Conservation laws and kinetic formulations rough fluxes and stochastic averaging lemma

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Introduction

We want to find $u(x,t)$ solution of

$$\begin{cases} du + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(u) \circ dW^i(t) = 0 & x \in \mathbb{R}^N,\ t \geq 0 \\ u(,t=0) = u^0(x) \end{cases}$$

$$\mathbf{A} = (A_1, ..., A_N) \in C^2(\mathbb{R}; \mathbb{R}^N), \qquad \text{(Flux)}$$

$$\mathbf{W} = (W^1, ..., W^N) \in C\Big([0,\infty); \mathbb{R}^N\Big),$$

two special cases being

$$\mathbf{W} = (B^1, ..., B^N) \qquad \text{(N-dimensional Brownian motion)}$$

$$\mathbf{W(t)} = (t, ..., t) \qquad \text{(Standard SCL, Kruzkov)}$$

Introduction

Theorem (Pathwise entropy solutions) There is a unique 'kinetic pathwise solution'

- for a given $\mathbf{W}$

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le \|u_2^0 - u_1^0\|_{L^1(\mathbb{R}^N)}.$$

- for two paths $\mathbf{W_i}$ and $u_i^0 \in BV(\mathbb{R}^N)$, then u_1 and u_2 satisfy

$$\begin{aligned}\|u_2(\cdot,t) - u_1(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le \; & \|u_2^0 - u_1^0\|_{L^1(\mathbb{R}^N)} \\ & + C|(\mathbf{W_1} - \mathbf{W_2})(t)| + C \sup_{s\in(0,t)} \left|(\mathbf{W_1} - \mathbf{W_2})(s)\right|\end{aligned}$$

Motivation

One motivation :

For $i = 1, ..., L$, the system of stochastic interacting agents

$$dX_t^i = \sigma\Big(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\Big) \circ d\mathbf{W}_t,$$

Uncertainty in drivers behaviour

Randomness in an oil well extension

Variability in nephrons arrangements

Our problem is the formal limit $L \to \infty$

Related works

Related works :

- Flandoli Stochastic perturbations

$$du + \text{div}(bu) + dB(t) \circ \nabla u = 0 \qquad \text{(Stratonovich)}$$

$$\Longleftrightarrow$$

$$du + \text{div}(bu) + dB(t).\nabla u = \Delta u \qquad \text{(Itō)}$$

Extensions to perturbations of Vlasov/Navier-Stokes style equations

- Feng & Nualart, Debussche, Vovelle, Hofmanova

$$du + \text{div} A(u) = F(u).dB(t)$$

Related works

- Lions & Souganidis : **Topological point of view**

$$du = F(D^2u; Du)dt + \sum_{i=1}^{m} H_i(Du) \circ dW_i$$

$$du = F(D^2u; Du)dt + \sum_{i=1}^{m} \Phi_i(u) \circ dW_i$$

Principles :

  - Pathwise
  - Use characteristics for short times (iterate-Trotter)

- Lyons, Fritz...
  - Rough paths.. $\frac{d}{dt}X(t) = \sigma(X(t))\dot{W}(t)$

Outlines

1. Hyperbolic equations and shocks
2. Difficulties related to $dW(t)$
3. How do we define a solution ?
4. Can one prove existence, uniqueness ?
5. The $x-$dependant case
6. Stochastic averaging lemmas

Hyperbolic equations and shocks

$$\begin{cases} \frac{\partial}{\partial t}\overline{u} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(\overline{u}) = 0 & x \in \mathbb{R}^N,\ t \geq 0 \\ \overline{u}(,t=0) = u^0(x) \end{cases}$$

- Generates shocks (discontinuities) : low regularity
- Entropy inequality selects a type of solution (unique)

$$\frac{\partial}{\partial t} S(\overline{u}) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \eta_i(\overline{u}) \leq 0$$

for all $S : \mathbb{R} \to \mathbb{R}$ convex. Example (Kruzkov)

$$S(u) = |u - k|, \qquad k \in \mathbb{R},$$

- Non reversible in time !

Hyperbolic equations and shocks

$$\begin{cases} \frac{\partial}{\partial t}\overline{u} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(\overline{u}) = 0 & x \in \mathbb{R}^N,\ t \geq 0 \\ \overline{u}(x, t = 0) = u^0(x) \end{cases}$$

- For $A(\cdot)$ convex (1 dimension)
  - Decreasing discontinuities are propagated as shocks
  - Increasing discontinuities are regularized
- We want to alternate $A(\cdot)$ convex and $A(\cdot)$ concave

Hyperbolic equations and shocks

$$\begin{cases} du + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(u) \circ dW^i(t) = 0 & x \in \mathbb{R}^N,\ t \geq 0 \\ u(,t=0) = u^0(x) \end{cases}$$

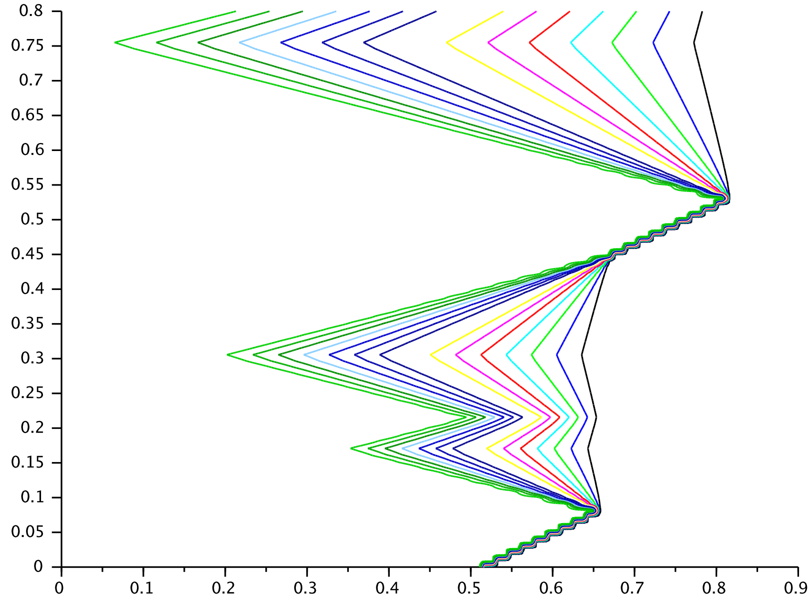
$$\frac{\partial}{\partial t} S(u) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \eta_i(u) \circ dW^i(t) \leq 0, \qquad \forall\ S(\cdot) \quad \text{convex.}$$

- Motivates the notation '$\circ$' as in Stratonowich form
- Irreversible in time. We cannot write in 1 dimension,

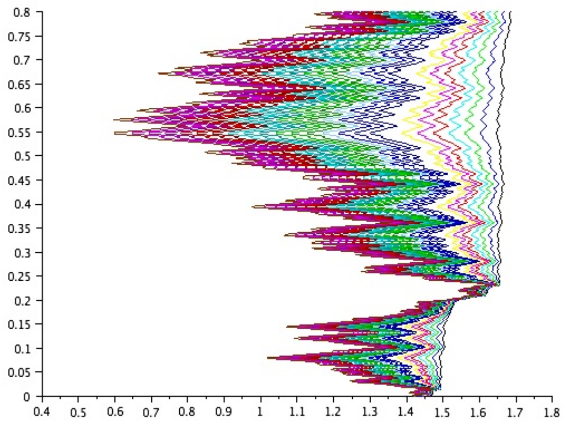
$$u(x,t) = \overline{u}(x, W(t))$$

$$d\overline{u}(x, W(t)) = -\frac{\partial}{\partial x}\ A(\overline{u}(x, W(t))) \circ dW^i(t)$$

Hyperbolic equations and shocks



- Usual method : BV estimates (might be correct in x, not in t)
- Compensated compactness (Murat-Tartar)
- Kinetic formulation (Lions, BP, Tadmor)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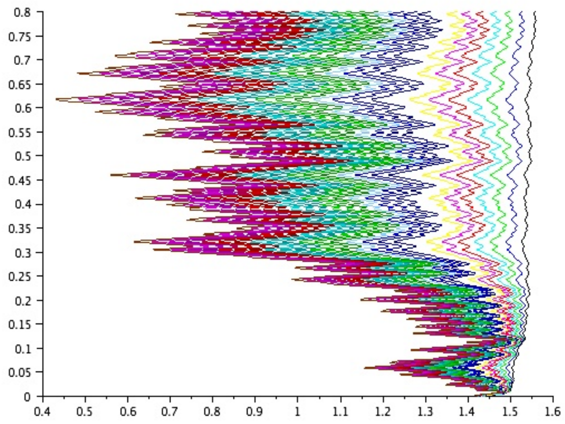
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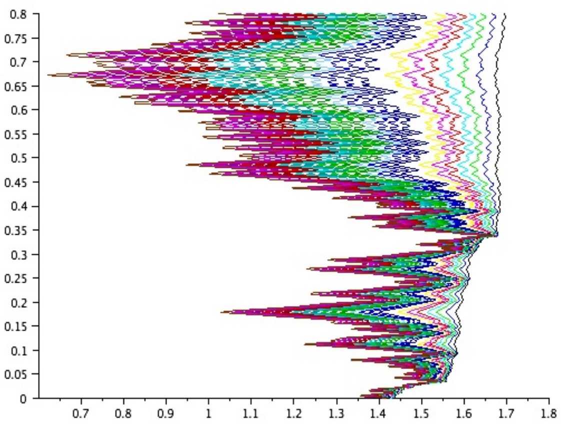
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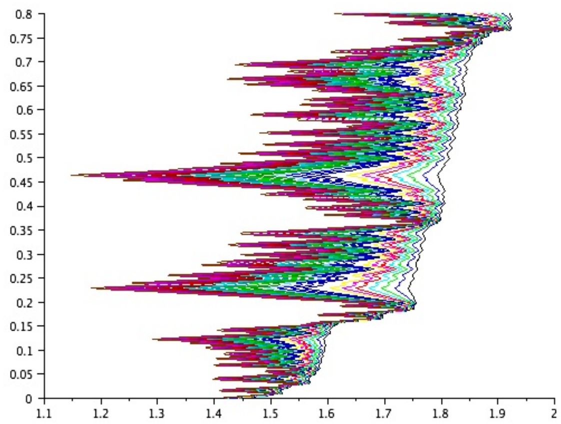
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What do we want ?

$$\begin{cases} du + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(u) \circ dW^i(t) = 0 & \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ u = u^0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}. \end{cases}$$

$$\mathbf{W} = (W^1, ..., W^N) \in C\Big([0,\infty); \mathbb{R}^N\Big), \qquad \mathbf{A} = (A_1, ..., A_N) \in C^2(\mathbb{R}; \mathbb{R}^N),$$

$$\mathbf{a(u)} = A' = (A_1'(u), ..., A_N'(u)), \qquad \text{(Velocity)}$$

- **Entropy dissipation :** For S convex

$$\begin{cases} dS(u) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \eta_i(u) \circ dW^i \leq 0, \\ \eta_i(u)' = a_i(u) S'(u) \qquad a_i = A_i' \end{cases}$$

(Stratonovich, no additional entropy dissipation)

What do we want ?

- If we use Itō formula we loose the entropy ! We take expectations

$$\frac{d}{dt}\mathbb{E}(u^2) = \mathbb{E}\Big(ua(u)^2(u_x)^2\Big)$$

No possible control of the RHS (shocks)

- For W continuous and $u(t) \in BV_x$, we cannot obtain BV in time

$$\frac{du}{dt} = \dot{W}(t)\,\frac{\partial}{\partial x}u(x,t)$$

No control.

- What does it mean to be a solution ?

How do we define a solution ?

As in Debussche & Vovelle, we use the kinetic formulation

$$\chi(x,\xi,t) = \chi(u(x,t),\xi) = \begin{cases} +1 & \text{if} \quad 0 \le \xi \le u(x,t), \\ -1 & \text{if} \quad u(x,t) \le \xi \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$S\big(u(x,t)\big) = \int_{\mathbb{R}} S'(\xi)\chi(x,\xi,t)d\xi$$

$$\begin{cases} d\chi + \sum_{i=1}^{N} a_i(\xi)\frac{\partial}{\partial x_i}\chi \circ dW^i = \frac{\partial}{\partial \xi} m dt \quad \text{in} \quad (x,\xi,t) \in \mathbb{R}^N \times \mathbb{R} \times (0,\infty), \\ m(x,\xi,t) \le 0 \end{cases}$$

Equivalent to the **Entropy dissipation**

How do we define a solution ?

We can define solutions along the characteristics

$$\dot{X}(t) = a(\xi)\dot{\mathbf{W}}, \qquad \dot{\xi} = 0,$$

$$\frac{d}{dt}\chi\Big(x - a(\xi)\mathbf{W}(t), \xi, t\Big) = \frac{\partial}{\partial \xi} m(x - a(\xi)\mathbf{W}(t), \xi, t)$$

These are globally defined (at variance with the case of H.-J. eq.)

- We show regularization/unique limit usi,g only on this formulation.
- The uniqueness proof based on kinetic formulation
- Continuity with respect to $\mathbf{W}$ in C^0

How do we define a solution ?

Definition. We 'regularize along the characteristics. Consider

$$\begin{cases} \rho^0 \in \mathcal{D}(\mathbb{R}^N) \quad \text{such that} \quad \rho^0 \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \rho^0(x)dx = 1, \\ \rho(y,x,\xi,t) = \rho^0\Big(y - x + \mathbf{a}(\xi)\mathbf{W}(t)\Big), \end{cases}$$

solves formally the linear transport equation

$$d\rho + \textstyle\sum_{i=1}^N a_i(\xi)\frac{\partial}{\partial x_i}\rho \circ dW^i = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R} \times (0,\infty),$$

and, hence,

$$d(\rho(y,x,\xi,t)\chi(x,\xi,t)) + \sum_{i=1}^N a_i(\xi)\frac{\partial}{\partial x_i}\rho\chi \circ dW^i = \rho(y,x,\xi,t)\frac{\partial}{\partial \xi}m(x,\xi,t)dt.$$

$$\frac{d}{dt}\int_{\mathbb{R}^N} \chi(x,\xi,t)\rho(y,x,\xi,t)dx = -\int_{\mathbb{R}^N} \frac{\partial}{\partial \xi}\rho(y,x,\xi,t)m(x,\xi,t)dx.$$

Can one prove existence, uniqueness ?

Theorem (Pathwise entropy solutions) There is a unique 'kinetic pathwise solution'

- for a given $\mathbf{W}$

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_2^0 - u_1^0\|_{L^1(\mathbb{R}^N)}.$$

- for two paths $\mathbf{W_i}$ and $u_i^0 \in BV(\mathbb{R}^N)$, then u_1 and u_2 satisfy

$$\begin{aligned}\|u_2(\cdot,t) - u_1(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq &\ \|u_2^0 - u_1^0\|_{L^1(\mathbb{R}^N)} \\ &+ |(\mathbf{W_1} - \mathbf{W_2})(t)|\ \|\mathbf{a}\|(|u_1^0|_{BV(\mathbb{R}^N)} + |u_2^0|_{BV(\mathbb{R}^N)}) \\ &+ \left(\sup_{s\in(0,t)} \left|(\mathbf{W_1} - \mathbf{W_2})(s)\right|\ \|\mathbf{a}'\| [\|u_1^0\|^2_{L^2(\mathbb{R}^N)} + \|u_2^0\|^2_{L^2(\mathbb{R}^N)}]\right)^{1/2}.\end{aligned}$$

Conclude...

Space dependent case

$$\begin{cases} du + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,u) \circ dW(t) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ u = u^0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}. \end{cases}$$

One $W(t)$ only !

Kinetic formulation A.-L. Dalibard

$$d\chi + \sum_{i=1}^{N} a_i(x,\xi) \frac{\partial}{\partial x_i} \chi \circ dW(t) - b(x,\xi) \frac{\partial}{\partial \xi} \chi \circ dW(t) = \frac{\partial}{\partial \xi} m dt$$

$$a_i(x,\xi) = \frac{\partial}{\partial x_i} A_i(x,\xi), \qquad b(x,\xi) = \sum_i \frac{\partial}{\partial \xi} A_i(x,\xi).$$

Space dependent case

We test against smooth ‘generalized convolution kernels’

$$d\rho + \sum_{i=1}^{N} a_i(x,\xi)\frac{\partial}{\partial x_i}\rho \circ dW(t) - b(x,\xi)\frac{\partial}{\partial \xi}\rho \circ dW(t) = 0.$$

And these are given by

$$\rho(x,\xi,t) = \widehat{\rho}(x,\xi,W(t)),$$

with

$$\frac{\partial}{\partial t}\widehat{\rho} + \sum_{i=1}^{N} a_i(x,\xi)\frac{\partial}{\partial x_i}\widehat{\rho} - b(x,\xi)\frac{\partial}{\partial \xi}\widehat{\rho} = 0.$$

Definition A stochastic kinetic solution is defined by

$$\frac{d}{dt}\int \rho(x,\xi,t)\,\chi(x,\xi,t)dxd\xi = -\int m(x,\xi,t)\frac{\partial}{\partial \xi}\rho(x,\xi,t).$$

Space dependent case

Theorem There is a unique stochastic kinetic solution and for a given $\mathbf{W}$

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq \|u_2^0 - u_1^0\|_{L^1(\mathbb{R}^N)}.$$

- Existence is through weak limits
- Continuous dependency on $W(t)$ is not proved
- Extension to multiple $W^i(t)$ by B. Guess. Characteristics

$$dX_i = a_i(x,\xi)dW^i(t), \qquad d\Xi(t) = -b(x,\xi)d\mathbf{W}(t)$$

Stochastic averaging lemmas

It is difficult to resist the idea to consider simply

$$\begin{cases} \frac{\partial}{\partial t} f(x,\xi,t) + \dot{B}(t) \circ \xi.\nabla_x f = g(x,\xi,t) \quad \text{in} \quad \mathbb{R}^{2d} \times (0,\infty), \\ f(0) = f^0 \quad \text{on} \quad \mathbb{R}^{2N}. \end{cases}$$

The notation for the flux means

$$\dot{B}(t) \circ \xi.\nabla_x f = \dot{B}(t) \sum_{i=1}^{N} \xi_i \frac{\partial f}{\partial x_i}.$$

And the Stratonovich solution

$$\frac{d}{dt} f\Big(x - B(t)\xi, \xi, t\Big) = g\Big(x - B(t)\xi, \xi, t\Big).$$

Stochastic averaging lemmas

$$\begin{cases} \frac{\partial}{\partial t} f(x,\xi,t) + \dot{B}(t) \circ \xi . \nabla_x f = g(x,\xi,t) \quad \text{in} \quad \mathbb{R}^{2N} \times (0,\infty), \\ f(0) = f^0 \quad \text{on} \quad \mathbb{R}^{2N}. \end{cases}$$

Kinetic averaging lemma aim to prove regularity for

$$\rho_\psi(x,t) = \int_{\mathbb{R}^N} \psi(\xi) f(x,\xi,t) d\xi$$

with ψ a smooth function with compact support.

Stochastic averaging lemmas

$$\begin{cases} \frac{\partial}{\partial t} f(x,\xi,t) + \xi.\nabla_x f = g(x,\xi,t) \quad \text{in} \quad \mathbb{R}^{2N} \times (0,\infty), \\ f(0) = f^0 \quad \text{on} \quad \mathbb{R}^{2N}. \end{cases}$$

Theorems (Deterministic averaging). Take $B(t) = t$.

For $g = 0$ and $\lambda \geq 0$

$$\|e^{-\lambda t} \rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq C(\psi)\ \|f^0\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$$

For $f^0 = 0$

$$\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq C\ \|g\|_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\ \|\ f\|_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}$$

For $f^0 = 0$ and $g = \text{div}_\xi\, h$, we have

$$\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/4}(\mathbb{R}^N)\right)} \leq C\ \|h\|^{1/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\ \|f\|^{3/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}$$

Stochastic averaging lemmas

Long story behind that : F. Golse, BP, R. Sentis (CRAS 1985), P.-L. Lions, Meyer, Gérard, Souganidis... Tadmor and Tao

The proof is based is inspired by the version in F. Bouchut and L. Desvillettes (no Fourier in time)

Stochastic averaging lemmas

Theorem (Comparison deterministic/stochastic).

1. For $g = 0$ and $\lambda \geq 0$ we have

$$\|e^{-\lambda t}\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq C(\psi)\ \|f^0\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}.$$

$$\mathbb{E}\|e^{-\lambda t}\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq \frac{C(supp\ \psi)}{\lambda^{1/2}}\|f^0\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}.$$

Stochastic averaging lemmas

Theorem (Comparison deterministic/stochastic).

2. For $f^0 = 0$ we have

$$\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq C\|g\|_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\|f\|_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}.$$

$$\mathbb{E}\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/2}(\mathbb{R}^N)\right)} \leq C\|g\|^{1/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\|f\|^{3/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}.$$

Stochastic averaging lemmas

Theorem (Comparison deterministic/stochastic).

3. For $f^0 = 0$ and $g = \mathrm{div}_\xi h$, we have

$$\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/4}(\mathbb{R}^N)\right)} \leq C\|h\|^{1/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\|f\|^{3/2}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}.$$

$$\mathbb{E}\|\rho_\psi\|^2_{L^2\left(R^+;\dot{H}^{1/3}(\mathbb{R}^N)\right)} \leq C\|h\|^{2/3}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}\|f\|^{4/3}_{L^2(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N)}.$$

Stochastic averaging lemmas

Idea of the proof.

$$\frac{\partial}{\partial t}\widehat{f}(k,\xi,t) + i\dot{B}(t)\circ k.\xi\widehat{f} = \widehat{g}.$$

$$\frac{\partial}{\partial t}\widehat{f}(k,\xi,t) + i\dot{B}(t)\circ k.\xi\widehat{f} + \lambda\widehat{f} = \widehat{g} + \lambda\widehat{f}.$$

$$\widehat{f}(k,\xi,t) = \widehat{f^0}(k,\xi)e^{-\lambda t - iB(t)k.\xi}$$

$$+ \int_0^t e^{-\lambda s}\Big[\widehat{g} + \lambda\widehat{f}](k,\xi,t-s)\Big]e^{ik.\xi\Big(B(t-s)-B(t)\Big)}ds$$

Stochastic averaging lemmas

$$|\widehat{\rho_\psi}(k,t)|^2 \leq 2\left|\int \psi \widehat{f^0}(k,\xi) e^{-\lambda t - iB(t)k.\xi} d\xi\right|^2$$

$$+2\left|\int_0^t \int e^{-\lambda s}\left[\psi \widehat{g} + \lambda \psi \widehat{f}\right](k,\xi,t-s) e^{ik.\xi\left(B(t-s)-B(t)\right)} ds d\xi\right|^2 .$$

For $g = 0$

$$\leq \mathbb{E} \int_{t=0}^{\infty} \int \psi \widehat{f^0}(k,\xi_1) \, \overline{\psi \widehat{f^0}(k,\xi_2)} e^{-2\lambda t - iB(t)k.(\xi_1 - \xi_2)} d\xi_2 d\xi_1 dt$$

Conclusion

In the non-degenerate case : $\xi \mapsto a(\xi)$ not locally contained in an hyperplane

we know regularizing effects based on the kinetic formulation.

For random conservation laws, they are certainly very different

HAPPY BIRTHDAY EITAN