Coarse-graining of collective dynamics models
A model for local body alignment

Pierre Degond

Imperial College London
pdegond@imperial.ac.uk (see http://sites.google.com/site/degond/)

Joint works with:
Amic Frouvelle (Dauphine), Sara Merino-Aceituno (Imperial),
Ariane Trescases (Cambridge)
Summary

1. Individual-based model
2. Mean-Field model
3. Self-Organized Quaternionic Hydrodynamics (SOHQ)
4. Comparison with SOH dynamics for Vicsek
5. Conclusion
1. Individual-Based Model
Collective dynamics & self-organization

Link micro to macro scales
- Lack of conservations
- Breakdown of chaos property

Phase transitions
- Symmetry-breaking
- Jamming
- Continuum to network

Micro-scale
- Individual agents obey simple rules
- No leader

Macro scale
- Emergence of large-scale coherent structures
- Not directly encoded in agent’s behavior

Continuum to network
Description of the system to model

Self-propelled agents which align with their neighbors
   Case 1: Alignment of their directions of motion (Vicsek)
   Case 2: Alignment of their full body attitude (new model)
**Vicsek model** [Vicsek, Czirók, Ben-Jacob, Cohen, Shochet, PRL 95]

**Individual-Based (aka particle) model**

- **self-propelled** \( \Rightarrow \) all particles have same constant speed \( a \)
- **align** with their neighbours up to a certain noise

\[
X_k(t) \in \mathbb{R}^d: \text{position of the } k\text{-th particle at time } t
\]
\[
V_k(t) \in S^{d-1}: \text{velocity orientation } (|V_k(t)| = 1)
\]

\[
\dot{X}_k(t) = aV_k(t)
\]
\[
dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2\tau} dB^k_t), \quad P_{V_k^\perp} = \text{Id} - V_k \otimes V_k
\]

\[
J_k = \sum_{j, \ |X_j - X_k| \leq R} V_j, \quad \bar{V}_k = \frac{J_k}{|J_k|}
\]

- \( \nu \) alignment frequency; \( \tau \) noise intensity
- \( J_k, \bar{V}_k \) neighbors’ mean velocity, mean orientation
- \( P_{V_k^\perp} \) projection on \( V_k^\perp \), maintains \( |V_k(t)| = 1 \)
- \( \circ \) indicates Stratonovitch SDE
Body attitude alignment model  [M3AS, to appear]

\( X_k(t) \in \mathbb{R}^d \): position of the \( k \)-th subject at time \( t \)

\( A_k(t) \in SO(d) \): rotation mapping reference frame \((e_1, \ldots, e_d)\) to subject’s body frame

\( A_k(t)e_1 \in S^{d-1} \): propulsion direction

\[
\dot{X}_k(t) = a A_k(t)e_1
\]

\[
dA_k(t) = P_{T_{A_k(t)}SO(d)} \circ (\nu \bar{A}_k dt + \sqrt{2\tau} dB^k_t),
\]

\[
M_k(t) = \sum_{j, |X_j - X_k| \leq R} A_j(t), \quad \bar{A}_k = PD(M_k(t))
\]

\( M_k \) arithmetic mean of neighbors’ \( A \) matrices

\( A = PD(M) \iff \exists S \) symmetric s.t. \( M = AS \) (polar decomp.)

\( P_{T_{A_k(t)}SO(d)} \) projection on the tangent \( T_{A_k(t)}SO(d) \), maintains \( A_k(t) \in SO(d) \)
Questions

Can we quantify the difference between the two models?

Is body-alignment just Vicsek for direction of motion with frame dynamic superimposed to it?

Or does body-alignment provide genuinely new dynamic? i.e. do gradients of body frames orientation influence direction of motion?

Not easy to answer with Individual-Based Model
   Goal: use coarse-grained model to answer this question
**Quaternions**

**Quaternions:**  \( q = q_0 + q_1 i + q_2 j + q_3 k, \ q_0, \ldots, q_3 \in \mathbb{R} \).

\[ i^2 = j^2 = k^2 = ijk = -1: \text{ division ring } \mathbb{H} \text{ (non commutative)} \]

\[ q = \text{Re}q + \text{Im}q \quad \text{with} \quad \text{Re}q = q_0, \quad \text{Im}q = q_1 i + q_2 j + q_3 k \]

\( \mathbb{R}^3 \ni \tilde{q} = (q_1, q_2, q_3) \approx q = q_1 i + q_2 j + q_3 k \in \{q \in \mathbb{H}, \text{Re}q = 0\} \)

Conjugate \( q^* = \text{Re}q - \text{Im}q \)

Scalar product \( p \cdot q = pq^* = \text{Re}p \text{Re}q + \text{Im}p \cdot \text{Im}q \)

**Unitary quaternions** \( \mathbb{H}_1 = \{q \in \mathbb{H}, qq^* = 1\} \approx \mathbb{S}^3 \)

\( \mathbb{H}_1 \ni q = \cos(\theta/2) + \sin(\theta/2)n, \ \theta \in [0, 2\pi), \ \tilde{n} \in \mathbb{S}^2 \)

The map \( \mathbb{R}^3 \ni \tilde{v} \rightarrow \text{Im}(qvq^*) \in \mathbb{R}^3 \) is rotation axis \( \tilde{n} \) angle \( \theta \)

Given \( A \in \text{SO}(3) \) encoded by \( q \) and \( -q \in \mathbb{H}_1 \)

\( A(q_1)A(q_2) = A(q_1q_2) \)
Quaternion representation (d=3) [arXiv:1701.01166]

\[ X_k(t) \in \mathbb{R}^d \]: position of the \( k \)-th subject at time \( t \)

\[ q_k(t) \in \mathbb{H}_1 \]: quaternion encoding rotation mapping reference frame \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) to subject's body frame

\[ \vec{e}_1(q_k(t)) = \text{Im}(q_k(t) e_1 q_k(t)^*) \in \mathbb{S}^{d-1} \]: propulsion direction

\[ \dot{X}_k(t) = a \vec{e}_1(q_k(t)) \]

\[ dq_k(t) = P_{q_k(t)}^\perp \circ (\nu F_k(t) dt + \sqrt{2\tau} dB^k_t), \]

\[ F_k(t) = (\bar{q}_k(t) \cdot q_k(t)) \bar{q}_k(t) \]

\[ \bar{q}_k(t) \] leading eigenvector of tensor

\[ Q_k(t) = \sum_{j, |X_j-X_k|\leq R} q_j(t) \otimes q_j(t) \]

\[ Q_k(t) \] de Gennes Q-tensor; \( \bar{q}_k(t) \) mean nematic alignment direction

Describes alignment of \( q_k \) with \( \bar{q}_k \) or \(-\bar{q}_k\)

\[ P_{q_k(t)}^\perp \] projection on \( q_k^\perp \), maintains \( q_k q_k^* = 1 \)

Similarity with polymer models

Quaternion dynamics identical to previous rotation matrix dynamics
2. Mean-Field model
Mean-field model

\[ f(x, q, t) = \text{particle probability density } x \in \mathbb{R}^3, q \in \mathbb{H}_1 \]
satisfies a Fokker-Planck equation

\[ \partial_t f + a \vec{e}_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = \tau \Delta_q f \]

\[ \mathcal{F}_f(x, q, t) = \nu P_{q\perp} ((\bar{q}_f(x, t) \cdot q)\bar{q}_f(x, t)), \quad P_{q\perp} = \text{Id} - q \otimes q \]

\[ \bar{q}_f(x, t) = \text{leading eigenvector of tensor} \]

\[ Q_f(x, t) = \int_{|x' - x| < R} \int_{\mathbb{H}_1} f(x', q', t) (q' \otimes q') dq' dx' \]

\[ Q_f(x, t) = Q\text{-tensor in a neighborhood of } x \]

\((\bar{q}_f(x, t) \cdot q)\bar{q}_f(x, t)\) provides nematic alignment of \(q\) with \(\bar{q}_f(x, t)\)

\[ \mathcal{F}_f(x, q, t)) = \text{projection of nematic alignment direction on } q\perp \]

\((x, q) \in \mathbb{R}^3 \times \mathbb{H}_1 ; \quad \nabla_q, \nabla_{q'}: \text{div and grad on } \mathbb{H}_1 \]

\[ \Delta_q \text{ Laplace-Beltrami operator on } \mathbb{H}_1 \approx \mathbb{S}^3 \]
Passage to dimensionless units

Highlights important physical scales & small parameters

Choose time scale $t_0$, space scale $x_0 = at_0$
Set $f$ scale $f_0 = 1/x_0^3$, $F$ scale $F_0 = 1/t_0$
Introduce dimensionless parameters $\bar{\nu} = \nu t_0$, $\bar{\tau} = \tau t_0$, $\bar{R} = \frac{R}{x_0}$
Change variables $x = x_0 x'$, $t = t_0 t'$, $f = f_0 f'$, $F = F_0 F'$

Get the scaled Fokker-Planck system (omitting the primes):

$$\partial_t f + \vec{e}_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = \bar{\tau} \Delta_q f$$

$$\mathcal{F}_f(x, q, t) = \bar{\nu} P_{q^\perp} \left( (\bar{q}_f(x, t) \cdot q) \bar{q}_f(x, t) \right), \quad P_{q^\perp} = \text{Id} - q \otimes q$$

$$\bar{q}_f(x, t) = \text{leading eigenvector of tensor } Q_f(x, t) = \int_{|x' - x| < \bar{R}} \int_{\mathbb{H}_1} f(x', q', t) (q' \otimes q') \, dq' \, dx'$$
Macroscoping scaling

Choice of $t_0$ such that $\bar{\tau} = \frac{1}{\varepsilon}, \varepsilon \ll 1$

Macroscopic scale:
there are many velocity diffusion events within one time unit

Assumption 1: $k := \frac{\bar{\nu}}{\bar{\tau}} = \mathcal{O}(1)$

Social interaction and diffusion act at the same scale
Implies $\bar{\nu}^{-1} = \mathcal{O}(\varepsilon)$, i.e. mean-free path is microscopic

Assumption 2: $\bar{R} = \varepsilon$

Interaction range is microscopic
and of the same order as mean-free path $\bar{\nu}^{-1}$
Possible variant: $\bar{R} = \mathcal{O}(\sqrt{\varepsilon})$: interaction range still small
but large compared to mean-free path. To be investigated later
Fokker-Planck under macroscopic scaling

With Assumption 2 \((\bar{R} = \mathcal{O}(\varepsilon))\)

Interaction is local at leading order: by Taylor expansion:

\[
Q_f = Q_f + \mathcal{O}(\varepsilon^2), \quad Q_f(x,t) = \int_{\mathbb{H}_1} f(x,q',t) (q' \otimes q') \, dq'
\]

\(Q_f(x,t) = \) local \(Q\)-tensor. From now on, neglect \(\mathcal{O}(\varepsilon^2)\) term

Fokker-Planck eq. in scaled variables

\[
\varepsilon (\partial_t f^\varepsilon + \vec{e}_1(q) \cdot \nabla_x f^\varepsilon) = -\nabla_q \cdot (F_{f^\varepsilon} f^\varepsilon) + \Delta_q f^\varepsilon
\]

\[
F_f(x,q,t) = k P_{q\perp} \left( (\bar{q}_f(x,t) \cdot q) \bar{q}_f(x,t) \right), \quad P_{q\perp} = \text{Id} - q \otimes q
\]

\[
\bar{q}_f(x,t) = \text{leading eigenvector of tensor}
\]

\[
Q_f(x,t) = \int_{\mathbb{H}_1} f(x,q',t) (q' \otimes q') \, dq'
\]

Coarse-grained model is obtained in the limit \(\varepsilon \to 0\)
3. Self-Organized Quaternionic Hydrodynamics (SOHQ)
Collision operator

Model can be written

$$\partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} C(f^\varepsilon)$$

with collision operator

$$C(f) = -\nabla_q \cdot (F_f f) + \Delta_q f$$

$$F_f = kP_{q\perp} ((\bar{q}_f \cdot q) \bar{q}_f)$$

$$\bar{q}_f$$ leading eigenvector of $$Q_f$$

$$Q_f = \int_{\mathbb{H}_1} f(q') (q' \otimes q') dq'$$

When $$\varepsilon \to 0$$, $$f^\varepsilon \to f$$ (formally) such that $$C(f) = 0$$

$$\Rightarrow$$ importance of the solutions of $$C(f) = 0$$ (equilibria)

$$C$$ acts on $$q$$-variable only ($$(x, t)$$ are just parameters)
Algebraic preliminaries

Force $F_f$ can be written: 

$$F_f(v) = \frac{k}{2} \nabla_q((\bar{q}_f \cdot q)^2)$$

Note $\bar{q}_f$ independent of $q$ ($(x,t)$ are fixed)

Rewrite:

$$C(f)(q) = \nabla_q \cdot \left[ -f \frac{k}{2} \nabla_q((\bar{q}_f \cdot q)^2) + \nabla_q f \right]$$

$$= \nabla_q \cdot \left[ f \nabla_q \left( -\frac{k}{2} (\bar{q}_f \cdot q)^2 + \ln f \right) \right]$$

Let $\bar{q} \in \mathbb{H}_1$ be given: Solutions of

$$\nabla_q \left( -\frac{k}{2} (\bar{q}_f \cdot q)^2 + \ln f \right) = 0$$

are proportional to:

$$f(v) = M_{\bar{q}}(q) := \frac{1}{Z} \exp \left( \frac{k}{2} (\bar{q} \cdot q)^2 \right)$$

with

$$\int_{\mathbb{H}_1} M_{\bar{q}}(q) \, dq = 1$$

'generalized' von Mises-Fisher (VMF) distribution
Again:

$$M_{\bar{q}}(q) := \frac{e^{k(q \cdot \bar{q})^2}}{\int_{H_1} e^{k(q' \cdot \bar{q})^2} dq'}$$

$k > 0$: concentration parameter; $\bar{q} \in H_1 \approx S^3$: orientation

**Order parameter:** $c_1(k)$ s.t. $\int_{H_1} M_{\bar{q}}(q) e_1(q) dq = c_1(k)e_1(\bar{q})$

$k \rightarrow c_1(k), \quad 0 \leq c_1(k) \leq 1$

Here:

concentration parameter $k$

and order parameter $c_1(k)$

are constant
Equilibria

Definition: equilibrium manifold \( \mathcal{E} = \{ f(q) \mid C(f) = 0 \} \)

Theorem: \( \mathcal{E} = \{ \rho M \bar{q} \text{ for arbitrary } \rho \in \mathbb{R}_+ \text{ and } \bar{q} \in \mathbb{H}_1 \} \)

Note: \( \dim \mathcal{E} = 4 \)

Proof: follows from entropy inequality:

\[
H(f) = \int C(f) \frac{f}{M_{\bar{q}f}} dq = - \int M_{\bar{q}f} \left| \nabla_q \left( \frac{f}{M_{\bar{q}f}} \right) \right|^2 \leq 0
\]

follows from \( C(f) = \nabla_q \cdot \left[ M_{\bar{q}f} \nabla_q \left( \frac{f}{M_{\bar{q}f}} \right) \right] \)

Then, \( C(f) = 0 \) implies \( H(f) = 0 \) and \( \frac{f}{M_{\bar{q}f}} = \text{Constant} \)
and \( f \) is of the form \( \rho M \bar{q} \)

Reciprocally, if \( f = \rho M \bar{q} \), then, \( \bar{q}_f = \bar{q} \) and \( C(f) = 0 \)
Use of equilibria

$$f^\varepsilon \to f \text{ as } \varepsilon \to 0 \text{ with } q \to f(x, q, t) \in \mathcal{E} \text{ for all } (x, t)$$

Implies that $$f(x, q, t) = \rho(x, t)\bar{M}_{\bar{q}(x, t)}(q)$$

Need to specify the dependence of $$\rho$$ and $$\bar{q}$$ on $$(x, t)$$

Requires 4 equations since $$(\rho, \bar{q}) \in \mathbb{R}_+ \times H_1 \approx \mathbb{R}_+ \times S^3$$ are determined by 4 independent real quantities

$$f$$ satisfies

$$\partial_t f + e_1(q) \cdot \nabla_x f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} C(f^\varepsilon)$$

Problem: $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} C(f^\varepsilon)$$ is not known

Trick:

Collision invariant
Collision invariant

is a function $\psi(q)$ such that $\int C(f) \psi \, dq = 0, \quad \forall f$

Form a linear vector space $CI$

Multiply eq. by $\psi$: $\varepsilon^{-1}$ term disappears

Find a conservation law:

$$\partial_t \left( \int_{\mathbb{H}_1} f(x, q, t) \psi(q) \, dq \right) + \nabla_x \cdot \left( \int_{\mathbb{H}_1} f(x, q, t) \psi(q) e_1(q) \, dq \right) = 0$$

Have used that $\partial_t$ or $\nabla_x$ and $\int \ldots dq$ can be interchanged

Limit fully determined if dim $CI = \text{dim } E = 4$

$CI = \text{Span}\{1\}$. Interaction preserves mass but no other quantity

Due to self-propulsion, no momentum conservation

$\text{dim } CI = 1 < \text{dim } E = 4$. Is the limit problem ill-posed?
Proof that $\psi(q) = 1$ is a CI?

Obvious. $C(f) = \nabla_q \cdot \left[ \ldots \right]$ is a divergence

By Stokes theorem on the sphere, $\int C(f) \, dq = 0$

Use of the CI $\psi(q) = 1$: Get the conservation law

$$\partial_t \left( \int_{\mathbb{H}_1} f(x, q, t) \, dq \right) + \nabla_x \cdot \left( \int_{\mathbb{H}_1} f(x, q, t) \, e_1(q) \, dq \right) = 0$$

With $f = \rho M\bar{q}$ we have

$$\int f(x, v, t) \, dv = \rho(x, t), \quad \int f(x, v, t) \, e_1(q) \, dq = c_1 \rho(x, t) e_1(\bar{q}(x, t))$$

We end up with the mass conservation eq.

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho e_1(\bar{q})) = 0$$
Given \( \bar{q}_0 \in H_1 \), Define \( \mathcal{R}_{\bar{q}_0}(f) = \nabla_q \cdot [M_{\bar{q}_0} \nabla_q (\frac{f}{M_{\bar{q}_0}})] \)

Note \( f \rightarrow \mathcal{R}_{\bar{q}_0}(f) \) is linear and \( C'(f) = \mathcal{R}_{\bar{q}_f}(f) \)

A function \( \psi_{\bar{q}_0}(q) \) is a GCI associated to \( \bar{q}_0 \), iff

\[
\int \mathcal{R}_{\bar{q}_0}(f) \psi_{\bar{q}_0} \, dq = 0, \quad \forall f \text{ such that } P_{\bar{q}_0}^+ \left[ \left( \int_{H_1} f(q) (q \otimes q) \, dq \right) \bar{q}_0 \right] = 0
\]

The set of GCI \( G_{\bar{q}_0} \) is a linear vector space

**Theorem:** Given \( \bar{q}_0 \in H_1 \), \( G_{\bar{q}_0} \) is the 4-dim vector space :

\[
G_{\bar{q}_0} = \{ q \mapsto \alpha + h(q \cdot \bar{q}_0) \beta \cdot q, \text{ with arbitrary } \alpha \in \mathbb{R} \text{ and } \beta \in H \text{ with } \beta \cdot \bar{q}_0 = 0 \}.
\]

Introduce \( r = q \cdot \bar{q}_0 \in [-1, 1] \). \( h \) is the unique solution in \( V \) of:

\[
-(1-r^2)^{-3/2} \exp \left( -\frac{k}{2} r^2 \right) \frac{d}{dr} \left[ (1-r^2)^{5/2} \exp \left( \frac{k}{2} r^2 \right) \frac{dh}{dr} \right] + (k r^2 + 3) h(r) = -r
\]

\[
V = \{ h \mid (1-r^2)^{3/4} h \in L^2(-1, 1), \quad (1-r^2)^{5/4} h' \in L^2(-1, 1) \}
\]

Furthermore, \( h \) is odd and non-positive for \( r \geq 0 \)
Use of GCI: equation for $\bar{q}(x, t)$

Use GCI $h(q \cdot \bar{q}_0) \beta \cdot q$ for $\beta \in \mathbb{H}$ with $\beta \cdot \bar{q}_0 = 0$

Equivalently, use the quaternion valued function

$$\psi_{\bar{q}_0}(q) = h(q \cdot \bar{q}_0) P_{\bar{q}_0} q$$

Multiply FP eq by GCI $\psi_{f \varepsilon}$: $O(\varepsilon^{-1})$ terms disappear

$$\int C(f) \tilde{\psi}_{f \varepsilon} \, dv = \int R_{\bar{q}_f}(f) \psi_{f \varepsilon} \, dq = 0 \quad \text{by property of GCI}$$

Gives:

$$\int \left( \partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon \right) \psi_{f \varepsilon} \, dq = 0$$

As $\varepsilon \to 0$: $f^\varepsilon \to \rho M_{\bar{q}}$ and $\psi_{f \varepsilon} \to \psi_{\bar{q}}$ Leads to:

$$\int \left( \partial_t (\rho M_{\bar{q}}) + e_1(q) \cdot \nabla_x (\rho M_{\bar{q}}) \right) \psi_{\bar{q}} \, dq = 0$$

Not a conservation equation

because of dependence of $\psi_{\bar{q}}$ upon $(x, t)$ through $\bar{q}$

$\partial_t$ or $\nabla_x$ and $\int \ldots dq$ cannot be interchanged
Equation for $\bar{q}(x, t)$

Takes the form:

$$
\rho \left( \partial_t \bar{q} + c_2 (e_1 (\bar{q}) \cdot \nabla_x) \bar{q} \right) + c_3 [e_1 (\bar{q}) \times \nabla_x \rho] \bar{q} \\
+ c_4 \rho \left[ (\nabla_{x, \text{rel}} \bar{q}) e_1 (\bar{q}) + (\nabla_{x, \text{rel}} \cdot \bar{q}) e_1 (\bar{q}) \right] \bar{q} = 0
$$

where

$$(\nabla_{x, \text{rel}} \bar{q}) = (\partial_{x_i, \text{rel}} \bar{q})_{i=1,2,3} = ((\partial_{x_i} \bar{q}) \bar{q}^*)_{i=1,2,3} \in \mathbb{H}_\text{Im}^3$$

$$(\nabla_{x, \text{rel}} \cdot \bar{q}) = \sum_{i=1,2,3} (\partial_{x_i, \text{rel}} \bar{q})_i = \sum_{i=1,2,3} ((\partial_{x_i} \bar{q}) \bar{q}^*)_i \in \mathbb{R}$$

$$\mathbb{H}_\text{Im} = \{ q \in \mathbb{H}, \ \text{Im} q = 0 \} \approx \mathbb{R}^3$$

$$(\partial_{x_i, \text{rel}} \bar{q})_j = j\text{-th component of } \partial_{x_i, \text{rel}} \bar{q}$$

$$(\nabla_{x, \text{rel}} \bar{q}) e_1 (\bar{q}) = ((\partial_{x_i, \text{rel}} \bar{q}) \cdot e_1 (\bar{q}))_{i=1,2,3}$$

Coefficients $c_2$ and $c_4$ depend on GCI $h$
Self-Organized Quaternionic Hydrodynamics (SOHQ)

System for density $\rho(x, t)$ and quaternion orientation $\bar{q}(x, t)$:

\[
\begin{align*}
\partial_t \rho + c_1 \nabla_x (\rho e_1(\bar{q})) &= 0 \\
\rho \left( \partial_t \bar{q} + c_2 (e_1(\bar{q}) \cdot \nabla_x) \bar{q} \right) + c_3 [e_1(\bar{q}) \times \nabla_x \rho] \bar{q} + c_4 \rho \left[ (\nabla_{x, \text{rel}} \bar{q}) e_1(\bar{q}) + (\nabla_{x, \text{rel}} \cdot \bar{q}) e_1(\bar{q}) \right] \bar{q} &= 0 \\
|\bar{q}| &= 1
\end{align*}
\]
4. Comparison with SOH dynamics for Vicsek
SOH model for the Vicsek dynamics

Vicsek mean-field model for $f^\varepsilon(x,v,t)$
position $x \in \mathbb{R}^3$, velocity orientation $v \in \mathbb{S}^2$

As $\varepsilon \to 0$, $f^\varepsilon(x,v,t) \to \rho(x,t)\mathcal{M}_{\Omega(x,t)}(v)$

$\rho(x,t) \geq 0$, $\Omega(x,t) \in \mathbb{S}^2$

$\mathcal{M}_\Omega(v) = \frac{1}{Z} \exp(k(\Omega \cdot v))$, $\int_{\mathbb{S}^2} \mathcal{M}_\Omega(v) \, dv = 1$

$(\rho(x,t), \Omega(x,t))$ solves SOH model:

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + c_3 \, P_{\Omega \perp} \, \nabla_x \rho = 0,$$

$|\Omega| = 1$

Similar to Compressible Euler eqs. of gas dynamics
System of hyperbolic eqs.

But major differences:
Geometric constraint $|\Omega| = 1$: requires $P_{\Omega \perp}$ to be maintained
System is non conservative due to the presence of $P_{\Omega \perp}$
$c_2 \neq c_1$: loss of Galilean invariance
\[ \bar{q}(x, t) \in H_1 \text{ encodes rotation } \Lambda(x, t) \in SO(3) \]
\[ \Lambda(x, t) \text{ describes agents' local average body attitude} \]
\[ \Omega(x, t) = \Lambda(x, t)e_1 = e_1(\bar{q}(x, t)) : \text{ direction of motion} \]
\[ u(x, t) = \Lambda(x, t)e_2 = e_2(\bar{q}(x, t)) : \text{ belly to back} \]
\[ v(x, t) = \Lambda(x, t)e_3 = e_3(\bar{q}(x, t)) : \text{ right to left wing} \]

SOHQ model equivalent to
\[ \partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0 \]
\[ \rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + P_{\Omega \perp} \left( c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v) \right) = 0 \]
\[ \rho \left( \partial_t u + c_2 (\Omega \cdot \nabla_x) u \right) - u \cdot \left( c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v) \right) \Omega + c_4 \rho \delta(\Omega, u, v) v = 0 \]
\[ \rho \left( \partial_t v + c_2 (\Omega \cdot \nabla_x) v \right) - v \cdot \left( c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v) \right) \Omega - c_4 \rho \delta(\Omega, u, v) u = 0 \]
\[ |\Omega| = |u| = |v| = 1, \quad \Omega \cdot u = u \cdot v = v \cdot \Omega = 0 \]

with \( r(\Omega, u, v) \) (for rotational) and \( \delta(\Omega, u, v) \) (for divergence):
\[ r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v \in \mathbb{R}^3 \]
\[ \delta(\Omega, u, v) u = \left[ (\Omega \cdot \nabla_x) u \right] \cdot v + \left[ (u \cdot \nabla_x) v \right] \cdot \Omega + \left[ (v \cdot \nabla_x) \Omega \right] \cdot u \in \mathbb{R} \]
Compare eqs for $\rho$ and $\Omega$:

**SOH: Coarse-grained Vicsek model**

$$
\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0 \\
\rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + c_3 P_{\Omega \perp} \nabla_x \rho = 0
$$

**SOHQ: Coarse-grained body orientation model**

$$
\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0 \\
\rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + P_{\Omega \perp} \left( c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v) \right) = 0
$$

Difference is the term

$$
r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v
$$

Shows how differences in body orientation affect direction of the flock
Answers

Can we quantify the difference between the Vicsek and body alignment models?
YES: by using coarse-grained models SOH and SOHQ respectively

Is body-alignment just Vicsek for direction of motion with frame dynamic superimposed to it?
Answer is ’NO’

Or does body-alignment provide genuinely new dynamic?
i.e. do gradients of body frames orientation influence direction of motion?
Answer is ’YES’
5. Conclusion
New collective dynamics model relying on full body alignment
body frame alignment ⇔ quaternion nematic alignment

Coarse-grained model is SOHQ
First order PDE for density and local average quaternion
describes dynamics of agents’ local mean body frame
dynamics genuinely ≠ from velocity alignment (Vicsek or SOH)

Perspectives
analysis of the model
rigorous proof of convergence
numerical simulations
Higher dimensions