$, which implies,

$$||\mu^K(t)||_\pi \leq \frac{H_m e^{\frac{2L_m+3N}{2\epsilon}t}}{K^m}. \tag{5.22}$$

■

Theorem 5.3. *Given $\phi(t,x,z)$, if for some integer $m > 0$, and positive constants C_f, C_ϕ, such that $||\partial_z^l f(0)||_\pi \leq C_f$, $\|\partial_z^l \nabla_{\boldsymbol{v}} f(0)\| \leq C_f$, $||\partial_z^l \nabla_x \phi||_{L^\infty} \leq C_\phi$, $||\partial_z^l \nabla_x^2 \phi||_{L^\infty} \leq C_\phi$, for $l = 0, \cdots, m$. Then the K-th order numerical approximation $\hat{f}^K$ converges to the solution f with an error,*

$$||f - \hat{f}^K||_\pi \leq \frac{O_m}{K^m}. \tag{5.23}$$

where $O_m = C_A D_m e^{\frac{G_m}{\epsilon}t} + H_m e^{\frac{2L_m+3N}{2\epsilon}t}$ is a finite positive constant depending on C_f, C_ϕ and ϵ.

Proof.

$$\begin{aligned} ||f - \hat{f}^K||_\pi \leq ||R^K||_\pi + ||\mu^K||_\pi &\leq \frac{C_A ||\partial_z^m f||_\pi}{K^m} + \frac{H_m e^{\frac{2L_m+3N}{2\epsilon}t}}{K^m} \\ &\leq \frac{C_A D_m e^{\frac{G_m}{\epsilon}t} + H_m e^{\frac{2L_m+3N}{2\epsilon}t}}{K^m}, \end{aligned} \tag{5.24}$$

The first inequality is because of the definition in (5.10), the second inequality is because of the error for projection and Theorem 5.2, the third inequality is because of Theorem 3.1. ■

Remark 5.4. *Theorem 5.3 shows that as $\epsilon \to 0$, one needs $K \gg O(e^{\frac{mt}{\epsilon}})$ to get a good accuracy. This motivates the development of the s-AP scheme in which one can take K independent of ϵ.*

6 The s-AP schemes

6.1 The High Field Limit of the gPC Method

We will first formally derive the high field limit of the gPC system (4.11). Integrating (4.11), and letting $\hat{\boldsymbol{j}}^K = \int_{\mathbb{R}} \hat{\boldsymbol{f}}^K \boldsymbol{v}^\top d\boldsymbol{v}$ be the flux, one gets,

$$\partial_t \hat{\boldsymbol{\rho}}^K + \nabla_{\boldsymbol{x}} \cdot \hat{\boldsymbol{j}}^K = 0, \tag{6.1}$$

then, multiplying $\boldsymbol{v}^\top$, the transpose of $\boldsymbol{v}$, to (6.4) and integrating it over $\boldsymbol{v}$ gives,

$$\left(\sum_{k=0}^{K} E_k \hat{\boldsymbol{\rho}}^K \nabla_{\boldsymbol{x}} \phi_k \right) + \hat{\boldsymbol{j}}^K = 0. \tag{6.2}$$

Plugging (6.2) into (6.1) yields the High-field limit system for the coefficient of $\hat{\rho}^K$ and $\hat{\phi}^K$,

$$\begin{cases} \partial_t \hat{\boldsymbol{\rho}}^K - \nabla_{\boldsymbol{x}} \cdot \left(\displaystyle\sum_{k=0}^{K} E_k \hat{\boldsymbol{\rho}}^K \nabla_{\boldsymbol{x}} \hat{\phi}_k \right) = 0, \\ - \triangle_x \hat{\boldsymbol{\phi}}^K = \hat{\boldsymbol{\rho}}^K - \hat{\boldsymbol{h}}^K. \end{cases} \tag{6.3}$$

This system is exactly the gPC system for the High-field limit with uncertainty (2.14), which shows that the gPC system is AP.

6.2 The fully discrete first order scheme

Here we'll give the VPFP system with uncertainty a fully discrete scheme when $N = 1$.

First we combine the stiff terms $\partial_v \left[\sum_{k=0}^{K} \left(\partial_x \hat{\phi}_k E_k \hat{\boldsymbol{f}}^K \right) \right]$ and $\partial_v \left(v \hat{\boldsymbol{f}}^K + \partial_v \hat{\boldsymbol{f}}^K \right)$, then

$$\begin{cases} \partial_t \hat{\boldsymbol{f}}^K + v \partial_x \hat{\boldsymbol{f}}^K = \dfrac{1}{\epsilon} \partial_v \left[\left(\displaystyle\sum_{k=0}^{K} \partial_x \hat{\phi}_k E_k + v I_K \right) \hat{\boldsymbol{f}}^K + \partial_v \hat{\boldsymbol{f}}^K \right], \\ - \partial_{xx} \boldsymbol{\phi}^K = \hat{\boldsymbol{\rho}}^K - \hat{\boldsymbol{h}}^K. \end{cases} \tag{6.4}$$

where I_K is $K \times K$ identity matrix.

Here we denote,

$$F = \sum_{k=0}^{K} \partial_x \hat{\phi}_k E_k, \quad P = F + v I_K, \quad A = -\frac{1}{2} |P|^2, \tag{6.5}$$

where

$$|P|^2 := P^\top P. \tag{6.6}$$

Let

$$M = \frac{1}{\sqrt{2\pi}} \mathrm{e}^A. \tag{6.7}$$

Concerning the properties of the matrix M, we give the following proposition.

Proposition 6.1. *Suppose M is defined in (6.7), then* *(a)* $\partial_v(M) = -PM$*;*

(b) M *is invertible, and* $M^{-1} = \sqrt{2\pi}e^{-A}$*,* $\partial_v M^{-1} = PM^{-1}$*;*

(c) M *and* M^{-1} *are both symmetric and positive definite;*

(d) $M(v_1)M(v_2)$ *is symmetric and positive definite for any* v_1, v_2 *and* $M(v_1)M(v_2) = M(v_2)M(v_1)$*;*

(e) $\int_{\mathbb{R}} M dv = I_K$*,* $\int_{\mathbb{R}} vM dv = F$*;*

(f) $MPM^{-1} = P$*.*

Proof. See the Appendix A.2 ∎

Back to system (6.4), where the stiff terms can be represented by $\partial_v \left[M\partial_v\left(M^{-1}\hat{\boldsymbol{f}}^K\right)\right]$ from Proposition 6.1 (a), (b), (g), thus (6.4) is equivalent to

$$\begin{cases} \partial_t \hat{\boldsymbol{f}}^K + v\partial_x \hat{\boldsymbol{f}}^K = \dfrac{1}{\epsilon}\partial_v \left[M\partial_v(M^{-1}\hat{\boldsymbol{f}}^K)\right], \\ -\partial_{xx}\boldsymbol{\phi}^K = \hat{\boldsymbol{\rho}}^K - \hat{\boldsymbol{h}}^K. \end{cases} \tag{6.8}$$

Denote $\hat{\boldsymbol{f}}_{ij}^n = \hat{\boldsymbol{f}}(t_n, x_i, v_j)$, $0 \le i \le N_x$, $-\frac{N_v}{2} \le j \le \frac{N_v}{2}$, $n \ge 0$. N_x, N_v(even) are numbers of mesh points in x and v directions respectively. Let $x_i = a + i\delta_x$, $v_j = j\delta_v$, $\hat{\boldsymbol{\rho}}_i^n = \delta_v \sum_{j=-N_v/2}^{N_v/2} \hat{\boldsymbol{f}}_{i,j}^n$ be the numerical approximation of density $\hat{\boldsymbol{\rho}}$. We choose N_v sufficiently large such that outside the velocity domain,

$$f|_{|v| \ge \frac{N_v}{2}\delta_v} \sim 0, \quad M|_{|v| \ge \frac{N_v}{2}\delta_v} \sim 0, \tag{6.9}$$

during the computational time.

We basically adopt the scheme in [15] for deterministic problem. The first order scheme is

$$\frac{\hat{\boldsymbol{f}}_{ij}^{n+1} - \hat{\boldsymbol{f}}_{ij}^n}{\delta_t} + \frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2},j}^n - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j}^n}{\delta_x} = \frac{1}{\epsilon}P(\hat{\boldsymbol{f}}_{ij}^{n+1}), \tag{6.10}$$

$$-\triangle_x \hat{\boldsymbol{\phi}}_{ij}^{n+1} = \hat{\boldsymbol{\rho}}_i^{n+1} - \hat{\boldsymbol{h}}_i^{n+1}, \tag{6.11}$$

where the upwind flux is used for spatial discretization,

$$\hat{\boldsymbol{f}}_{i+\frac{1}{2},j}^n = \frac{v_j + |v_j|}{2}\hat{\boldsymbol{f}}_{i,j}^n + \frac{v_j - |v_j|}{2}\hat{\boldsymbol{f}}_{i+1,j}^n. \tag{6.12}$$

$P(\hat{\boldsymbol{f}}_{ij}^{n+1})$ is the discretization form of $\mathcal{P}(\hat{\boldsymbol{f}}) = \partial_v[M\partial_v(M^{-1}\hat{\boldsymbol{f}})]$, which is defined as,

$$\begin{aligned} P(\hat{\boldsymbol{f}}_j) =& \frac{1}{\delta_v}\left[M_{j+1/2}[\partial_v(M^{-1}\hat{\boldsymbol{f}})]_{j+1/2} - M_{j-1/2}[\partial_v(M^{-1}\hat{\boldsymbol{f}})]_{j-1/2}\right] \\ =& \frac{1}{\delta_v^2}\left[M_{j+1}^{1/2}M_j^{1/2}(M_{j+1}^{-1}\hat{\boldsymbol{f}}_{j+1} - M_j^{-1}\hat{\boldsymbol{f}}_j) - M_j^{1/2}M_{j-1}^{1/2}(M_j^{-1}\hat{\boldsymbol{f}}_j - M_{j-1}^{-1}\hat{\boldsymbol{f}}_{j-1})\right] \\ =& \frac{M_j^{1/2}}{\delta_v^2}\left[M_{j+1}^{-1/2}\hat{\boldsymbol{f}}_{j+1} - (M_{j+1}^{1/2} + M_{j-1}^{1/2})M_j^{-1/2}(M_j^{-1/2}\hat{\boldsymbol{f}}_j) + M_{j-1}^{-1/2}\hat{\boldsymbol{f}}_{j-1}\right]. \end{aligned} \tag{6.13}$$

The algorithm is implemented as following:

- Step 1. Summing (6.10) over j. Since the RHS vanishes, one gets,

$$\frac{\hat{\boldsymbol{\rho}}_i^{n+1} - \hat{\boldsymbol{\rho}}_i^n}{\delta_t} + \frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\delta_x} = 0, \tag{6.14}$$

where $F_{i+\frac{1}{2}}^n = \delta_v \sum_j f_{i+\frac{1}{2},j}^n$. This gives $\hat{\boldsymbol{\rho}}_i^{n+1}$.

- Step 2. By using a Poisson solver, one gets $\hat{\boldsymbol{\phi}}_i^{n+1}$ from (6.11), which in term gives M_{ij}^{n+1} as,

$$M_{i,j}^{n+1} = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}|\sum_{k=0}^{K} \frac{(\hat{\phi}_k)_{i+1}^{n+1} - (\hat{\phi}_k)_{i-1}^{n+1}}{2\delta_x} E_k + v_j I_K|^2\right). \tag{6.15}$$

- Step 3. Since $F_i^n = \sum_{k=0}^{K} \frac{(\hat{\phi}_k)_{i+1}^n - (\hat{\phi}_k)_{i-1}^n}{\delta_x} E_k$ can be decomposed as $F_i^n = Q_i^n \Lambda_i^n (Q_i^n)^\top$, where Q_i^n is an orthogonal matrix, $\Lambda_i^n = \text{diag}(\lambda_0, \cdots, \lambda_K)_i^n$ is a diagonal matrix. Then $M_{ij}^n = Q_i^n e^{-\frac{1}{2}(v_j + \Lambda_i^n)^2} (Q_i^n)^\top$, therefore Let $\boldsymbol{\Lambda}_{ij}^n = e^{-\frac{1}{4}(v_j + \Lambda_i^n)^2}$, (6.13) can be written as,

$$P(\hat{\boldsymbol{f}}_{ij}^{n+1}) = \left\{ \frac{Q\Lambda_j}{\epsilon \triangle v^2} \left[\Lambda_{j+1}^{-1} Q^\top \hat{\boldsymbol{f}}_{j+1} - (\Lambda_{j+1} + \Lambda_{j-1}) \Lambda_j^{-1} \Lambda_j^{-1} Q^\top \hat{\boldsymbol{f}}_j + \Lambda_{j-1}^{-1} Q^\top \hat{\boldsymbol{f}}_{j-1} \right] \right\}_i^{n+1} \tag{6.16}$$

Multiply $(\Lambda_{ij}^{n+1})^{-1} (Q_i^{n+1})^\top$ to (6.10), and let $\hat{\boldsymbol{g}}_{ij}^{n+1} = (\Lambda_{ij}^{n+1})^{-1} (Q_i^{n+1})^\top \hat{\boldsymbol{f}}_{ij}^{n+1}$, one has,

$$\begin{aligned} &\hat{\boldsymbol{g}}_{i,j+1}^{n+1} - \left[(\Lambda_{i,j+1}^{n+1} + \Lambda_{i,j-1}) \Lambda_{ij}^{-1} + \frac{\epsilon \triangle v^2}{\triangle t} \right] \hat{\boldsymbol{g}}_{ij}^{n+1} + \hat{\boldsymbol{g}}_{i,j-1}^{n+1} \\ =&\epsilon \delta_v^2 (\Lambda_{ij}^{n+1})^{-1} (Q_i^{n+1})^\top \left(\frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2},j}^n - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j}^n}{\delta_x} - \frac{\hat{\boldsymbol{f}}_{ij}^n}{\delta_t} \right) \end{aligned} \tag{6.17}$$

Let $\boldsymbol{b}_{ij}^n = (\Lambda_{ij}^{n+1})^{-1} (Q_i^{n+1})^\top \left(\frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2},j}^n - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j}^n}{\delta_x} - \frac{\hat{\boldsymbol{f}}_{ij}^n}{\delta_t} \right)$, then one has a scalar solver for each component $\hat{g}_k^{n+1}$ of $\hat{\boldsymbol{g}}^{n+1}$, $k = 0, \cdots, K$,

$$(g_k)_{i,j+1}^{n+1} - \left[\frac{(m_k)_{i,j+1}^{n+1}}{(m_k)_{ij}^{n+1}} + \frac{(m_k)_{i,j-1}^{n+1}}{(m_k)_{ij}^{n+1}} + \frac{e \triangle v^2}{\triangle t} \right] (g_k)_{ij}^{n+1} + (g_k)_{i,j-1}^{n+1} = (b_k)_{ij}^n, \tag{6.18}$$

where $(m_k)_{ij}^{n+1} = e^{-\frac{|v_j + (\lambda_k)_i^{n+1}|^2}{4}}$, which has been proved in [18] that the linear system for $(g_k)_i^{n+1}$ is positive definite, so one can invert it by conjugate gradient method.

Remark 6.2. *Instead of using $M_{j+\frac{1}{2}} = M_j^{\frac{1}{2}} M_{j+1}^{\frac{1}{2}}$, one can also use $M_{j+1/2} = \frac{M_{j+1} + M_j}{2}$. By setting $g_{i,j} = \boldsymbol{\Lambda}_{i,j}^{-2} Q_i^\top f_{i,j}$, thus for fixed i, n, (6.13) will become,*

$$\begin{aligned} P(\hat{\boldsymbol{f}}_j) =& \frac{1}{\delta_v^2} \left[\frac{M_{j+1} + M_j}{2} (M_{j+1}^{-1} \hat{\boldsymbol{f}}_{j+1} - M_j^{-1} \hat{\boldsymbol{f}}_j) - \frac{M_j + M_{j-1}}{2} (M_j^{-1} \hat{\boldsymbol{f}}_j - M_{j-1}^{-1} \hat{\boldsymbol{f}}_{j-1}) \right] \\ =& \frac{Q}{2\delta_v^2} \left[(\boldsymbol{\Lambda}_{j+1}^2 + \boldsymbol{\Lambda}_j^2) \hat{\boldsymbol{g}}_{j+1} - (\boldsymbol{\Lambda}_{j+1}^2 + 2\boldsymbol{\Lambda}_j^2 + \boldsymbol{\Lambda}_{j-1}^2) \hat{\boldsymbol{g}}_j + (\boldsymbol{\Lambda}_j^2 + \boldsymbol{\Lambda}_{j-1}^2) \hat{\boldsymbol{g}}_{j-1} \right] \end{aligned} \tag{6.19}$$

Thus, (6.17) becomes,

$$
\begin{aligned}
&\left[(\mathbf{\Lambda}_{j+1}^2+\mathbf{\Lambda}_j^2)\hat{\boldsymbol{g}}_{j+1}-(\mathbf{\Lambda}_{j+1}^2+(2+\delta_v^2)\mathbf{\Lambda}_j^2+\mathbf{\Lambda}_{j-1}^2)\hat{\boldsymbol{g}}_j+(\mathbf{\Lambda}_j^2+\mathbf{\Lambda}_{j-1}^2)\hat{\boldsymbol{g}}_{j-1}\right]_i^{n+1}\\
=&2\delta_v^2 Q^\top\left[\frac{-\hat{\boldsymbol{f}}_{ij}^n}{\delta_t}+\frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2},j}^n-\hat{\boldsymbol{f}}_{i-\frac{1}{2},j}^n}{\delta_x}\right]
\end{aligned}
\tag{6.20}
$$

which can be decomposed to a scalar solver for each component of $\hat{\boldsymbol{g}}^{n+1}$. Besides, it is easy to see the coefficient in (6.20) is diagonally dominated matrix with negative diagonal entries, so it is a negative definite matrix.

6.3 The s-AP property

6.3.1 Mass Conservation

Since $\mathcal{P}(\hat{\boldsymbol{f}})$ has the property of mass conservation, its discretization $P(\hat{\boldsymbol{f}})$ should have the same property. Let

$$
K_j = M_{j+1}^{1/2}M_j^{1/2}\left(M_{j+1}^{-1}\hat{\boldsymbol{f}}_{j+1}-M_j^{-1}\hat{\boldsymbol{f}}_j\right),
\tag{6.21}
$$

then, by (6.13),

$$
\begin{aligned}
\sum_j P(\hat{\boldsymbol{f}}_j)=&\sum_j\frac{1}{\delta_v^2}[K_j-K_{j-1}]=\frac{1}{\delta_v^2}\sum_j K_j-\frac{1}{\delta_v^2}\sum_j K_{j-1}\\
=&\frac{1}{\delta_v^2}\sum_j K_j-\frac{1}{\delta_v^2}\sum_j K_j=0.
\end{aligned}
\tag{6.22}
$$

Thus, summing (6.10), one can get the scheme for $\hat{\boldsymbol{\rho}}^{n+1}$, (6.14), which also implies $\sum_i\hat{\boldsymbol{\rho}}_i^{n+1}=\sum_i\hat{\boldsymbol{\rho}}_i^n$.

6.3.2 The formal proof of s-AP

Here we want to prove the scheme is stochastic asymptotic preserving, that is for fixed $\delta_t,\delta_x,\delta_v$, when $\epsilon\to 0$, it automatically becomes a gPC-SG approximation for the high field limit.

Lemma 6.3. *In scheme (6.10), $\hat{\boldsymbol{f}}_{ij}^n\to M_{ij}^n\hat{\boldsymbol{c}}_i^n$, as $\epsilon\to 0$, where $\hat{\boldsymbol{c}}_i^n$ is independent of j.*

Proof. For fixed i, n, let $\epsilon\to 0$, multiply v_j to (6.10) and sum it over j, one gets,

$$
0=\sum_j v_j P(\hat{\boldsymbol{f}}_j)=\frac{1}{\delta_v^2}\sum_j v_j[K_j-K_{j-1}]=\frac{1}{\delta_v^2}\sum_j\delta_v K_j,
\tag{6.23}
$$

which is equivalent to,

$$
\sum_j K_j=0.
\tag{6.24}
$$

Letting $\epsilon\to 0$, (6.10) also implies $P(\hat{\boldsymbol{f}}_j)=0$ for $\forall j$, or equivalently,

$$
\frac{1}{\delta_v^2}(K_j-K_{j-1})=0\quad\text{for }\forall j.
\tag{6.25}
$$

This implies,

$$K_j = \boldsymbol{c} \quad \text{for } \forall j, \tag{6.26}$$

where $\boldsymbol{c}$ is a constant depending on i and n.

From (6.24), (6.26), one has $K_j \equiv 0$. By the definition of K_j in (6.21), this implies,

$$(M^n_{i,j+1})^{-1}\hat{\boldsymbol{f}}^n_{i,j+1} - (M^n_{ij})^{-1}\hat{\boldsymbol{f}}^n_{ij} = 0, \tag{6.27}$$

therefore,

$$(M^n_{ij})^{-1}\hat{\boldsymbol{f}}^n_{ij} = \hat{\boldsymbol{c}}^n_i \quad for \ \forall\ j, \tag{6.28}$$

this gives,

$$\hat{\boldsymbol{f}}^n_{ij} = M^n_{ij}\hat{\boldsymbol{c}}^n_i.$$

■

Lemma 6.4. *If $\hat{\boldsymbol{f}}^n_{ij} = M^n_{ij}\hat{\boldsymbol{c}}^n_i$, where $\hat{\boldsymbol{c}}^n_i$ is a constant vector, then $\hat{\boldsymbol{c}}^n_i = \hat{\boldsymbol{\rho}}^n_i + O(\delta_v^2)$.*

Proof. As defined in section 6.2,

$$\delta_v \sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} \hat{\boldsymbol{f}}_{ij} = \hat{\boldsymbol{\rho}}_i = (\delta_v \sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} M_{ij})\hat{\boldsymbol{c}}_i. \tag{6.29}$$

Since for fixed i, n, $M_{ij} = \frac{1}{\sqrt{2\pi}}\exp(-\frac{|F_i+v_jI|^2}{2})$, where F is a constant symmetric matrix for each i. So there exists a unity matrix Q, and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_K)$, s.t. $F = Q^\top \Lambda Q$. Thus,

$$M_j = \frac{1}{\sqrt{2\pi}}Q^\top e^{-\frac{\Lambda^2+2v_j\Lambda+v_j^2 I}{2}}Q = \frac{1}{\sqrt{2\pi}}Q^\top \mathrm{diag}\left(e^{-\frac{(\lambda_1+v_j)^2}{2}}, \cdots, e^{-\frac{(\lambda_n+v_j)^2}{2}}\right)Q. \tag{6.30}$$

Use the trapezoidal rule and assumption (6.9),

$$\begin{aligned}1 &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda_i+v)^2}{2}} dv \\ &= \sum_{j=-\frac{N_v}{2}+1}^{j=\frac{N_v}{2}-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda_i+v_j)^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\lambda_i + v_{-\frac{N_v}{2}})^2\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\lambda_i + v_{\frac{N_v}{2}})^2\right) + O(\delta_v^2),\end{aligned} \tag{6.31}$$

Again by assumption (6.9), $\frac{1}{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\lambda_i + v_{-\frac{N_v}{2}})^2\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\lambda_i + v_{\frac{N_v}{2}})^2\right) \le O(\delta_v^2)$, so (6.31) implies,

$$\sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda_i+v_j)^2}{2}} + O(\delta_v^2),$$

so

$$\begin{aligned}\delta_v \sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} M_j =& Q^\top \mathrm{diag}\left(\delta_v \sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda_1+v_j)^2}{2}}, \cdots, \delta_v \sum_{j=-\frac{N_v}{2}}^{j=\frac{N_v}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\lambda_n+v_j)^2}{2}}\right)Q \\ =& Q^\top(1+O(\delta_v^2))IQ = \left(1+O(\delta_v^2)\right)I. \end{aligned} \tag{6.32}$$

(6.33)

Therefore,

$$(\delta_v \sum_j M_j)^{-1} = \frac{1}{1+O(\delta_v^2)} I = \left(1+O(\delta_v^2)\right) I. \tag{6.34}$$

So by (6.29) and (6.34), one gets $\hat{\boldsymbol{c}}_i^n = \hat{\boldsymbol{\rho}}_i^n + O(\delta_v^2)$. ■

Theorem 6.5. *The first order scheme defined as (6.10) - (6.12) is s-AP. That is, when $\epsilon \to 0$, the limit of the first order scheme coincides with the gPC-SG discretization of high field limit (2.14).*

Proof. From Lemma 6.4 and 6.3, as $\epsilon \to 0$,

$$\hat{\boldsymbol{f}}_{ij}^n \to M_{ij}^n \left(\hat{\boldsymbol{\rho}}_i^n + O(\delta_v^2)\right). \tag{6.35}$$

Thus,

$$\begin{aligned} F^+ &= \int_{\mathbb{R}} \frac{v+|v|}{2} M(v)dv = \int_0^\infty vM(v)dv = \int_0^\infty \frac{1}{\sqrt{2\pi}}(vI_K+F)e^A dv - \int_0^\infty FM(v)dv \\ &= \int_{-\infty}^{-\frac{|F|^2}{2}} \frac{1}{\sqrt{2\pi}} e^A dA - F\int_F^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{|P|^2}{2}} dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{|F|^2}{2}} - F\mathrm{erf}(F), \end{aligned} \tag{6.36}$$

where $\mathrm{erf}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{|t|^2}{2}} dt$, F, P, A is defined in (6.5).
Similarly,

$$F^- = \int_{\mathbb{R}} \frac{v-|v|}{2} M(v)dv = \int_{-\infty}^0 vM(v)dv = -\frac{1}{\sqrt{2\pi}} e^{-\frac{|F|^2}{2}} - F\mathrm{erf}(-F). \tag{6.37}$$

Then $F_{i+\frac{1}{2}}^n$ defined in (6.14) becomes

$$F_{i+\frac{1}{2}}^n = (F^+\hat{\boldsymbol{\rho}})_i^n + (F^-\hat{\boldsymbol{\rho}})_{i+1}^n + O(\delta_v^2), \tag{6.38}$$

which is exactly the numerical flux of the kinetic scheme for (6.3) by ([7], ch3). So as $\epsilon \to 0$ (6.14) becomes the forward Euler in time and kinetic scheme in space for the resulting system of the high field limit equation with uncertainty (2.14), which completes the proof for s-AP property. ■

6.4 A second order scheme

Using backward difference formula for time discretization [9], and MUSCL scheme for space discretization, the second order scheme is given by

$$\frac{3\hat{\boldsymbol{f}}_{ij}^{n+1} - 4\hat{\boldsymbol{f}}_{ij}^n + \hat{\boldsymbol{f}}_{ij}^{n-1}}{2\delta_t} + 2v\partial_x \hat{\boldsymbol{f}}_{ij}^n - v\partial_x \hat{\boldsymbol{f}}_{ij}^{n-1} = \frac{1}{\epsilon} P(\hat{\boldsymbol{f}}_{ij}^{n+1}), \tag{6.39}$$

$$-\triangle_x \hat{\boldsymbol{\phi}}_{ij}^{n+1} = \hat{\boldsymbol{\rho}}_i^{n+1} - \hat{\boldsymbol{h}}_i^{n+1} \text{ (by Poisson Solver).} \tag{6.40}$$

Here,

$$v_j \partial_x \hat{\boldsymbol{f}}_{ij} = v_j \frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2},j} - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j}}{\delta_x}, \quad \text{and} \quad \begin{cases} \hat{\boldsymbol{f}}_{i+\frac{1}{2},j} = \hat{\boldsymbol{f}}_{i,j} + \frac{1}{2}\psi(\theta_{i+\frac{1}{2}}^+)(\hat{\boldsymbol{f}}_{i+1} - \hat{\boldsymbol{f}}_i) & v_j > 0, \\ \hat{\boldsymbol{f}}_{i+\frac{1}{2},j} = \hat{\boldsymbol{f}}_{i+1,j} - \frac{1}{2}\psi(\theta_{i+\frac{1}{2}}^+)(\hat{\boldsymbol{f}}_{i+1} - \hat{\boldsymbol{f}}_i) & v_j < 0. \end{cases} \tag{6.41}$$

Where $\theta^+_{i+\frac{1}{2}} = \frac{\hat{\boldsymbol{f}}_i - \hat{\boldsymbol{f}}_{i-1}}{\hat{\boldsymbol{f}}_{i-1} - \hat{\boldsymbol{f}}_i}$ and $\theta^-_{i+\frac{1}{2}} = \frac{\hat{\boldsymbol{f}}_{i+2} - \hat{\boldsymbol{f}}_{i+1}}{\hat{\boldsymbol{f}}_{i+1} - \hat{\boldsymbol{f}}_i}$ are smooth indicators, $\psi = \max(0, \min(1, \theta))$ is the slope limiter function [19].

The AP property can be similarly established as the first order scheme, so we omit the details here.

7 Numerical Examples

We solve the one-dimensional VPFP system with uncertainty,

$$\begin{cases} \partial_t f + v\partial_x f - \dfrac{1}{\epsilon}\partial_x\phi\partial_v f = \dfrac{1}{\epsilon}\partial_v[vf + \partial_v f], \\ -(1+\lambda_2 z_2)\partial_{xx}\phi = \rho - h, \quad x \in [x_0, x_I], v \in \mathbb{R}, \end{cases} \tag{7.1}$$

with periodic function $\phi(t, x, z_1)$ satisfying,

$$\phi(t, x_0, z) = \phi(t, x_I, z) = 0. \tag{7.2}$$

and only in Section 7.3.2, $\lambda_2 \neq 0$. Initial conditions are given by,

$$\rho_0 = \rho_0(x, \lambda_1 z_1), \qquad f_0 = f_0(x, v, \lambda_1 z_1), \tag{7.3}$$

and the given positive charged background $h(x, \boldsymbol{z})$ satisfies the global neutrality relation.

Here $\boldsymbol{z} = (z_1, z_2)$ are two independent random variables following the uniform distribution $U[a, b]$.

Given the gPC coefficients $\hat{f}_m$, $(m = 0, 1, \cdots, K)$ of the numerical approximation $\hat{f}^K$, the statistical quantities such as expectation, standard deviation are retrieved as,

$$\mathbb{E}[\hat{f}^K] = \hat{f}_0, \quad \mathbb{S}[\hat{f}^K] = \sqrt{\sum_{m=1}^{K} \hat{f}_m^2}. \tag{7.4}$$

7.1 The Order of Convergence

This section is devoted to check the spectral convergence. The initial data is given by an C^∞ function in $\boldsymbol{z} \sim U[0, 1]$, and periodic in x:

$$\rho_0(x, z) = 2 + \sin(x)e^z, \quad f_0 = \frac{\rho_0(x, z)}{\sqrt{2\pi}} e^{-\frac{|v + \partial_x\phi(x, \boldsymbol{z})|^2}{2}}, \quad x \in (0, 2\pi). \tag{7.5}$$

In order to satisfy the global neutrality relation for the background charge h, i.e., equation (2.3), we set,

$$h_0(x) = 2 + \sin(x)z, \quad \text{periodic in } x \in (0, 2\pi) \tag{7.6}$$

Define the l_1-error for the expectation and standard deviation of the approximation solution $\hat{f}^K$,

$$\text{error}_{\mathbb{E}} = \delta_x\delta_v \sum_{i,j} |\mathbb{E}f_{ij} - \mathbb{E}\hat{f}^K_{ij}|, \quad \text{error}_{\mathbb{S}} = \delta_x\delta_v \sum_{i,j} |\mathbb{S}f_{ij} - \mathbb{S}\hat{f}^K_{ij}|, \tag{7.7}$$

where f, the reference solution, is calculated by the Stochastic Collocation method [22] with 20 Legendre quadrature points and mesh size $\delta_x = \frac{2\pi}{1000}$, $\delta_t = \frac{\delta_x}{15}$, $\delta_v = \frac{12}{400}$, while $\hat{f}^K$ is the numerical solution by the K-th order gPC-SG and the same mesh size as the reference solution.

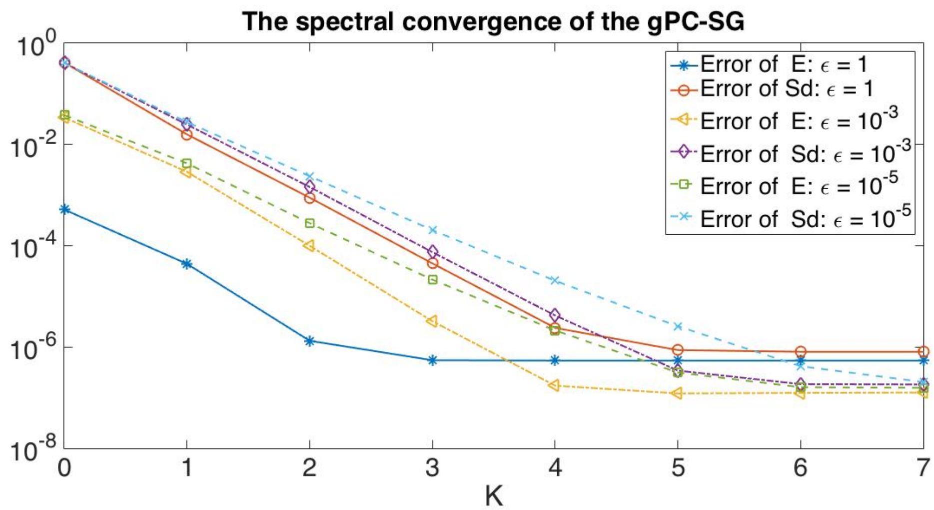

Figure 1: Example 7.1: Error of the numerical solution at $T = 0.01$ defined in (7.7) when $\epsilon = 1$, 10^{-3}, 10^{-5}. We take $\delta_x = \frac{2\pi}{1000}$, $v \in [-6, 6]$, $\delta_v = \frac{12}{1000}$, $\delta_t = \frac{\delta_x}{15}$, $0 \le K \le 8$.

Figure 1 is the l_1-error in terms of gPC order K for $\epsilon = 1$, 10^{-3}, 10^{-5} respectively with fixed δ_x, δ_v and δt. It shows exponential decay in K, until the errors due to spatial, temporal and velocity discretizations dominate. Furthermore, the amplitudes of the errors increase as ϵ decreases, but are within the estimated numerical approximation errors.

7.2 The asymptotic preserving property

This section is devoted to check the asymptotic preserving property of the scheme. We take the equilibrium initial data, and non-equilibrium initial data respectively.
The certain part of the initial data in this example is same as section 3.2 in [15].

$$\rho_0(x, v, \boldsymbol{z}) = \frac{\sqrt{2\pi}}{2}(2 + \cos(2\pi x)) + \lambda_1 z_1, \quad h(x, \boldsymbol{z}) = \frac{5.0132}{1.2661} e^{cos(2\pi x)} + 0.1 z_1, \quad x \in [0, 1]. \tag{7.8}$$

For equilibrium initial condition, f_0 is given by,

$$f_0(x, v, \boldsymbol{z}) = \frac{\rho_0(x, \boldsymbol{z})}{\sqrt{2\pi}} e^{-\frac{|v + \partial_x \phi|^2}{2}}, \quad \text{periodic in } x \in [0, 1], \tag{7.9}$$

while for the non-equilibrium initial data, f_0 is given by,

$$f_0(x, v, \boldsymbol{z}) = \frac{\rho_0(x, \boldsymbol{z})}{2\sqrt{2\pi}} \left(e^{-\frac{|v+1.5|^2}{2}} + e^{-\frac{|v-1.5|^2}{2}} \right), \quad \text{periodic in } x \in [0, 1]. \tag{7.10}$$

We study the evolution of the difference between f and equilibrium $M_{eq} = \frac{\rho}{\sqrt{(2\pi)}} e^{-\frac{|v + \partial_x \phi|^2}{2}}$, with respect to different ϵ as shown in Figure 2. Here the difference is defined as,

$$\text{difference} = ||\mathbb{E} f - \mathbb{E} M_{eq}||_1 = \delta_x \delta_v \sum_{i,j} |\mathbb{E} f_{ij} - \mathbb{E}(M_{eq})_{ij}| \tag{7.11}$$

Figure 2 shows the time evolution of the difference defined in (7.11) with different ϵ. One can see no matter whether the initial data is equilibrium or non-equilibrium, the s-AP method will push f towards the local Maxwellian quickly, and this is how [5] defined strong AP property.

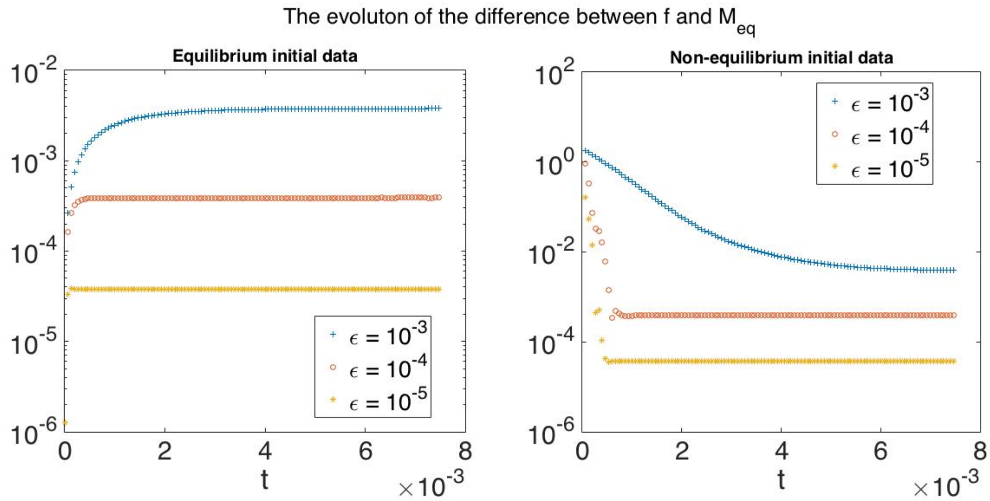

Figure 2: Example 7.2: The l_1-norm of $\mathbb{E}(f - M_{eq})$. We take $x \in (0,1), Nx = 1000, v \in [-6,6], Nv = 400, t \in [0, 0.01]$, $\delta_t = \delta_x/15$ and $\epsilon = 10^{-3}, 10^{-4}, 10^{-5}, K = 4$. The left figure: second order scheme with equilibrium initial data defined as (7.9); The right figure: second order scheme with non-equilibrium initial data defined as (7.10).

7.3 Statistical Quantities

In this section, we will see the expectation and standard deviation of $\rho(t, x, \boldsymbol{z})$, $E(t, x, \boldsymbol{z})$, $j(t, x, \boldsymbol{z})$ for different cases.

7.3.1 Mixing regimes

In the first case, we compare the second order gPC-SG method with the reference solution (Calculated with 20 Legendre quadrature points and mesh size $\delta_x = 1/1000$, $\delta_t = \frac{\delta_x}{15}$, $\delta_v = \frac{12}{400}$). The mixing regime is defined as following,

$$\epsilon(x) = \begin{cases} 10^{-3} + \dfrac{1}{2}\left(\tanh(5 - 10x) + \tanh(5 + 10x)\right), & x \le 0.3, \\ 10^{-3}, & x > 0.3. \end{cases} \tag{7.12}$$

So it contains both the kinetic and high field regimes. See Figure 3

The initial condition is given by,

$$\rho_0 = \frac{\sqrt{2\pi}}{6}(2 + \sin(\pi x)) + 0.1 z_1, \quad f_0 = \frac{\rho_0(x, \boldsymbol{z})}{\sqrt{2\pi}} e^{-\frac{|v + \partial_x \phi(x, \boldsymbol{z})|^2}{2}}, \quad \text{periodic in } x \in (-1, 1). \tag{7.13}$$

with

$$h_0 = \frac{1.6711}{2.5322} e^{\cos(\pi x)} + 0.1 z_1. \tag{7.14}$$

Where the certain part of the initial data is given in [15] Section 3.3. The time evolution of the expectation and standard deviation for ρ, j, E at $T = 0.1, 0.2, 0.3$ are shown in Figure 4.

Figure 4 shows the expectation and deviation of ρ, j and ϕ at time $T = 0.1, 0.2, 0.3$. One can see the statistic quantities of gPC-SG matches well with the reference solution.

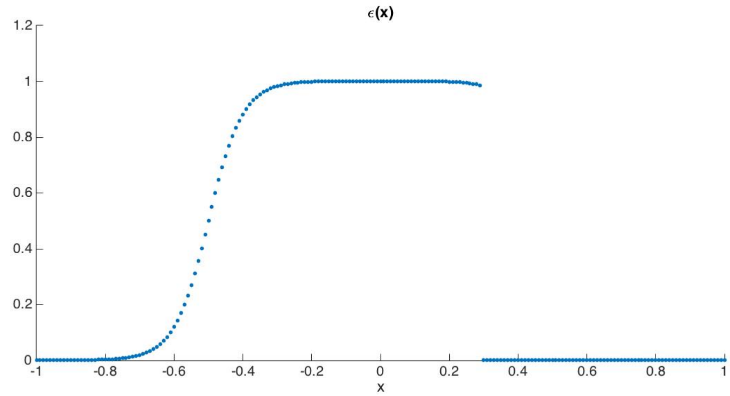

Figure 3: $\epsilon(x)$ given in (7.12)

7.3.2 Piecewise Constant Initial Data

In the second case, we test the second order scheme with periodic piecewise constant initial data defined as following, where the certain part is same as [15] Section 3.4.

$$\begin{cases} (\rho_0, h_0) = (\frac{1}{8}, \frac{1}{4}) + \lambda_1 z_1, & 0 \leq x < \frac{1}{4}, \\ (\rho_0, h_0) = (\frac{1}{2}, \frac{1}{8}) + \lambda_1 z_1, & \frac{1}{4} \leq x < \frac{3}{4}, \\ (\rho_0, h_0) = (\frac{1}{8}, \frac{1}{2}) + \lambda_1 z_1, & \frac{3}{4} \leq x < 1, \end{cases} \quad f_0 = \frac{\rho_0(x, \boldsymbol{z})}{\sqrt{2\pi}} e^{-\frac{|v+\phi_x(x,\boldsymbol{z})|^2}{2}}, \quad \epsilon = 10^{-3}. \tag{7.15}$$

In order to test how the random variables affect the final result, we compare two cases,

1. $\lambda_2 = 0, \quad \lambda_1 = 0.1;$ v.s. $\lambda_2 = 0, \quad \lambda_1 = 0.2.$
2. $\lambda_2 = 0, \quad \lambda_1 = 0.1;$ v.s. $\lambda_2 = 0.2, \quad \lambda_1 = 0.1.$

Figure 5 shows the comparison of the first case at $T = 0.2$. As the coefficient of z_1 getting bigger, the expectation remains the same, while the standard deviation becomes bigger and it increases in the same order as the coefficient.

Figure 6 shows the comparison of the second case at $T = 0.2$. One can tell that the randomness in the poisson equation doesn't have a significant effect on density, while it does affect the electric field.

A Appendices

A.1 The proof of Lemma 3.2

Proof. 1. The conclusion holds for $l = m - 1$, since from the last line of (3.10),

$$0 < A^m_{m-1} = \frac{C_\phi^2}{2} m^2 \leq b \left(\frac{m!}{(m-1)!} \right)^2. \tag{A.1}$$

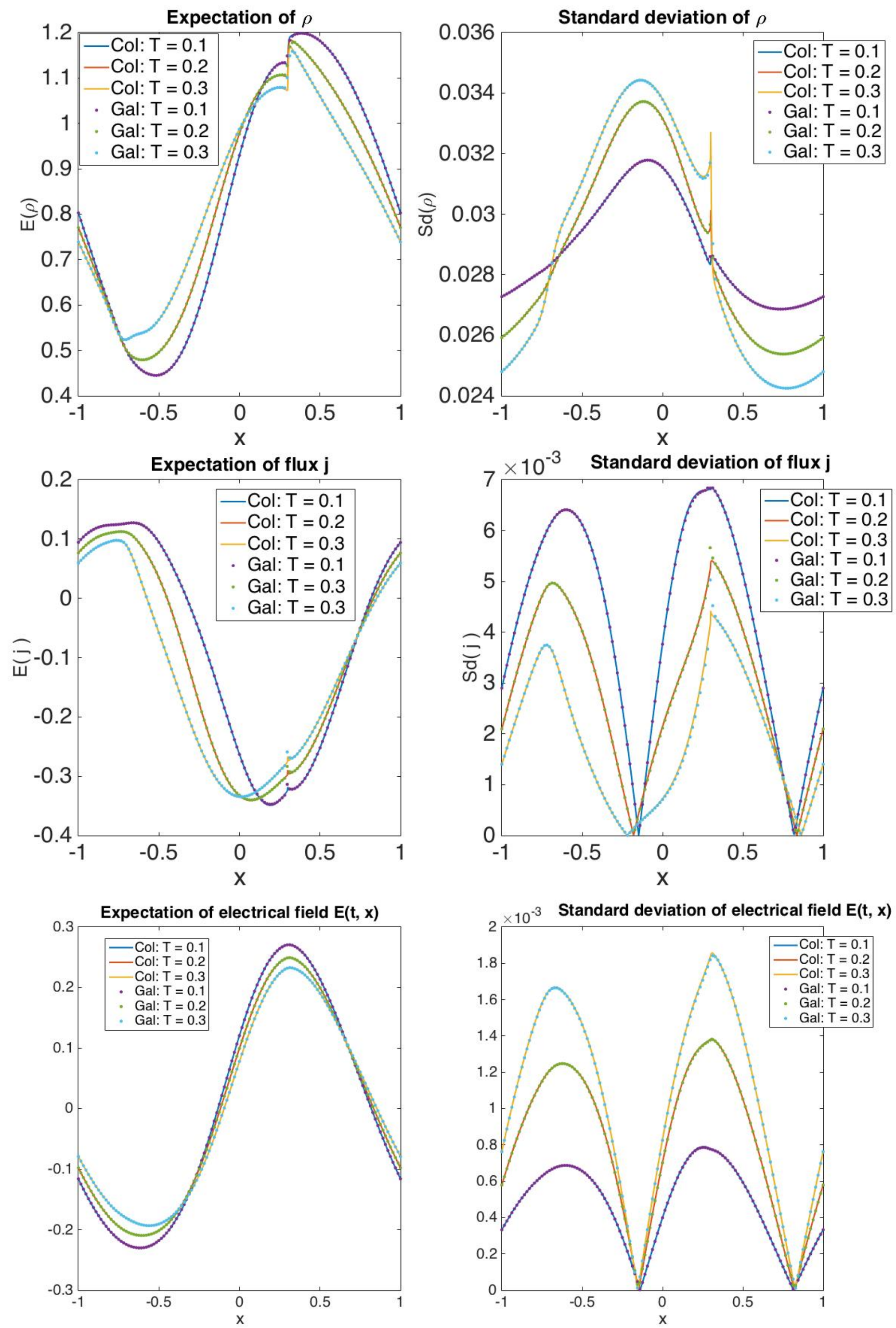

Figure 4: Example 7.3.1. The dot lines represent for the result obtained by gPC-SG: $N_x = 128$, $v \in [-6, 6]$, $N_v = 64$, and $\delta_t = \frac{\delta_x}{15}$, $K = 5$. The solid lines are reference solution with $Nx = 1000$, $N_v = 400$, $\delta_t = \frac{\delta_x}{15}$ and 20 Gaussian quadrature points.

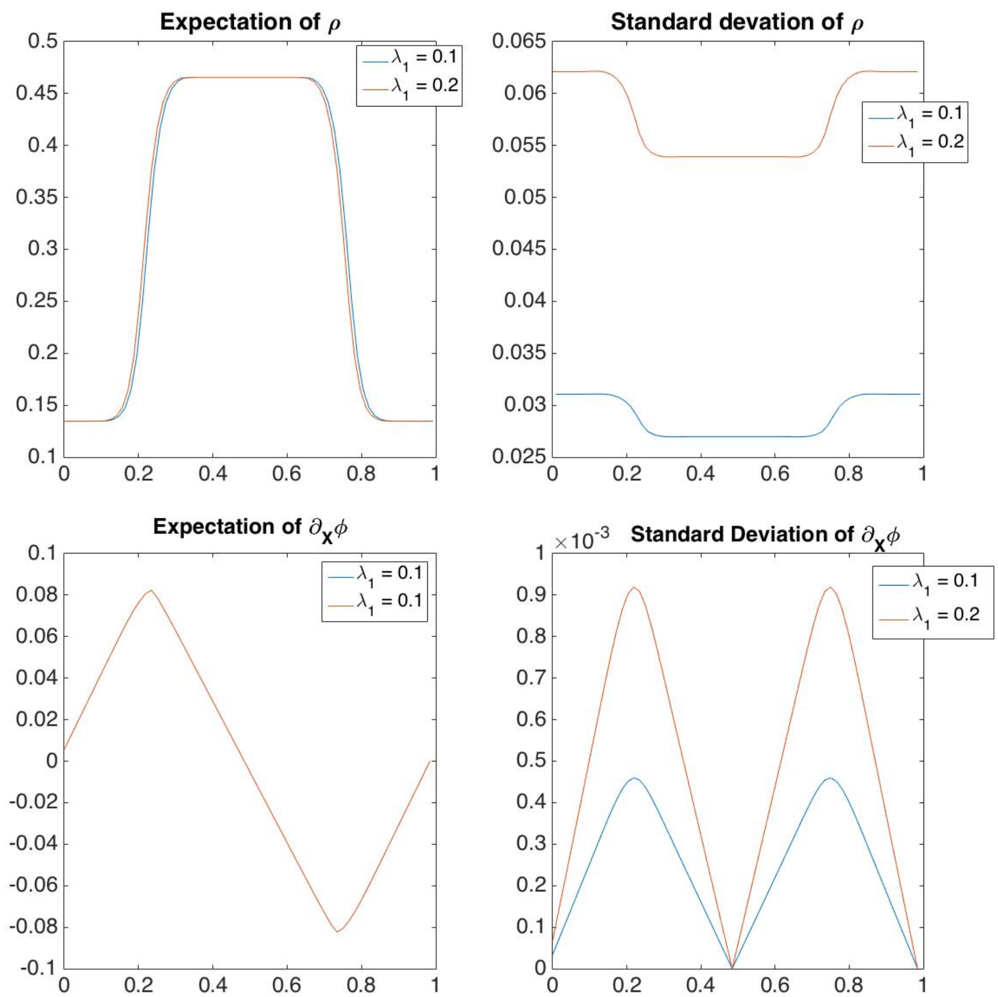

Figure 5: Example 7.3.2: The dash line is the expectation of two cases, while the solid line is obtained by $\mathbb{E} \pm \mathrm{Sd}$. $N_x = 64$, $v \in [-6, 6]$, $N_v = 100$, and $\delta_t = \frac{\delta_x}{15}$, $K = 5$.

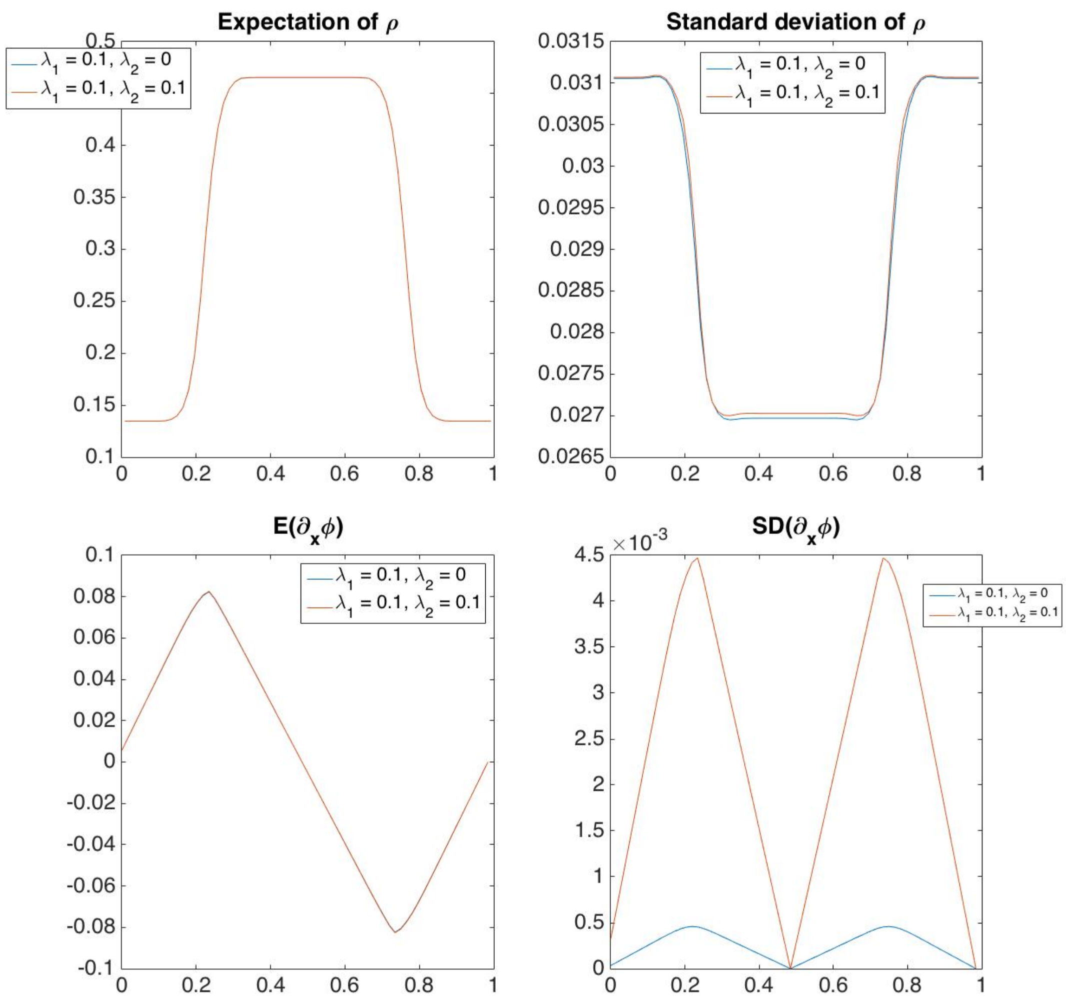

Figure 6: Example 7.3.2: The dash line is the expectation of the two cases, while the solid line is obtained by $\mathbb{E} \pm \mathrm{Sd}$. $N_x = 100$, $v \in [-6, 6]$, $N_v = 100$, and $\delta_t = \frac{\delta_x}{15}$, $K = 5$.

2. Assume the conclusion holds for $l = k+1, \cdots, m-1$, then one has,

$$\begin{aligned} 0 < A_k^m &= \frac{C_\phi^2}{2}\left(\sum_{i=k+1}^{m-1} \binom{i}{k}^2 A_i^m + \binom{m}{k}^2\right) \le \frac{b}{2}\left(\sum_{i=k+1}^{m-1} b^{m-i}\left(\frac{i!}{k!(i-k)!}\right)^2\left(\frac{m!}{i!}\right)^2 + \left(\frac{m!}{k!(m-k)!}\right)^2\right) \\ &= \frac{1}{2}\left(\frac{m!}{k!}\right)^2 \sum_{i=k+1}^{m} b^{m+1-i}\left(\frac{1}{(i-k)!}\right)^2 = b^{m-k}\left(\frac{m!}{k!}\right)^2 \sum_{i=1}^{m-k} \frac{b^{1-i}}{2}\left(\frac{1}{i!}\right)^2 \\ &\le b^{m-k}\left(\frac{m!}{k!}\right)^2 \sum_{i=0}^{\infty} \frac{1}{2}\left(\frac{1}{b^i 4^i}\right) \le b^{m-k}\left(\frac{m!}{k!}\right)^2. \end{aligned} \tag{A.2}$$

■

A.2 The proof of Proposition 6.1

Proof. To prove (a), By the definition of $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$, one has,

$$\partial_v M = \sum_{n=1}^{\infty} \frac{1}{n!} \partial_v (A^n). \tag{A.3}$$

One notes $\partial_v A = -P$, which implies, $(\partial_v A)\, A = A\, (\partial_v A)$. Therefore,

$$\partial_v (A^n) = \sum_{i=1}^{n} A^{n-i} (\partial_v A) A^{i-1} = (\partial_v A) \sum_{i=1}^{n} A^{n-i} A^{i-1} = n(\partial_v A) A^{n-1}. \tag{A.4}$$

Thus,

$$\partial_v M = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (-P) A^{n-1} = -PM. \tag{A.5}$$

To prove (b), as long as matrices A and B are commutative, then $e^A e^B = e^{A+B}$. Since $e^A e^{-A} = e^0 = I$, the inverse of M exists and is

$$M^{-1} = \exp(-A). \tag{A.6}$$

To prove (c), since P is a symmetric matrix, there exists a unity matrix Q and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_K)$, such that $P = Q^\top \Lambda Q$, so $|P|^2 = Q^\top \Lambda^2 Q$. Since

$$M = e^{-\frac{|P|^2}{2}} = Q^\top e^{-\frac{\Lambda^2}{2}} Q,$$

the eigenvalues of M are $e^{-\frac{\lambda_m^2}{2}} > 0$, $m = 1, \ldots, M$. The proof for M^{-1} is similar.

To prove (d), let $P_1 = \sum_{k=0}^{K} \partial_x \phi_k E_k + v_1 I_K$, $P_2 = \sum_{k=0}^{K} \partial_x \phi_k E_k + v_2 I_K$, then it is easy to check $P_1 P_2 = P_2 P_1$, hence $|P_1|^2 |P_2|^2 = |P_2|^2 |P_1|^2$, which means $\frac{|P_1|^2}{2}$ and $\frac{|P_1|^2}{2}$ are commutative. Thus

$$M(v_1) M(v_2) = e^{-\frac{|P_1|^2}{2} - \frac{|P_2|^2}{2}}. \tag{A.7}$$

is symmetric. Since if the matrices A, B are positive definite and AB is symmetric, then AB is still positive definite. Therefore, we conclude $M(v_1)M(v_2)$ is still positive definite.

The commutativity can be easily obtained from (A.7).

To prove (e), since F is a symmetric matrix, there exists a unity matrix Q and a diagonal matrix Λ such that $F = Q^\top \Lambda Q$, so one can represent $|P|^2 = Q^\top(\Lambda^2 + v^2 I + 2v\Lambda)Q$. Thus,

$$\int M dv = Q^\top \left(\int \exp(-\frac{\Lambda^2 + v^2 I + 2v\Lambda}{2}) dv \right) Q = Q^\top \left(\sqrt{2\pi} I \right) Q = \sqrt{2\pi} I. \tag{A.8}$$

Similarly, we can derive,

$$\int_{\mathbb{R}} \frac{v}{\sqrt{2\pi}} M dv = F. \tag{A.9}$$

To prove (f),

$$\begin{aligned} MPM^{-1} &= (Q^\top e^{-\frac{1}{2}\Lambda^2} Q) Q^\top \Lambda Q (Q^\top e^{\frac{1}{2}\Lambda^2} Q) = Q^\top e^{-\frac{1}{2}\Lambda^2} \Lambda e^{\frac{1}{2}\Lambda^2} Q \\ &= Q^\top \Lambda e^{-\frac{1}{2}\Lambda^2 + \frac{1}{2}\Lambda^2} Q = Q^\top \Lambda Q = P. \end{aligned} \tag{A.10}$$

■

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