Coarse-graining of collective dynamics models

A model for local body alignment

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Summary

1. Individual-based model
2. Mean-Field model
3. Self-Organized Quaternionic Hydrodynamics (SOHQ)
4. Comparison with SOH dynamics for Vicsek
5. Conclusion

1. Individual-Based Model

Micro–scale

Individual agents obey simple rules

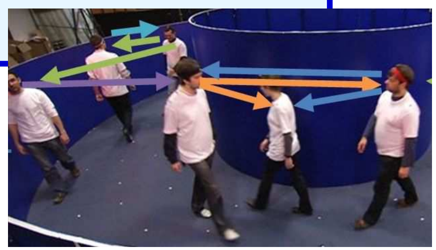
no leader

? →

Macro scale

Emergence of large–scale coherent structures

Not directly encoded in agent's behavior



Link micro to macro scales

- Lack of conservations
- Breakdown of chaos property

Phase transitions

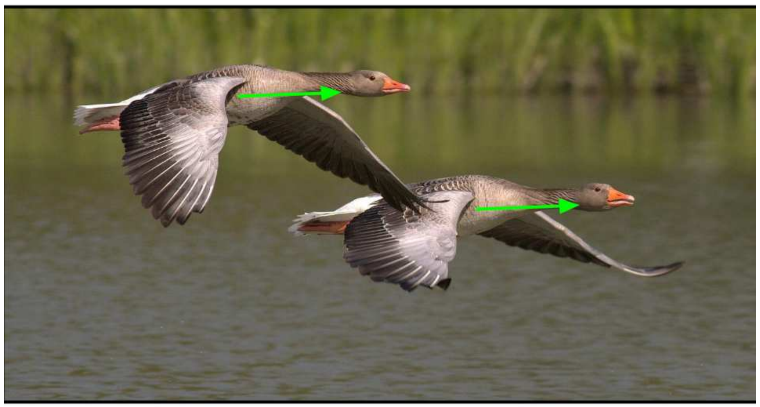
- Symmetry-breaking
- Jamming
- Continuum to network

Description of the system to model

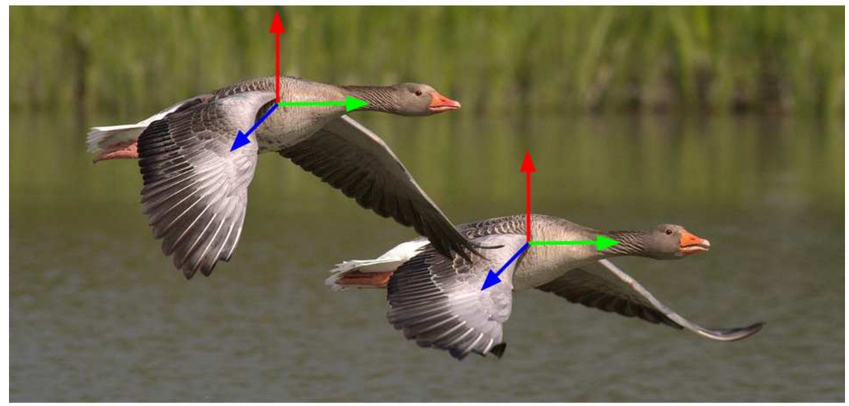
Self-propelled agents which align with their neighbors

Case 1: Alignment of their directions of motion (Vicsek)

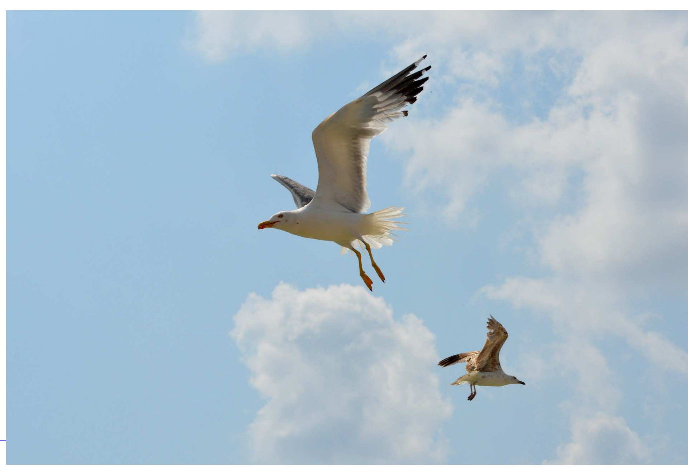
Case 2: Alignment of their full body attitude (new model)

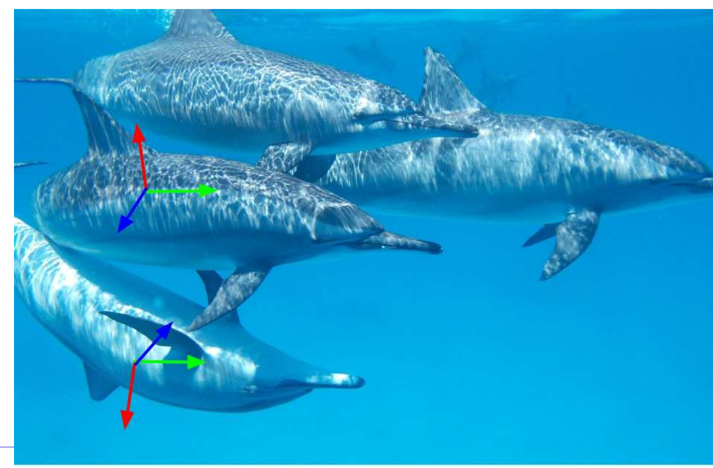


Vicsek model



Body attitude alignment





Individual-Based (aka particle) model

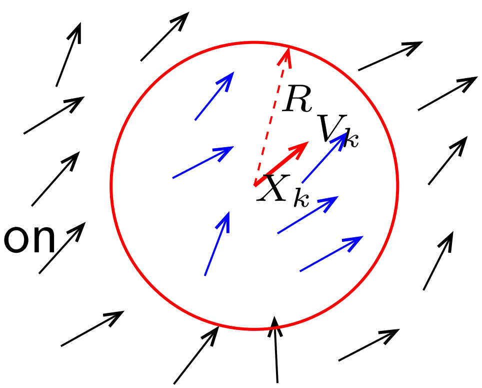
self-propelled $\Rightarrow$ all particles have same constant speed a
align with their neighbours up to a certain noise

$X_k(t) \in \mathbb{R}^d$: position of the k-th particle at time t
$V_k(t) \in \mathbb{S}^{d-1}$: velocity orientation ($|V_k(t)| = 1$)

$$\dot{X}_k(t) = aV_k(t)$$

$$dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2\tau}\, dB_t^k), \quad P_{V_k^\perp} = \text{Id} - V_k \otimes V_k$$

$$\mathcal{J}_k = \sum_{j,\, |X_j - X_k| \le R} V_j, \quad \bar{V}_k = \frac{\mathcal{J}_k}{|\mathcal{J}_k|}$$

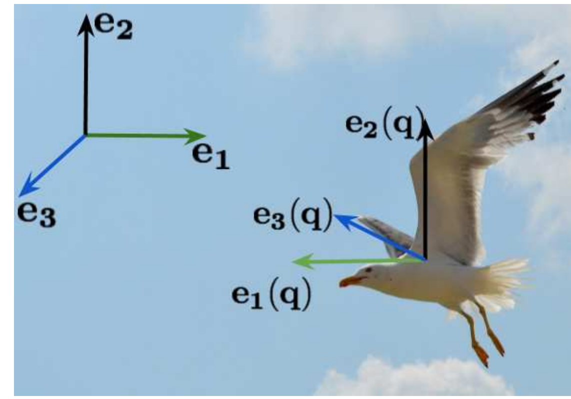


ν alignment frequency; τ noise intensity
$\mathcal{J}_k$, $\bar{V}_k$ neighbors' mean velocity, mean orientation
$P_{V_k^\perp}$ projection on $V_k^\perp$, maintains $|V_k(t)| = 1$
$\circ$ indicates Stratonovich SDE

$X_k(t) \in \mathbb{R}^d$: position of the k-th subject at time t

$A_k(t) \in \mathsf{SO}(d)$: rotation mapping reference frame $(e_1, \dots, e_d)$ to subject's body frame

$A_k(t)e_1 \in \mathbb{S}^{d-1}$: propulsion direction



$$\dot{X}_k(t) = a A_k(t) e_1$$

$$dA_k(t) = P_{T_{A_k(t)}\mathsf{SO}(d)} \circ (\nu \bar{A}_k dt + \sqrt{2\tau}\, dB_t^k),$$

$$M_k(t) = \sum_{j, |X_j - X_k| \le R} A_j(t), \quad \bar{A}_k = \mathsf{PD}(M_k(t))$$

M_k arithmetic mean of neighbors' A matrices

$A = \mathsf{PD}(M) \Leftrightarrow \exists S$ symmetric s.t. $M = AS$ (polar decomp.)

$P_{T_{A_k(t)}\mathsf{SO}(d)}$ projection on the tangent $T_{A_k(t)}\mathsf{SO}(d)$, maintains $A_k(t) \in \mathsf{SO}(d)$

Questions

Can we quantify the difference between the two models ?

Is body-alignment just Vicsek for direction of motion
with frame dynamic superimposed to it ?

Or does body-alignment provide genuinely new dynamic ?
i.e. do gradients of body frames orientation influence
direction of motion ?

Not easy to answer with Individual-Based Model
Goal: use coarse-grained model to answer this question

Quaternions: $q = q_0 + q_1 i + q_2 j + q_3 k$, $q_0, \dots, q_3 \in \mathbb{R}$.

$i^2 = j^2 = k^2 = ijk = -1$: division ring $\mathbb{H}$ (non commutative)

$q = \text{Re}q + \text{Im}q$ with $\text{Re}q = q_0$, $\text{Im}q = q_1 i + q_2 j + q_3 k$

$\mathbb{R}^3 \ni \vec{q} = (q_1, q_2, q_3) \approx q = q_1 i + q_2 j + q_3 k \in \{q \in \mathbb{H}, \text{Re}q = 0\}$

Conjugate $q^* = \text{Re}q - \text{Im}q$

Scalar product $p \cdot q = pq^* = \text{Re}p\,\text{Re}q + \text{Im}p \cdot \text{Im}q$

Unitary quaternions $\mathbb{H}_1 = \{q \in \mathbb{H}, qq^* = 1\} \approx \mathbb{S}^3$

$\mathbb{H}_1 \ni q = \cos(\theta/2) + \sin(\theta/2)n$, $\theta \in [0, 2\pi)$, $\vec{n} \in \mathbb{S}^2$

The map $\mathbb{R}^3 \ni \vec{v} \to \text{Im}(qvq^*) \in \mathbb{R}^3$ is rotation axis n angle θ

Given $A \in \text{SO}(3)$ encoded by q and $-q \in \mathbb{H}_1$

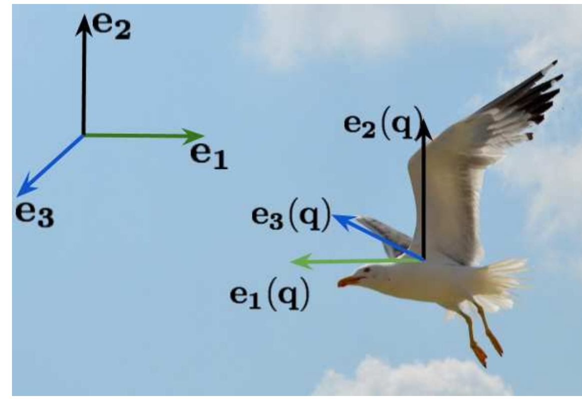
$A(q_1)A(q_2) = A(q_1 q_2)$

$X_k(t) \in \mathbb{R}^d$: position of the k-th subject at time t
$q_k(t) \in \mathbb{H}_1$: quaternion encoding rotation mapping reference frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to subject's body frame
$\vec{e}_1(q_k(t)) = \mathrm{Im}(q_k(t)\, e_1\, q_k(t)^*) \in \mathbb{S}^{d-1}$: propulsion direction

$$\dot{X}_k(t) = a\vec{e}_1(q_k(t))$$

$$dq_k(t) = P_{q_k(t)^\perp} \circ (\nu F_k(t)dt + \sqrt{2\tau}\, dB_t^k),$$

$$F_k(t) = \big(\bar{q}_k(t) \cdot q_k(t)\big)\, \bar{q}_k(t)$$



$\bar{q}_k(t)$ leading eigenvector of tensor

$$Q_k(t) = \sum_{j,\, |X_j - X_k| \le R} q_j(t) \otimes q_j(t)$$

$Q_k(t)$ de Gennes Q-tensor; $\bar{q}_k(t)$ mean nematic alignment direction
Describes alignment of q_k with $\bar{q}_k$ or $-\bar{q}_k$
$P_{q_k(t)^\perp}$ projection on $q_k^\perp$, maintains $q_k q_k^* = 1$
Similarity with polymer models
Quaternion dynamics identical to previous rotation matrix dynamics

2. Mean-Field model

Mean-field model

$f(x,q,t) =$ particle probability density $x \in \mathbb{R}^3$, $q \in \mathbb{H}_1$

satisfies a Fokker-Planck equation

$$\partial_t f + a\vec{e}_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = \tau \Delta_q f$$

$$\mathcal{F}_f(x,q,t) = \nu P_{q^\perp}\big((\bar{q}_f(x,t) \cdot q)\bar{q}_f(x,t)\big), \quad P_{q^\perp} = \mathrm{Id} - q \otimes q$$

$\bar{q}_f(x,t) =$ leading eigenvector of tensor

$$\mathcal{Q}_f(x,t) = \int_{|x'-x|<R} \int_{\mathbb{H}_1} f(x',q',t)\,(q' \otimes q')\,dq'\,dx'$$

$\mathcal{Q}_f(x,t) =$ Q-tensor in a neighborhood of x

$(\bar{q}_f(x,t) \cdot q)\bar{q}_f(x,t)$ provides nematic alignment of q with $\bar{q}_f(x,t)$

$\mathcal{F}_f(x,q,t)) =$ projection of nematic alignment direction on $q^\perp$

$(x,q) \in \mathbb{R}^3 \times \mathbb{H}_1$; $\nabla_q \cdot$, ∇_q: div and grad on $\mathbb{H}_1$

Δ_q Laplace-Beltrami operator on $\mathbb{H}_1 \approx \mathbb{S}^3$

Passage to dimensionless units

Highlights important physical scales & small parameters

- Choose time scale t_0, space scale $x_0 = a t_0$
- Set f scale $f_0 = 1/x_0^3$, F scale $\mathcal{F}_0 = 1/t_0$
- Introduce dimensionless parameters $\bar\nu = \nu t_0$, $\bar\tau = \tau t_0$, $\bar R = \frac{R}{x_0}$
- Change variables $x = x_0 x'$, $t = t_0 t'$, $f = f_0 f'$, $\mathcal{F} = \mathcal{F}_0 \mathcal{F}'$

Get the scaled Fokker-Planck system (omitting the primes):

$$\partial_t f + \vec e_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = \bar\tau \Delta_q f$$

$$\mathcal{F}_f(x,q,t) = \bar\nu P_{q^\perp}\big((\bar q_f(x,t) \cdot q) \bar q_f(x,t) \big), \quad P_{q^\perp} = \mathrm{Id} - q \otimes q$$

$\bar q_f(x,t) =$ leading eigenvector of tensor

$$\mathcal{Q}_f(x,t) = \int_{|x'-x|<\bar R} \int_{\mathbb{H}_1} f(x',q',t)\,(q' \otimes q')\, dq'\, dx'$$

Macroscoping scaling

Choice of t_0 such that $\bar{\tau} = \frac{1}{\varepsilon}$, $\varepsilon \ll 1$

Macroscopic scale:

there are many velocity diffusion events within one time unit

Assumption 1: $k := \frac{\bar{\nu}}{\bar{\tau}} = \mathcal{O}(1)$

Social interaction and diffusion act at the same scale

Implies $\bar{\nu}^{-1} = \mathcal{O}(\varepsilon)$, i.e. mean-free path is microscopic

Assumption 2: $\bar{R} = \varepsilon$

Interaction range is microscopic

and of the same order as mean-free path $\bar{\nu}^{-1}$

Possible variant: $\bar{R} = \mathcal{O}(\sqrt{\varepsilon})$: interaction range still small

but large compared to mean-free path. To be investigated later

Fokker-Planck under macroscopic scaling

With Assumption 2 ($\bar{R} = \mathcal{O}(\varepsilon)$)

Interaction is local at leading order: by Taylor expansion:

$$\mathcal{Q}_f = Q_f + \mathcal{O}(\varepsilon^2), \quad Q_f(x,t) = \int_{\mathbb{H}_1} f(x,q',t)\,(q' \otimes q')\,dq'$$

$Q_f(x,t) =$ local Q-tensor. From now on, neglect $\mathcal{O}(\varepsilon^2)$ term

Fokker-Planck eq. in scaled variables

$$\varepsilon\big(\partial_t f^\varepsilon + \vec{e}_1(q) \cdot \nabla_x f^\varepsilon\big) = -\nabla_q \cdot (F_{f^\varepsilon} f^\varepsilon) + \Delta_q f^\varepsilon$$

$$F_f(x,q,t) = k P_{q^\perp}\big((\bar{q}_f(x,t) \cdot q)\bar{q}_f(x,t)\big), \quad P_{q^\perp} = \mathrm{Id} - q \otimes q$$

$\bar{q}_f(x,t) =$ leading eigenvector of tensor

$$Q_f(x,t) = \int_{\mathbb{H}_1} f(x,q',t)\,(q' \otimes q')\,dq'$$

Coarse-grained model is obtained in the limit $\varepsilon \to 0$

3. Self-Organized Quaternionic Hydrodynamics (SOHQ)

Model can be written

$$\partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} C(f^\varepsilon)$$

with collision operator

$$C(f) = -\nabla_q \cdot (F_f\, f) + \Delta_q f$$

$$F_f = k P_{q^\perp}\big((\bar{q}_f \cdot q) \bar{q}_f \big)$$

$\bar{q}_f$ leading eigenvector of Q_f

$$Q_f = \int_{\mathbb{H}_1} f(q')\, (q' \otimes q')\, dq'$$

When $\varepsilon \to 0$, $f^\varepsilon \to f$ (formally) such that $C(f) = 0$

$\Rightarrow$ importance of the solutions of $C(f) = 0$ (equilibria)

C acts on q-variable only ((x, t) are just parameters)

Force F_f can be written: $F_f(v) = \frac{k}{2} \nabla_q \big((\bar{q}_f \cdot q)^2 \big)$

Note $\bar{q}_f$ independent of q ((x,t) are fixed)

Rewrite:

$$C(f)(q) = \nabla_q \cdot \big[- f \, \frac{k}{2} \nabla_q \big((\bar{q}_f \cdot q)^2 \big) + \nabla_q f \big]$$
$$= \nabla_q \cdot \big[f \, \nabla_q \big(- \frac{k}{2} \, (\bar{q}_f \cdot q)^2 + \ln f \big) \big]$$

Let $\bar{q} \in \mathbb{H}_1$ be given: Solutions of $\nabla_q \big(- \frac{k}{2} \, (\bar{q}_f \cdot q)^2 + \ln f \big) = 0$ are proportional to :

$$f(v) = M_{\bar{q}}(q) := \frac{1}{Z} \exp \big(\frac{k}{2} (\bar{q} \cdot q)^2 \big) \text{ with } \int_{\mathbb{H}_1} M_{\bar{q}}(q) \, dq = 1$$

'generalized' von Mises-Fisher (VMF) distribution

VMF distribution

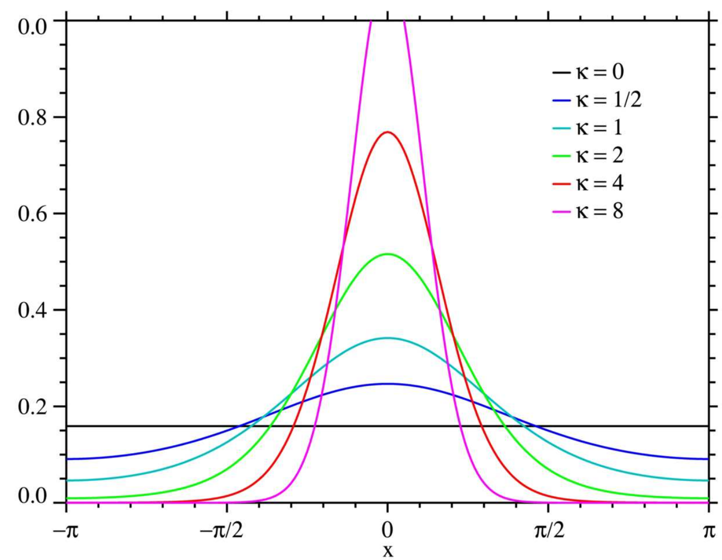
Again:

$$M_{\bar{q}}(q) := \frac{e^{\frac{k}{2}(q\cdot\bar{q})^2}}{\int_{\mathbb{H}_1} e^{\frac{k}{2}(q'\cdot\bar{q})^2}\, dq'}$$

$k > 0$: concentration parameter; $\bar{q} \in \mathbb{H}_1 \approx \mathbb{S}^3$: orientation

Order parameter: $c_1(k)$ s.t. $\int_{\mathbb{H}_1} M_{\bar{q}}(q)\, e_1(q)\, dq = c_1(k) e_1(\bar{q})$

$k \overset{\nearrow}{\to} c_1(k), \quad 0 \le c_1(k) \le 1$



Here:

concentration parameter k
and order parameter $c_1(k)$
are constant

Definition: equilibrium manifold $\mathcal{E} = \{f(q) \,|\, C(f) = 0\}$

Theorem: $\mathcal{E} = \{ \rho M_{\bar q}$ for arbitrary $\rho \in \mathbb{R}_+$ and $\bar q \in \mathbb{H}_1 \}$

Note: dim $\mathcal{E} = 4$

Proof: follows from entropy inequality:

$$H(f) = \int C(f) \frac{f}{M_{\bar q_f}} \, dq = - \int M_{\bar q_f} \Big| \nabla_q \Big(\frac{f}{M_{\bar q_f}} \Big) \Big|^2 \leq 0$$

follows from $C(f) = \nabla_q \cdot \Big[M_{\bar q_f} \nabla_q \Big(\frac{f}{M_{\bar q_f}} \Big) \Big]$

Then, $C(f) = 0$ implies $H(f) = 0$ and $\frac{f}{M_{\bar q_f}} =$ Constant

and f is of the form $\rho M_{\bar q}$

Reciprocally, if $f = \rho M_{\bar q}$, then, $\bar q_f = \bar q$ and $C(f) = 0$

Use of equilibria

$f^\varepsilon \to f$ as $\varepsilon \to 0$ with $q \to f(x,q,t) \in \mathcal{E}$ for all (x,t)

Implies that $f(x,q,t) = \rho(x,t) M_{\bar q(x,t)}(q)$

Need to specify the dependence of ρ and $\bar q$ on (x,t)

Requires 4 equations since $(\rho, \bar q) \in \mathbb{R}_+ \times \mathbb{H}_1 \approx \mathbb{R}_+ \times \mathbb{S}^3$ are determined by 4 independent real quantities

f satisfies

$$\partial_t f + e_1(q) \cdot \nabla_x f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} C(f^\varepsilon)$$

Problem: $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} C(f^\varepsilon)$ is not known

Trick:

Collision invariant

is a function $\psi(q)$ such that $\int C(f)\psi\, dq = 0, \quad \forall f$

Form a linear vector space $\mathcal{CI}$

Multiply eq. by ψ: ε^{-1} term disappears

Find a conservation law:

$$\partial_t \Big(\int_{\mathbb{H}_1} f(x,q,t)\, \psi(q)\, dq \Big) + \nabla_x \cdot \Big(\int_{\mathbb{H}_1} f(x,q,t)\, \psi(q)\, e_1(q)\, dq \Big) = 0$$

Have used that ∂_t or ∇_x and $\int \ldots dq$ can be interchanged

Limit fully determined if dim $\mathcal{CI} =$ dim $\mathcal{E} = 4$

$\mathcal{CI} =$ Span$\{1\}$. Interaction preserves mass but no other quantity

Due to self-propulsion, no momentum conservation

dim $\mathcal{CI} = 1 <$ dim $\mathcal{E} = 4$. Is the limit problem ill-posed ?

Use of CI: mass conservation eq.

Proof that $\psi(q) = 1$ is a CI ?

Obvious. $C(f) = \nabla_q \cdot \big[\dots \big]$ is a divergence

By Stokes theorem on the sphere, $\int C(f)\, dq = 0$

Use of the CI $\psi(q) = 1$: Get the conservation law

$$\partial_t \Big(\int_{\mathbb{H}_1} f(x,q,t)\, dq \Big) + \nabla_x \cdot \Big(\int_{\mathbb{H}_1} f(x,q,t)\, e_1(q)\, dq \Big) = 0$$

With $f = \rho M_{\bar{q}}$ we have

$$\int f(x,v,t)\, dv = \rho(x,t), \quad \int f(x,v,t)\, e_1(q)\, dq = c_1 \rho(x,t) e_1\big(\bar{q}(x,t)\big)$$

We end up with the mass conservation eq.

$$\partial_t \rho + c_1 \nabla_x \cdot \big(\rho e_1(\bar{q})\big) = 0$$

Given $\bar{q}_0 \in \mathbb{H}_1$, Define $\mathcal{R}_{\bar{q}_0}(f) = \nabla_q \cdot \left[M_{\bar{q}_0} \nabla_q \left(\frac{f}{M_{\bar{q}_0}} \right) \right]$

Note $f \to \mathcal{R}_{\bar{q}_0}(f)$ is linear and $C(f) = \mathcal{R}_{\bar{q}_f}(f)$

A function $\psi_{\bar{q}_0}(q)$ is a GCI associated to $\bar{q}_0$, iff

$$\int \mathcal{R}_{\bar{q}_0}(f) \psi_{\bar{q}_0} \, dq = 0, \quad \forall f \text{ such that } P_{q_0^\perp} \left[\left(\int_{\mathbb{H}_1} f(q) \, (q \otimes q) \, dq \right) \bar{q}_0 \right] = 0$$

The set of GCI $\mathcal{G}_{\bar{q}_0}$ is a linear vector space

Theorem: Given $\bar{q}_0 \in \mathbb{H}_1$, $\mathcal{G}_{\bar{q}_0}$ is the 4-dim vector space :

$$\mathcal{G}_{\bar{q}_0} = \{ q \mapsto \alpha + h(q \cdot \bar{q}_0) \, \beta \cdot q, \text{ with arbitrary } \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{H} \text{ with } \beta \cdot \bar{q}_0 = 0 \}.$$

Introduce $r = q \cdot \bar{q}_0 \in [-1, 1]$. h is the unique solution in V of:

$$-(1-r^2)^{-3/2} \exp\left(-\frac{k}{2} r^2\right) \frac{d}{dr} \left[(1-r^2)^{5/2} \exp\left(\frac{k}{2} r^2\right) \frac{dh}{dr} \right] + (k\, r^2 + 3) \, h(r) = -r$$

$$V = \{ h \mid (1 - r^2)^{3/4} h \in L^2(-1, 1), \ (1 - r^2)^{5/4} h' \in L^2(-1, 1) \}$$

Furthemore, h is odd and non-positive for $r \geq 0$

Use of GCI: equation for $\bar{q}(x,t)$

Use GCI $\quad h(q \cdot \bar{q}_0)\beta \cdot q$ for $\beta \in \mathbb{H}$ with $\beta \cdot \bar{q}_0 = 0$

Equivalently, use the quaternion valued function

$\psi_{\bar{q}_0}(q) = h(q \cdot \bar{q}_0) P_{\bar{q}_0^\perp} q$

Multiply FP eq by GCI $\psi_{\bar{q}_{f^\varepsilon}}$: $O(\varepsilon^{-1})$ terms disappear

$$\int C(f)\, \vec{\psi}_{\bar{q}_f}\, dv = \int \mathcal{R}_{\bar{q}_f}(f)\, \psi_{\bar{q}_f}\, dq = 0 \quad \text{by property of GCI}$$

Gives: $$\int (\partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon)\, \psi_{\bar{q}_{f^\varepsilon}}\, dq = 0$$

As $\varepsilon \to 0$: $f^\varepsilon \to \rho M_{\bar{q}}$ and $\psi_{\bar{q}_{f^\varepsilon}} \to \psi_{\bar{q}}$ Leads to:

$$\int \big(\partial_t(\rho M_{\bar{q}}) + e_1(q) \cdot \nabla_x(\rho M_{\bar{q}})\big)\, \psi_{\bar{q}}\, dq = 0$$

Not a conservation equation

because of dependence of $\psi_{\bar{q}}$ upon (x,t) through $\bar{q}$

∂_t or ∇_x and $\int \ldots dq$ cannot be interchanged

Equation for $\bar q(x,t)$

Takes the form:

$$\rho\Big(\partial_t \bar q + c_2\big(e_1(\bar q)\cdot\nabla_x\big)\bar q\Big) + c_3[e_1(\bar q)\times\nabla_x\rho]\,\bar q$$
$$+c_4\rho\big[(\nabla_{x,\mathrm{rel}}\bar q)e_1(\bar q) + (\nabla_{x,\mathrm{rel}}\cdot\bar q)e_1(\bar q)\big]\,\bar q = 0$$

where

$$(\nabla_{x,\mathrm{rel}}\bar q) = (\partial_{x_i,\mathrm{rel}}\bar q)_{i=1,2,3} = \big((\partial_{x_i}\bar q)\bar q^*\big)_{i=1,2,3} \in \mathbb{H}^3_{\mathrm{Im}}$$
$$(\nabla_{x,\mathrm{rel}}\cdot\bar q) = \sum_{i=1,2,3}(\partial_{x_i,\mathrm{rel}}\bar q)_i = \sum_{i=1,2,3}\big((\partial_{x_i}\bar q)\bar q^*\big)_i \in \mathbb{R}$$

$\mathbb{H}_{\mathrm{Im}} = \{q\in\mathbb{H},\ \mathrm{Im}q = 0\} \approx \mathbb{R}^3$

$(\partial_{x_i,\mathrm{rel}}\bar q)_j$ = j-th component of $\partial_{x_i,\mathrm{rel}}\bar q$

$(\nabla_{x,\mathrm{rel}}\bar q)e_1(\bar q) = \big((\partial_{x_i,\mathrm{rel}}\bar q)\cdot e_1(\bar q)\big)_{i=1,2,3}$

Coefficients c_2 and c_4 depend on GCI h

Resulting system: SOHQ

Self-Organized Quaternionic Hydrodynamics (SOHQ)

System for density $\rho(x,t)$ and quaternion orientation $\bar{q}(x,t)$:

$$\partial_t \rho + c_1 \nabla_x \big(\rho e_1(\bar{q})\big) = 0$$

$$\rho\Big(\partial_t \bar{q} + c_2\big(e_1(\bar{q}) \cdot \nabla_x\big)\bar{q}\Big) + c_3[e_1(\bar{q}) \times \nabla_x \rho]\,\bar{q}$$
$$+ c_4 \rho\big[(\nabla_{x,\mathrm{rel}} \bar{q}) e_1(\bar{q}) + (\nabla_{x,\mathrm{rel}} \cdot \bar{q}) e_1(\bar{q})\big]\bar{q} = 0$$

$$|\bar{q}| = 1$$

4. Comparison with SOH dynamics for Vicsek

Vicsek mean-field model for $f^\varepsilon(x,v,t)$

position $x \in \mathbb{R}^3$, velocity orientation $v \in \mathbb{S}^2$

As $\varepsilon \to 0$, $f^\varepsilon(x,v,t) \to \rho(x,t) \mathcal{M}_{\Omega(x,t)}(v)$

$\rho(x,t) \geq 0$, $\Omega(x,t) \in \mathbb{S}^2$

$\mathcal{M}_\Omega(v) = \frac{1}{Z} \exp\big(k(\Omega \cdot v)\big)$, $\quad \int_{\mathbb{S}^2} \mathcal{M}_\Omega(v)\, dv = 1$

$(\rho(x,t), \Omega(x,t))$ solves SOH model:

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \big(\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega\big) + c_3 \, P_{\Omega^\perp} \nabla_x \rho = 0, \quad P_{\Omega^\perp} = \mathrm{Id} - \Omega \otimes \Omega$$

$$|\Omega| = 1$$

Similar to Compressible Euler eqs. of gas dynamics

System of hyperbolic eqs.

But major differences:

Geometric constraint $|\Omega| = 1$: requires $P_{\Omega^\perp}$ to be maintained

System is non conservative due to the presence of $P_{\Omega^\perp}$

$c_2 \neq c_1$: loss of Galilean invariance

SOHQ model in frame representation

$\bar{q}(x,t) \in \mathbb{H}_1$ encodes rotation $\Lambda(x,t) \in SO(3)$

$\Lambda(x,t)$ describes agents' local average body attitude

$\Omega(x,t) = \Lambda(x,t) e_1 = e_1(\bar{q}(x,t))$: direction of motion
$u(x,t) = \Lambda(x,t) e_2 = e_2(\bar{q}(x,t))$: belly to back
$v(x,t) = \Lambda(x,t) e_3 = e_3(\bar{q}(x,t))$: right to left wing

SOHQ model equivalent to

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \big(\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \big) + P_{\Omega^\perp} \big(c_3 \nabla_x \rho - c_4 \rho \, r(\Omega, u, v) \big) = 0$$

$$\rho \big(\partial_t u + c_2 (\Omega \cdot \nabla_x) u \big) - u \cdot \big(c_3 \nabla_x \rho - c_4 \rho \, r(\Omega, u, v) \big) \Omega + c_4 \rho \, \delta(\Omega, u, v) v = 0$$

$$\rho \big(\partial_t v + c_2 (\Omega \cdot \nabla_x) v \big) - v \cdot \big(c_3 \nabla_x \rho - c_4 \rho \, r(\Omega, u, v) \big) \Omega - c_4 \rho \, \delta(\Omega, u, v) u = 0$$

$$|\Omega| = |u| = |v| = 1, \quad \Omega \cdot u = u \cdot v = v \cdot \Omega = 0$$

with $r(\Omega, u, v)$ (for rotational) and $\delta(\Omega, u, v)$ (for divergence):

$$r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v \ \in \mathbb{R}^3$$

$$\delta(\Omega, u, v) u = \big[(\Omega \cdot \nabla_x) u \big] \cdot v + \big[(u \cdot \nabla_x) v \big] \cdot \Omega + \big[(v \cdot \nabla_x) \Omega \big] \cdot u \ \in \mathbb{R}$$

Compare eqs for ρ and Ω:

SOH: Coarse-grained Vicsek model

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \left(\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + \; c_3 \, P_{\Omega^\perp} \nabla_x \rho = 0$$

SOHQ: Coarse-grained body orientation model

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \left(\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega \right) + \; P_{\Omega^\perp} \left(c_3 \nabla_x \rho - c_4 \rho \, r(\Omega, u, v) \right) = 0$$

Difference is the term

$$r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v$$

Shows how differences in body orientation affect direction of the flock

Answers

Can we quantify the difference between the Vicsek and body alignment models ?

YES: by using coarse-grained models SOH and SOHQ respectively

Is body-alignment just Vicsek for direction of motion with frame dynamic superimposed to it ?

Answer is 'NO'

Or does body-alignment provide genuinely new dynamic ?
i.e. do gradients of body frames orientation influence direction of motion ?

Answer is 'YES'

5. Conclusion

Summary & Perspectives

New collective dynamics model relying on full body alignment

- body frame alignment $\Leftrightarrow$ quaternion nematic alignment

Coarse-grained model is SOHQ

- First order PDE for density and local average quaternion
- describes dynamics of agents' local mean body frame
- dynamics genuinely $\neq$ from velocity alignment (Vicsek or SOH)

Perspectives

- analysis of the model
- rigorous proof of convergence
- numerical simulations
- Higher dimensions