Limits of stochastic binary interactions on a dense graph

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Introduction

Homogeneous Boltzmann-like equations are used to model the evolution of wealth, opinions or information among a population of interacting agents.

The agent dynamics are given by an N-dimensional pure jump Markov Process:

- Interaction times: Poisson arrival times.
- Interaction network: complete graph
- Pairwise interactions $(v, v^*) \to (F_1(v, v^*), F_2(v, v^*))$

In the infinite agent limit:

$$\frac{d}{dt}\int f(v)\phi(v)\,dv = \lambda \int f(v)f(u)\left[\phi(v') + \phi(u') - \phi(v) - \phi(u)\right]\,du\,dv$$

This does not take into account a possible underlying network structure constraining agent interactions.

Goal:
To derive a kinetic equation for stochastic binary interactions on an arbitrary graph.
Remark: We have restricted ourselves to dense graphs, for which:

- The interaction frequency between every pair of agents tends to 0, so propagation of chaos is possible
- The theory of *graphons* may be employed

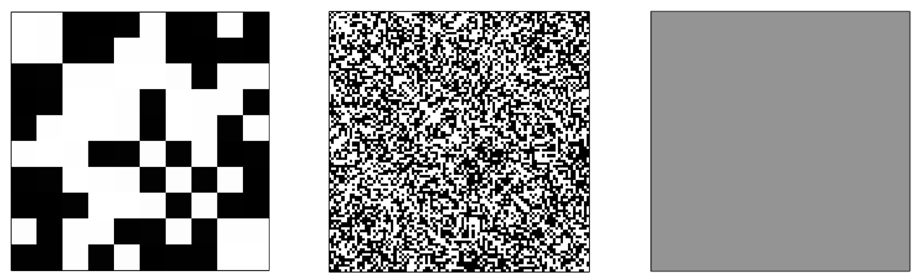
Graph Limits

Graphon: symmetric measurable function

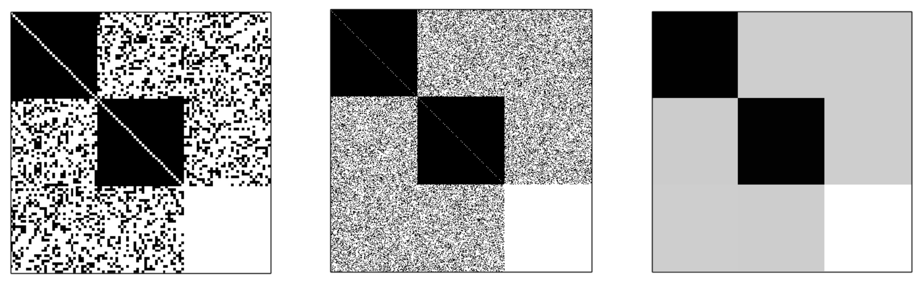
$$U : [0, 1]^2 \mapsto [0, 1]$$

Informally, U is limit of the adjacency matrices of a growing graph sequence [7]

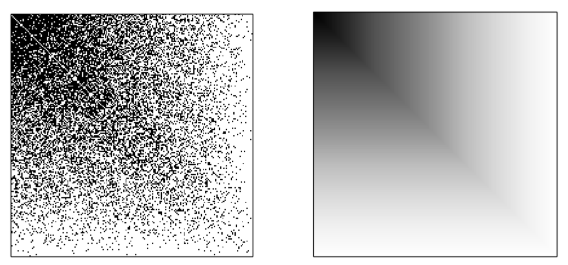
Example: Erdos-Renyi graph $G(n, p)$. The edge density is constant



Example: Stochastic blockmodel: An edge between nodes in blocks i,j is placed with probability $u_{ij}[0, 1]$.



Example: Randomly grown uniform attachment graph. At the nth iteration add a node and connect every pair of nodes with probability 1/n. Graphon: $U(w_1, w_2) = 1 - \max(w_1, w_2)$



Kinetic Equation for Graphons

Associate each agent with a node i, and map

$$i \mapsto 1/n := w_i.$$

Density: Let $f(x, w, t)$ be the joint pdf of states x_i and "network location" $w_i \in [0, 1]$.
Remark: agents' network locations are uniformly distributed so $\int f(x, w)dx = 1$.

Dynamics: when an agent activates, it samples an agent from its neighbors uniformly at random, and both agents update their states. The interaction frequency between agents in different network locations is given by:

$$K(w_1, w_2) := \frac{U(w_1, w_2)}{\int_0^1 U(w_1, w_2)\,dw_2}$$

For (possibly random) interactions $(x_i, x_j) \to (x_i', x_j')$, the weak form of the kinetic equation is:

$$\frac{d}{dt}\int f(x, w)\phi(x)\,dx = \frac{\lambda}{2}\Big\langle \int_0^1 K(w, w_2) \int_{\mathbb{R}^2} f(x, w)f(y, w_2)[\phi(x') - \phi(x)]\,dy\,dx\,dw_2 + \int_0^1 K(w_2, w) \int_{\mathbb{R}^2} f(x, w_2)f(y, w)[\phi(y') - \phi(y)]\,dy\,dx\,dw_2 \Big\rangle$$

where the expectation is with respect to the law of (x_i', x_j').

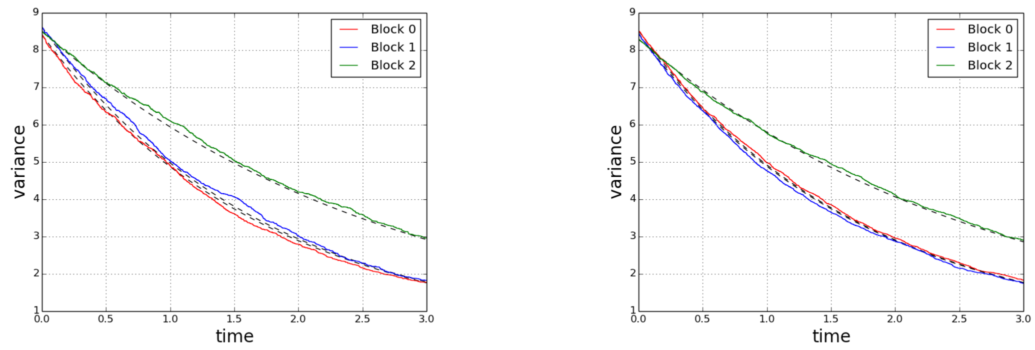


Figure 1: Evolution of variances for stochastic blockmodel and interaction $(x', y') = (0.3x + 0.5y, 0.2x + 0.7y)$, on networks with 6000 and 12000 nodes

Extension to Directed Graphs

The limit object for directed graphs [4] consists of four functions:

$$W_{00}, W_{01}, W_{10}, W_{11} : [0, 1]^2 \mapsto [0, 1],$$

where $W_{ij}(x, y) = W_{ji}(y, x)$ and $\sum_{ij} W_{ij} = 1$.

$W_{01}(x, y)$ (resp. $W_{10}(x, y)$) is the density of outgoing (resp. incoming) edges from network location x to y (resp. y to x). Assume $W_{11}(x, y) = 0$ for simplicity and the interaction:

$$(x_i', x_j') = \begin{cases} (F_1(x_i, x_j), F_2(x_i, x_j)) & \text{if } (i, j) \in E, (j, i) \notin E \\ (G_1(x_i, x_j), G_2(x_i, x_j)) & \text{if } (j, i) \in E, (i, j) \notin E \end{cases}$$

for arbitrary functions F_1, F_2, G_1, G_2. The kinetic equation is:

$$\begin{aligned} \frac{d}{dt}\int f(x, w)\phi(x)\,dx = & \\ & \int_0^1 K_{01}(w, w_2) \int_{\mathbb{R}^2} f(x, w)f(y, w_2)\left[\phi(F_1(x, y)) - \phi(x)\right]\,dy\,dx\,dw_2 \\ + & \int_0^1 K_{10}(w, w_2) \int_{\mathbb{R}^2} f(x, w)f(y, w_2)\left[\phi(G_1(x, y)) - \phi(x)\right]\,dy\,dx\,dw_2 \\ + & \int_0^1 K_{10}(w_2, w) \int_{\mathbb{R}^2} f(x, w_2)f(y, w)\left[\phi(G_2(x, y)) - \phi(y)\right]\,dy\,dx\,dw_2 \\ + & \int_0^1 K_{01}(w_2, w) \int_{\mathbb{R}^2} f(x, w_2)f(y, w)\left[\phi(F_2(x, y)) - \phi(y)\right]\,dy\,dx\,dw_2 \end{aligned}$$

Arnab Chatterjee introduced a wealth exchange model [3] for which $W_{01}(w, w_2) = p\mathbb{1}_{w<w_2} + (1 - p)\mathbb{1}_{w>w_2}$, $W_{10}(w, w_2) = 1 - W_{01}(w, w_2)$

Applications

Wealth exchange models:
Agents meet to trade and one agent will gain or lose wealth. Various interaction rules have been proposed, including models with debt and taxation [2]. Examples:

- Slanina's "inelastic scattering":

$$\begin{pmatrix} x_i' \\ x_j' \end{pmatrix} = \begin{pmatrix} 1 - \beta + \nu & \beta \\ \beta & 1 - \beta + \nu \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

where β is the fraction of money exchanged and ν is external incoming wealth.
- Chakraborti and Chakrabarti's "random wealth exchange with saving" (CC):

$$\begin{pmatrix} x_i' \\ x_j' \end{pmatrix} = \begin{pmatrix} \lambda + \epsilon(1 - \lambda) & \epsilon(1 - \lambda) \\ (1 - \epsilon)(1 - \lambda) & \lambda + (1 - \epsilon)(1 - \lambda) \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

where $\epsilon \sim \mathcal{U}[0, 1]$. λ is the fraction of money saved by each agent

Opinion dynamics models:
Under the Deffuant-Weisbuch model [6], agents average their states when are close enough:

$$(x', y') = \begin{cases} (\frac{x+y}{2}, \frac{x+y}{2}) & \text{if } |x - y| < \epsilon \\ (x, y) & \text{otherwise} \end{cases}$$

Remark: also used for community detection on networks.

Over-the-Counter Markets (OTC) financial markets
Duffie et. al introduced a model for information percolation in OTC markets. Assuming perfect Bayesian decision-makers, one can map the dynamics to the interaction $(x, y) \mapsto (x + y, x + y)$. Incorporating behavioral biases, this is generally a linear interaction, for $a, b, c, d \in [0, 1]$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The Bitcoin Transaction Network

- A direct way to test agent-based wealth exchange models
- 60 million transactions over a 5 year period
- Frequently interacting people define a network
- Results in multi-species model with (generally) stochastic laws
- Example: i buys bitcoins at exchange rate r_t,

$$(x_i', y_i') = ((1 - \alpha)x_i, y_i + (r_t + \epsilon)\alpha x_i)$$
$$(x_j', y_j') = (x_j + \alpha x_i, y_j - (r_t + \epsilon)\alpha x_i)$$

where (x_i, y_i) are dollars and bitcoins, $\alpha \in [0, 1]$, and $RV.\epsilon$

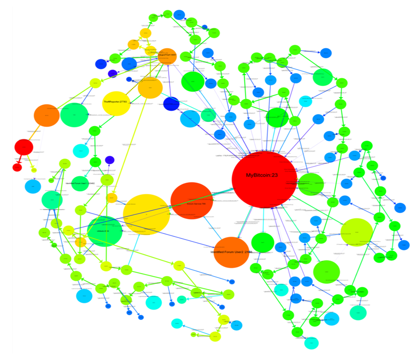


Figure: Fergal Reid and Martin Harrigan

Conclusions and Ongoing Work

We have derived and numerically verified a kinetic equation for binary stochastic interactions on dense graphs. Work in progress:

- Proving convergence of the particle system to the kinetic equation
- Existence, uniqueness and stability of self-similar solutions [1]
- Use the Bitcoin transaction network data to test wealth exchange models

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