Kinetic Theory and Stochastic Differential Games Toward a Systems Sociology Approach

Application to criminality dynamics

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Modeling and Control in Social Dynamics

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New Trends in Behavioral Economy and Sociology

Preliminary Reasonings

- The dynamics of social and economic systems are necessarily based on individual behaviors, by which single subjects express, either consciously or unconsciously, a particular strategy, which is heterogeneously distributed. The latter is often based not only on their own individual purposes, but also on those they attribute to other agents.

- In the last few years, a radical philosophical change has been undertaken in social and economic disciplines. An interplay among Economics, Psychology, and Sociology has taken place, thanks to a new cognitive approach no longer grounded on the traditional assumption of rational socio-economic behavior. Starting from the concept of bounded rationality, the idea of Economics as a subject highly affected by individual (rational or irrational) behaviors, reactions, and interactions has begun to impose itself.

- A key experimental feature of such systems is that interaction among heterogeneous individuals often produces unexpected outcomes, which were absent at the individual level, and are commonly termed emergent behaviors. Mathematical sciences can significantly contribute to a deeper understanding of the relationships between individual behaviors and the collective social outcomes they spontaneously generate.

New Trends in Behavioral Economy and Sociology

Selected bibliography in Economical and Social Sciences

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New Trends in Behavioral Economy and Sociology

Additional Reasonings

• Socio-economic systems can be described as ensembles of several living entities, namely **active particles**, able to develop **behavioral strategies**, by which they interact with each other. Their strategies are **heterogeneously distributed** and **change in time** in consequence of the interactions, since active particles can update them by **learning from past experiences**.

• Social systems exhibit various complexity features. In particular, **interactions among individuals need not have an additive linear character**. As a consequence, the global impact of a given number of entities (field entities) over a single one (test entity) cannot be assumed to merely consist in the linear superposition of the actions exerted individually by single field entities. **This nonlinear feature represents a serious conceptual difficulty** to the derivation, and subsequent analysis, of mathematical models for that type of systems.

Additional Reasonings

• The new point of view presents economics as an evolving complex system, where interactions among heterogeneous individuals and the interplay among different dynamics can produce even unpredictable emerging outcomes.

N.N. Taleb, **The Black Swan: The Impact of the Highly Improbable**, 2007.

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New Trends in Behavioral Economy and Sociology

Complex Systems, Game Theory, Networks

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• H. Gintis, **Game Theory Evolving**, 2nd Ed., Princeton University Press, Princeton NJ, (2009).

• D. Helbing, **Quantitative Sociodynamics. Stochastic Methods and Models of Social Interaction Processes**. Springer Berlin Heidelberg, 2nd edition, (2010).

• N. Bellomo, D. Knopoff, and J. Soler, On the difficult interplay between life "complexity" and mathematical sciences, *Math. Models Methods Appl. Sci.*, 23, 1861-1913, (2013).

• A. Albert and A.-L. Barabási, Statistical mechanics of complex networks, *Rev. Mod. Phys.*, 74, 47-97 (2002)

• A. Barrat, M. Bathélemy, and A. Vespignani, **The Structure and Dynamics of Networks**, Princeton University Press, (2006).

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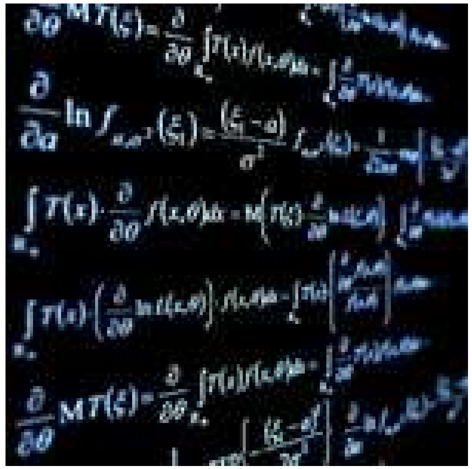
2. **Mathematical Tools: Kinetic Theory and Evolutive Stochastic Games**



3. **Modeling on the Onset and Dynamics of Criminality**
4. **Simulations Perspectives**

Mathematical Tools

Strategy

- Understanding the links between the dynamics of living systems and their complexity features;
- Derivation a general mathematical structure, consistent with the aforesaid features, with the aim of offering the conceptual framework toward the derivation of specific models;
- Design of specific models corresponding to well defined classes of systems by implementing the structure by suitable micro-scale models of individual-based interactions;
- Validation of models by quantitative comparison of the dynamics predicted by them with that one resulting from empirical data. Moreover, emerging behaviors that are repeated should be reproduced at a qualitative level.
- Develop the validation process by investigating the ability of the model to predict rare events, the so-called "black swan".

Mathematical Tools

Representation

• The description of the overall state of the system is delivered by the ***generalized one-particle distribution function***

$$f_i = f_i(t, u) \; : \qquad [0, T] \times D_u \; \to \mathbb{R}_+,$$

such that $f_i(t, u)\, du$ denotes the number of ***active particles*** whose state, at time t, is in the interval $[u, u + du]$ of the i-th ***functional subsystem***.

• u is the ***activity variable*** which can also be a vector.

Particles *play a game* at each interaction with an output that technically depends on their strategy often related to their well being. The output of the game generally is not deterministic even when a causality principle is identified.

The following particles are involved: (i) **test particles** (assumed to be representative of the whole system) whose distribution function is $f_i = f_i(t, u)$, **field particles** whose distribution function is $f_k = f_k(t, u^*)$, and **candidate particles**, whose distribution function is $f_h = f_h(t, u_*)$, which are candidate to take the state of the test particle after interaction with the field particles..

Mathematical Tools

Stochastic Games

1. **Competitive (dissent):** One of the interacting particle increases its status by taking advantage of the other, obliging the latter to decrease it. Therefore the competition brings advantage to only one of the two.

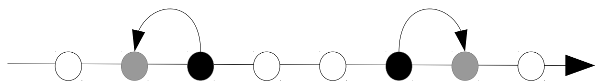
2. **Cooperative (consensus):** The interacting particles exchange their status, one by increasing it and the other one by decreasing it. Therefore, the interacting active particles show a trend to share their micro-state.

3. **Learning:** One of the two modifies, independently from the other, the micro-state, in the sense that learns by reducing the distance between them.

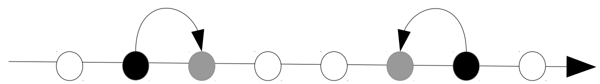
4. **Hiding-chasing:** One of the two attempts to increase the overall distance from the other, which attempts to reduce it.

5. **Transition:** Interactions produce a transition from one functional subsystem to the other.

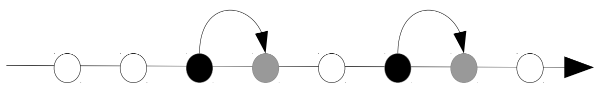
Mathematical Tools



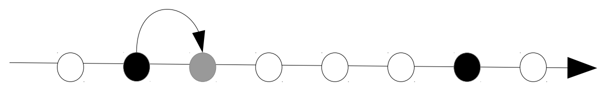
(a) Competition



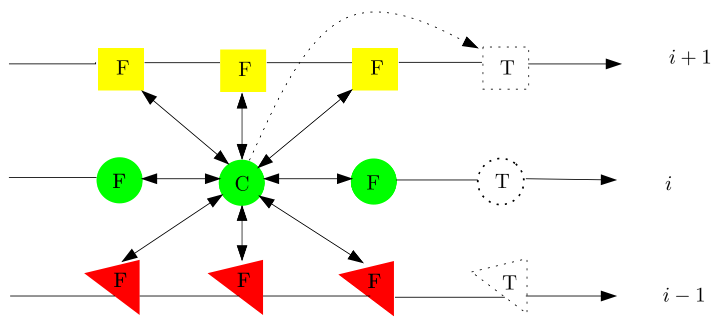
(b) Cooperation



(c) Hiding-chasing



(d) Learning



Mathematical Tools

Search of a Mathematical Structure

H.1. Candidate or test particles interact with the field particles in the interaction domain Ω. Interactions are weighted by the ***interaction rates*** $\eta_{hk}[\mathbf{f}]$ and $\mu_{hk}[\mathbf{f}]$ supposed to depend on the local distribution function in the position of the field particles.

H.2. A candidate particle modifies its state according to the probability density: $\mathcal{C}_{hk}^{i}[\mathbf{f}](u_* \to u|u_*, u)$, which denotes the probability density that a candidate particles of the h-subsystems with state u_* reaches the state u in the i-th subsystem after an interaction with the field particles of the k-subsystems with state u^*.

Balance within the space of microscopic states and Structures

Variation rate of the number of active particles

$$= \textit{Inlet flux rate caused by conservative interactions}$$

$$- \textit{Outlet flux rate caused by conservative interactions},$$

where the inlet flux includes the dynamics of mutations.

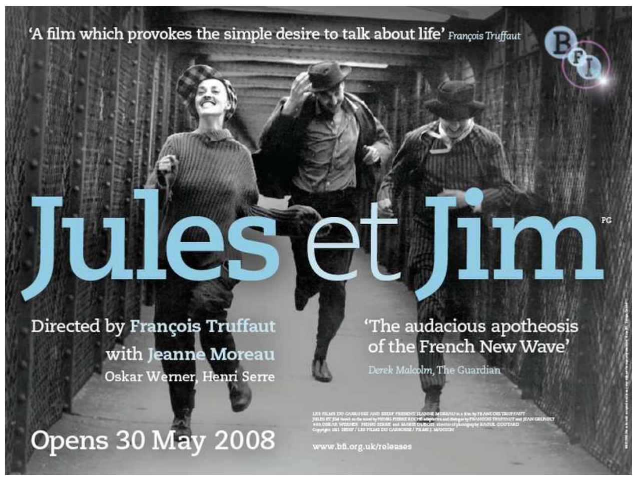
Mathematical Tools

Mathematical Structures - Nonlinear interactions and the interplay of different dynamics can generate the black swan

$$\partial_t f_i(t,u) = \left(\mathcal{C}_i[\mathbf{f}] - \mathcal{L}_i[\mathbf{f}]\right)(t,u)$$

$$= \sum_{h,k=1}^{n} \int_{D_u \times D_u} \eta_{hk}[\mathbf{f}](u_*,u^*)\mathcal{C}_{hk}^{i}[\mathbf{f}]\left(u_* \to u | u_*, u^*\right) f_h(t,u^*) f_k(t,u^*)\, du_*\, du^*$$

$$- f_i(t,u) \sum_{k=1}^{n} \int_{D_u} \eta_{ik}[\mathbf{f}](u,u^*)\, f_k(t,u^*)\, du^*,$$



Mathematical Tools

Summarizing

- The overall system is partitioned into functional subsystems, whose elements, called active particles, have the ability to collectively develop a common strategy;
- The strategy is heterogeneously distributed among the components and corresponds to an individual state, defined activity, of the active particles;
- The state of each functional subsystem is defined by a probability distribution over the activity variable;
- Active particles interact within the same functional subsystem as well as with particles of other subsystems, and with agents from the outer environment;
- Interactions generally are nonlinearly additive and are modeled as stochastic games, meaning that the outcome of a single interaction event can be known only in probability;
- The evolution of the probability distribution is obtained by a balance of particles within elementary volumes of the space of microscopic states, the inflow and outflow of particles being related to the aforementioned interactions.

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Modeling on the Onset and Dynamics of Criminality

Functional subsystems, representation, and structure

$i = 1$ Normal citizens, whose microscopic state is identified by their wealth, which constitutes the attraction for the eventual perpetration of criminal acts.

$i = 2$ Criminals, whose microscopic state is given by their criminal ability, namely their ability to succeed in the perpetration of illegal acts.

$i = 3$ Detectives who chase criminals according to their individual ability.

Functional subsystem	**Micro-state**
$i = 1$, citizens	$u \in D_1$, wealth
$i = 2$, criminals	$u \in D_2$, criminal ability
$i = 3$, detectives	$u \in D_3$, experience/prestige

Table 1: Microscopic variable for each functional subsystem.

Modeling on the Onset and Dynamics of Criminality

The representation of the system is delivered by the distribution functions

$$f_i : [0, T] \times D_i \to \mathbb{R}_+, \qquad \mathrm{i} = 1, 2, 3,$$

where $f_i(t, u)\, du$ denotes, under suitable integrability conditions, the number of active particles of functional subsystem i whose state, at time t, is in the interval $[u, u + du]$. Therefore the *size* of group i is

$$n_i(t) = \int_{D_i} f_i(t, u)\, du, \quad i = 1, 2, 3,$$

while, the *total size* of the population is given by

$$N(t) = \sum_{i=1}^{3} n_i(t) \cong N_0,$$

which is assumed to remain constant in time. By normalizing with respect to $N(0)$, f_i defines the fraction of individuals belonging to a certain functional subsystem for each time t.

Modeling on the Onset and Dynamics of Criminality

$$
\begin{aligned}
&\partial_t f_i(t,u) = J_i[\mathbf{f}](t,u) = \\
&= \sum_{h,k=1}^{3} \int_{D_h}\int_{D_k} \eta_{hk}(u_*,u^*)\mathcal{B}^i_{hk}(u_* \to u|u_*,u^*) f_h(t,u_*) f_k(t,u^*)\, du_*\, du^* \\
&\quad - f_i(t,u)\sum_{k=1}^{3}\int_{D_k} \eta_{ik}(u,u^*)\, f_k(t,u^*)\, du^* \\
&\quad + \int_{D_i} \mu_i(u_*,\mathbb{E}_i)\mathcal{M}_i(u_* \to u|u_*,\mathbb{E}_i) f_i(t,u_*) du_* \\
&\quad - \mu_i(u,\mathbb{E}_i) f_i(t,u),
\end{aligned}
$$

where $\eta_{hk}(u_*,u^*)$ and $\mu_h(u_*,\mathbb{E}_h)$ are, respectively, the encounter rate of individual based interactions and that between a candidate h-particle and the mean activity. Moreover, $\mathcal{B}^i_{hk}(u_* \to u|u_*,u^*)$ and $\mathcal{M}_h(u_* \to u|u_*,\mathbb{E}_h)$ are, respectively, the probability density for the state transition of individual based interactions and that between a candidate h-particle and the mean activity.

Modeling on the Onset and Dynamics of Criminality

Interaction	**Qualitative description**	η
$\boxed{1} \leftrightarrow \textcircled{1}$	Closer social states tend to interact more frequently	$\eta_{11}(u_*, u^*) = \eta^0 \left(1 - \lvert u_* - u^* \rvert\right)$
$\boxed{2} \leftrightarrow \textcircled{2}$	Experienced lawbreakers are more expected to expose themselves	$\eta_{22}(u_*, u^*) = \eta^0 (u_* + u^*)$
$\boxed{2} \leftrightarrow \textcircled{3}$	Experienced detectives are more likely to *hunt* less experienced criminals	$\eta_{23}(u_*, u^*) = \eta^0 \big((1 - u_*) + u^*\big)$
$\boxed{3} \leftrightarrow \textcircled{2}$		$\eta_{32}(u_*, u^*) = \eta^0 \big(u_* + (1 - u^*)\big)$

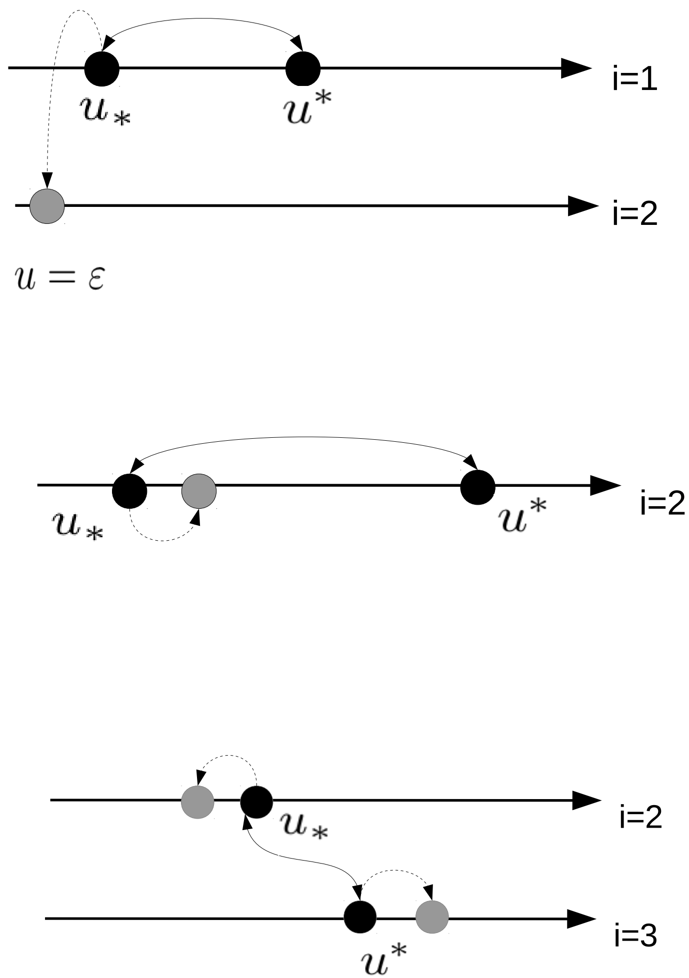
Table 2: Non-trivial interactions between a h-candidate particle (represented by a square) with state u_* and a k-field particle (represented by a circle) with state u^*.

Modeling on the Onset and Dynamics of Criminality

Interaction	**Qualitative description**	μ
$\boxed{2} \leftrightarrow \mathbb{E}_2$	Experienced criminals are more expected to expose themselves	$\mu_2(u_*, \mathbb{E}_2) = \mu^0 \lvert u_* - \mathbb{E}_2 \rvert$
$\boxed{3} \leftrightarrow \mathbb{E}_3$	Detectives interact with with the mean value through the mean micro-state distance	$\mu_3(u_*, \mathbb{E}_3) = \mu^0 \lvert u_* - \mathbb{E}_3 \rvert$

Table 3: Non-trivial interactions between a h-candidate particle (represented by a square) with activity u_* and the mean activity value $\mathbb{E}_h$.

Modeling on the Onset and Dynamics of Criminality



Modeling on the Onset and Dynamics of Criminality

Parameters involved in the table of games	
α_T	Susceptibility of citizens to become criminals
α_B	Susceptibility of criminals to reach back the state of normal citizen
β	Learning dynamics among criminals
γ	Motivation/efficacy of security forces to catch criminals
λ	Learning dynamics among detectives

Modeling on the Onset and Dynamics of Criminality

$$\partial_t f_1(t,u) = -\alpha_T(1-u)f_1(t,u)\int_0^1 \eta_{11}(u,u^*)u^* f_1(t,u^*)du^*$$
$$+\frac{1}{\varepsilon}\alpha_B \chi_{[0,\varepsilon)}(u)\int_0^1\int_0^1 \eta_{23}(u_*,u^*)(1-u_*)u^* f_2(t,u_*)f_3(t,u^*)du_*du^*,$$

$$\partial_t f_2(t,u) = \frac{1}{\varepsilon}\alpha_T \chi_{[0,\varepsilon)}(u)\int_0^1\int_0^1 \eta_{11}(u_*,u^*)(1-u_*)u^* f_1(t,u_*)f_1(t,u^*)du_*du^*$$
$$+\int_0^1 \chi_{[\beta u^*,1]}(u)\frac{1}{1-\beta u^*}\eta_{22}\left(\frac{u-\beta u^*}{1-\beta u^*},u^*\right) f_2\left(t,\frac{u-\beta u^*}{1-\beta u^*}\right) f_2(t,u^*)du^*$$
$$+\int_0^1 \chi_{[0,1-\gamma u^*]}(u)\frac{1}{1-\gamma u^*}\eta_{23}\left(\frac{u}{1-\gamma u^*},u^*\right)\left[1-\alpha_B\left(1-\frac{u}{1-\gamma u^*}\right)u^*\right]$$
$$\times f_2\left(t,\frac{u}{1-\gamma u^*}\right) f_3(t,u^*)du^*$$
$$-f_2(t,u)\sum_{k=2}^{3}\int_0^1 \eta_{2k}(u,u^*)f_k(t,u^*)du^*$$
$$+\frac{1}{\beta}\chi_{[(1-\beta)\mathbb{E}_2,\beta+(1-\beta)\mathbb{E}_2]}(u)\mu_2\left(\frac{u-(1-\beta)\mathbb{E}_2}{\beta},\mathbb{E}_2\right) f_2\left(t,\frac{u-(1-\beta)\mathbb{E}_2}{\beta}\right)$$
$$-\mu_2(u,\mathbb{E}_2)f_2(t,u),$$

Modeling on the Onset and Dynamics of Criminality

$$
\begin{aligned}
\partial_t f_3(t,u) &= \int_0^1 \chi_{[\gamma u^*,1]}(u)\frac{1}{1-\gamma u^*}\eta_{32}\left(\frac{u-\gamma u^*}{1-\gamma u^*},u^*\right)\\
&\times f_3\left(t,\frac{u-\gamma u^*}{1-\gamma u^*}\right) f_2(t,u^*)du^* - f_3(t,u)\int_0^1 \eta_{32}(u,u^*)f_2(t,u^*)du^*\\
&+\frac{1}{\lambda}\chi_{[(1-\lambda)\mathbb{E}_3,\lambda+(1-\lambda)\mathbb{E}_3]}(u)\mu_3\left(\frac{u-(1-\lambda)\mathbb{E}_3}{\lambda},\mathbb{E}_3\right) f_3\left(t,\frac{u-(1-\lambda)\mathbb{E}_3}{\lambda}\right)\\
&-\mu_3(u,\mathbb{E}_3)f_3(t,u).
\end{aligned}
$$

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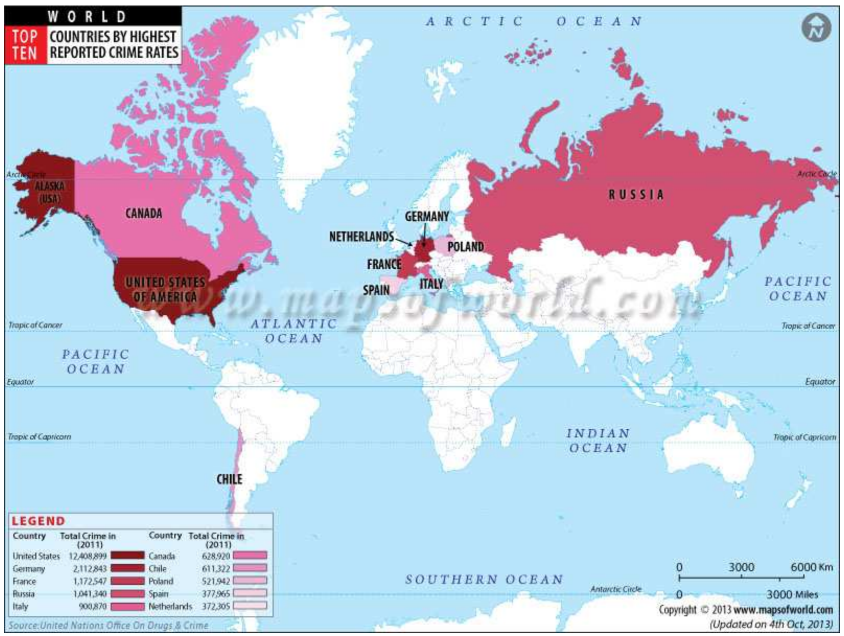


Case Studies and Trends

Case 1 - Role of the mean wealth

Decreasing mean wealth of the society	$\Longrightarrow$	Increasing number of criminals Increasing criminal ability

Case 2 - Role of the shape of wealth distribution

Poor society	Equal distribution	$\Longrightarrow$	Slow growth in the number of criminals
	Unequal distribution	$\Longrightarrow$	Fast growth in the number of criminals
Rich society	Equal distribution	$\Longrightarrow$	Fast decrease in the number of criminals
	Unequal distribution	$\Longrightarrow$	Slow decrease in the number of criminals

Modeling on the Onset and Dynamics of Criminality

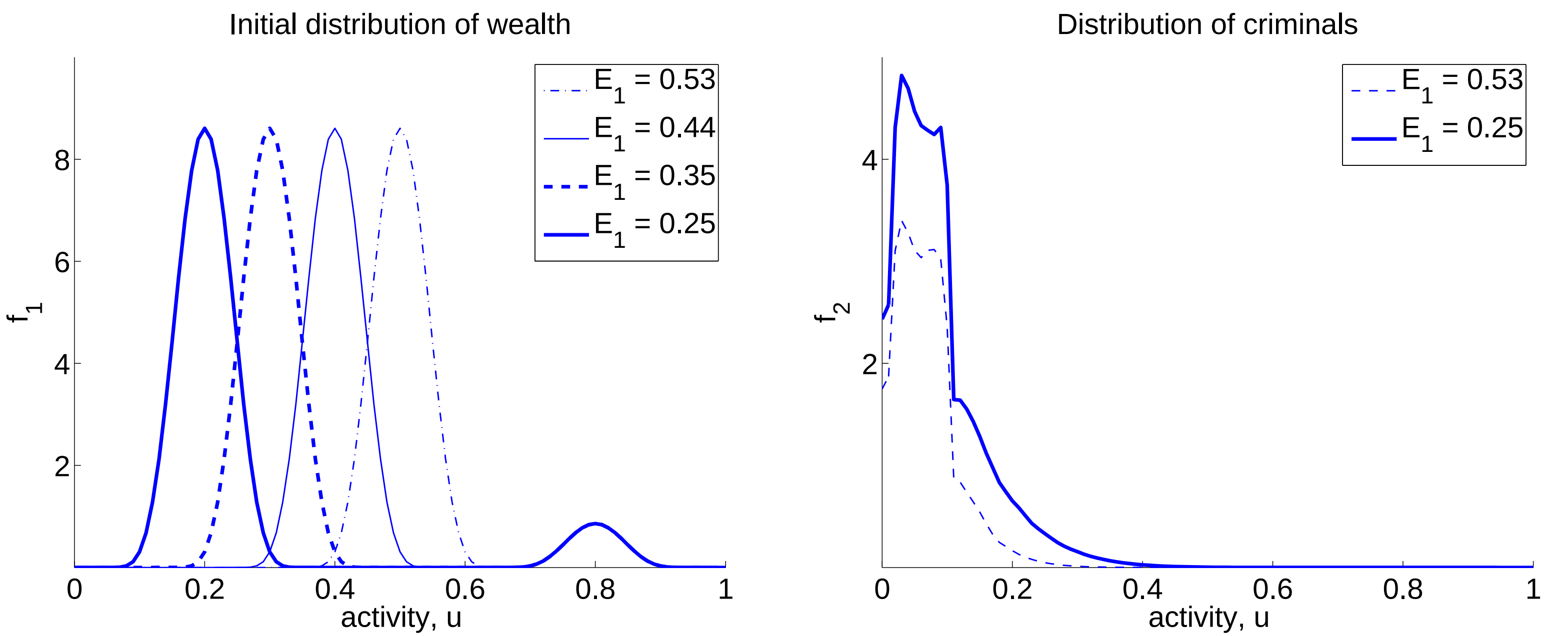
Case 3 - Role of the number of detective

Large number of detectives	$\Longrightarrow$	Decreasing number of criminals

Case 4 - Role of parameters α_T and γ

Low susceptibility to criminality	$\Longrightarrow$ $\Longrightarrow$	Number of criminals under control Criminal ability under control
Increasing ability of detectives	$\Longrightarrow$ $\Longrightarrow$	Decreasing number of criminals (with small sensitivity) Decreasing criminal ability

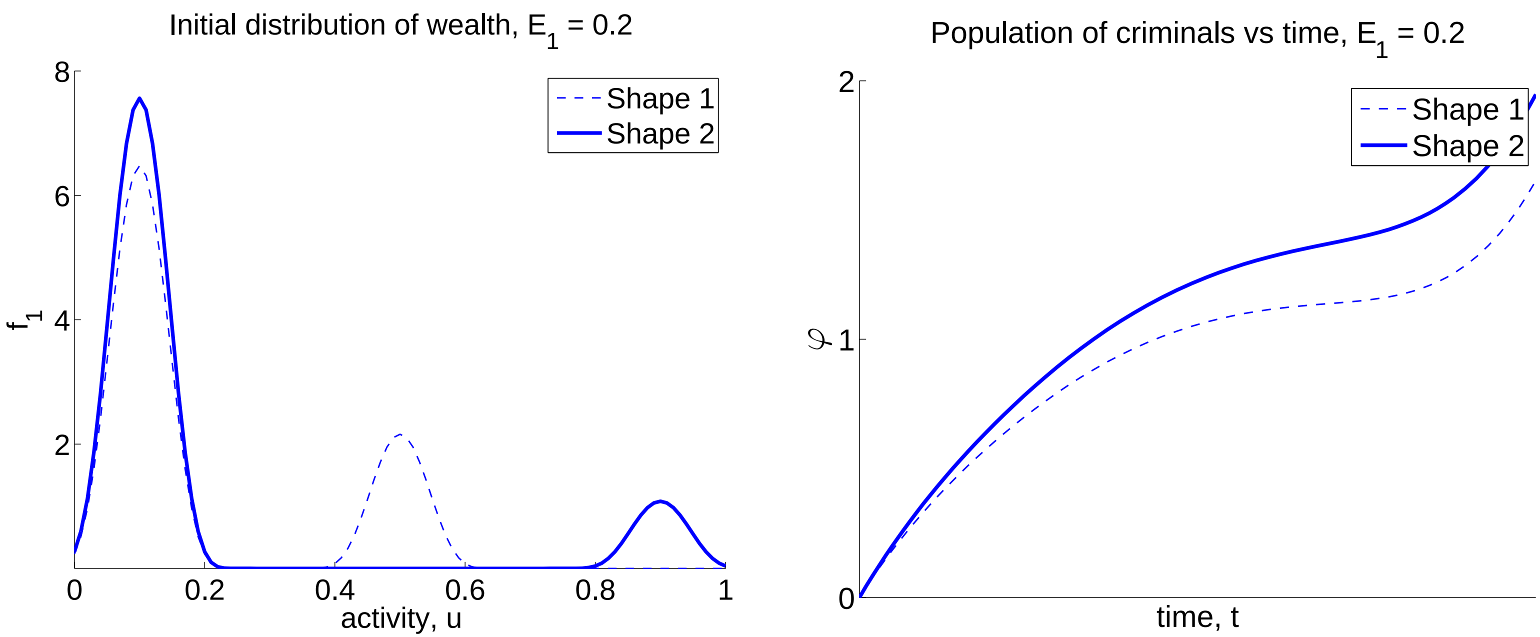
Simulations - Case 1



(a) (b)

Figure 1: (a) Initial wealth distributions used for the simulations, corresponding to different mean wealth values. All of them consist in a fixed small rich cluster and a large poorer cluster centered in different points of the activity domain. (b) Large time distribution of criminals for two of the selected mean wealth values.

Simulations - Case 2

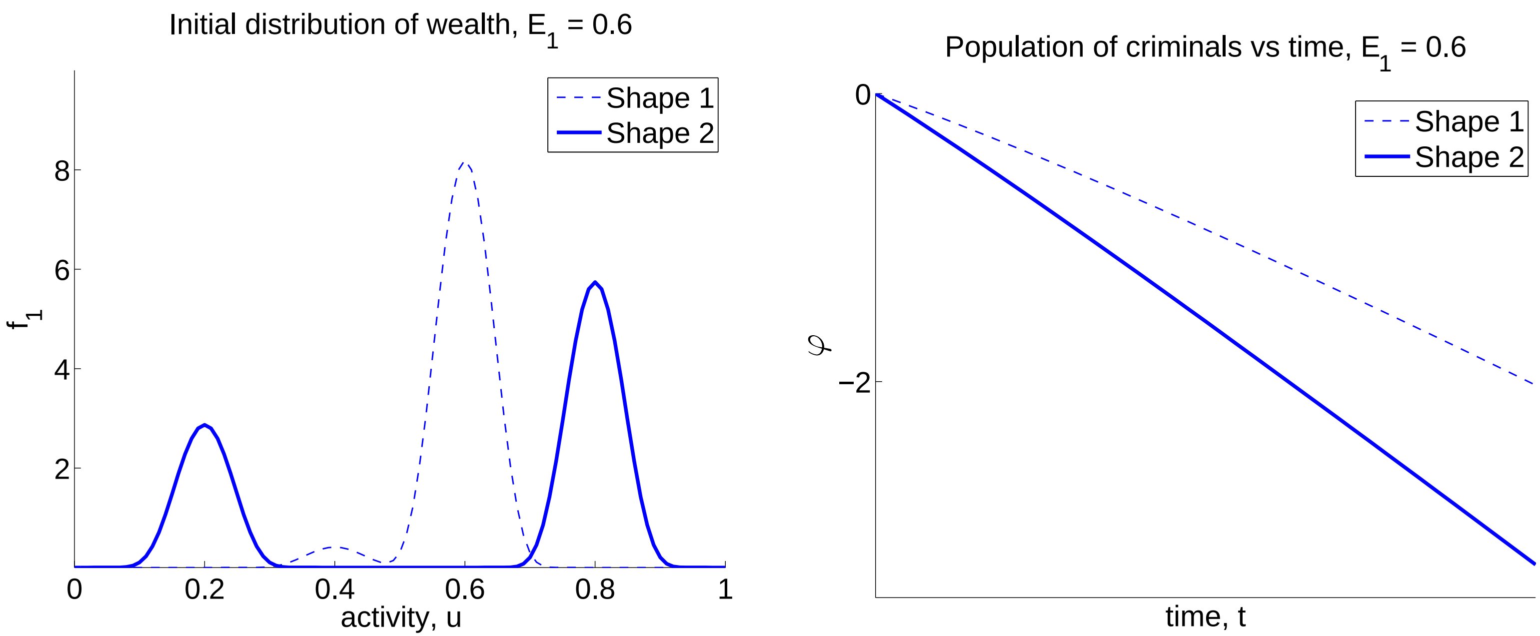


(a) (b)

Figure 2: (a) Two wealth distributions lead to different growths of the population of criminals (b). The more unequal distribution generates a greater increase.

$$\varphi(t) = \frac{n_2(t) - n_2(0)}{n_2(0)} \times 10^2,$$

Simulations - Case 2



(c) (d)

Figure 3: (c) Two wealth distributions for a rich society with $\mathbb{E}_1 = 0.6$ that generate, for the same set of parameters, a reduction in the number of criminals (d).

Simulations - Case 3

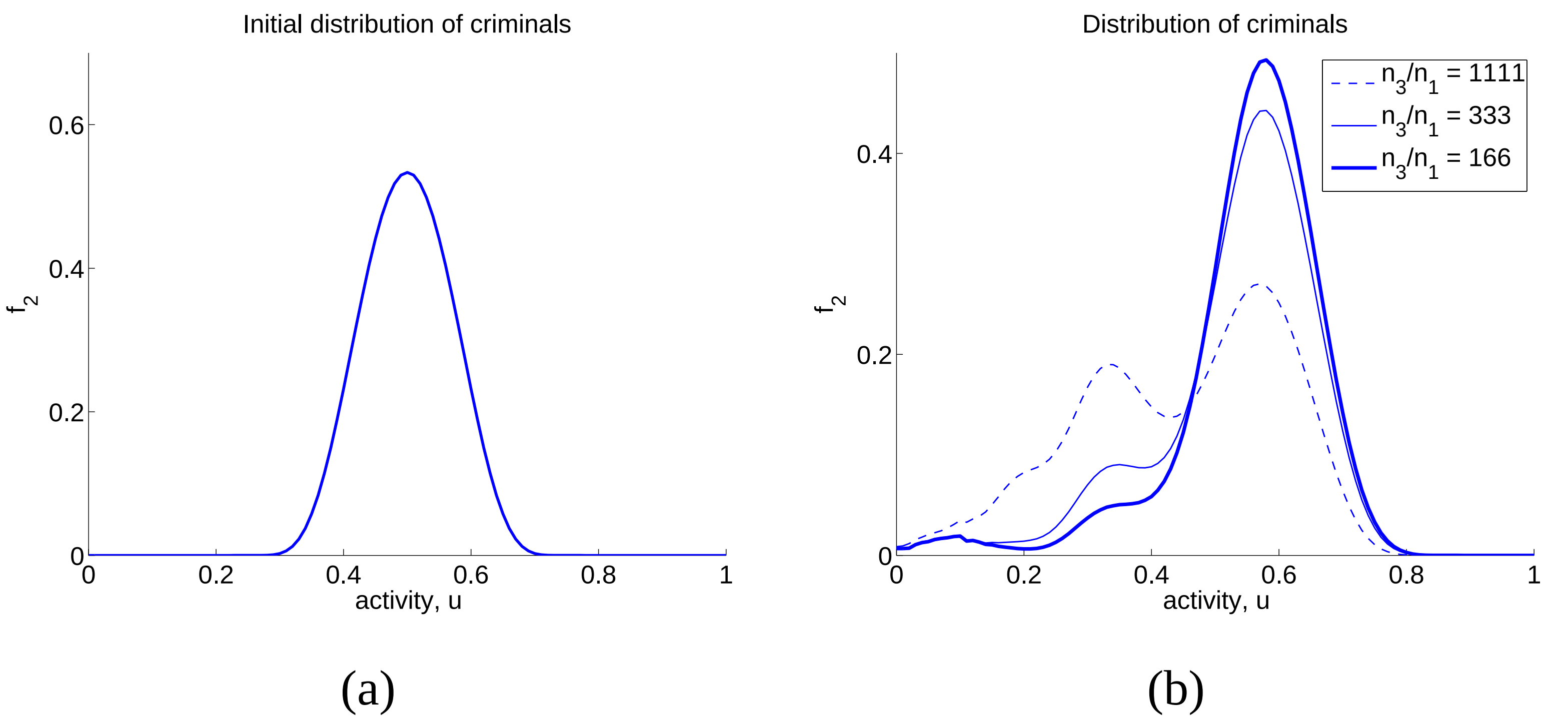
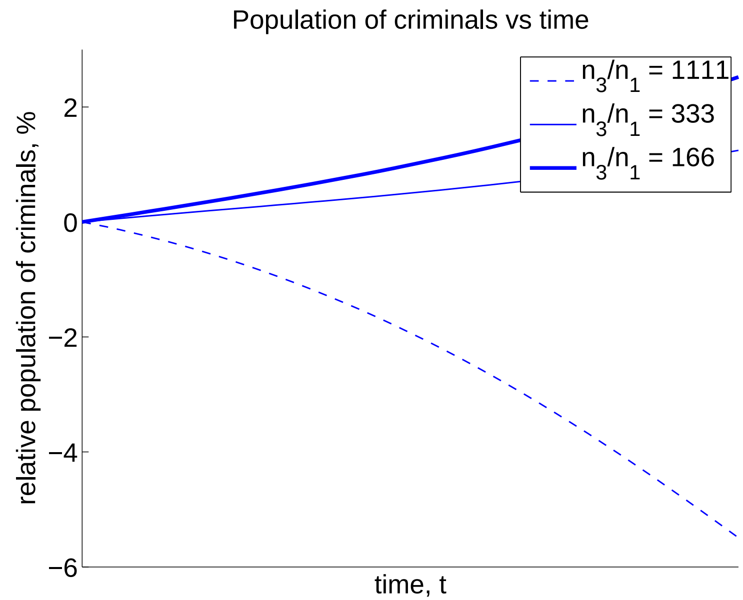
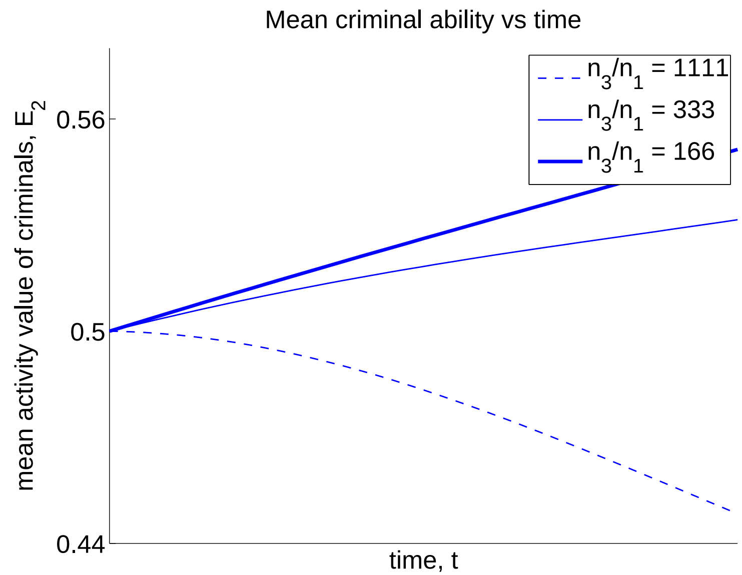


Figure 4: (a) Initial distribution of criminals. (b) Large time distribution of criminals for different number of security agents per 100,000 citizens.

Simulations - Case 3



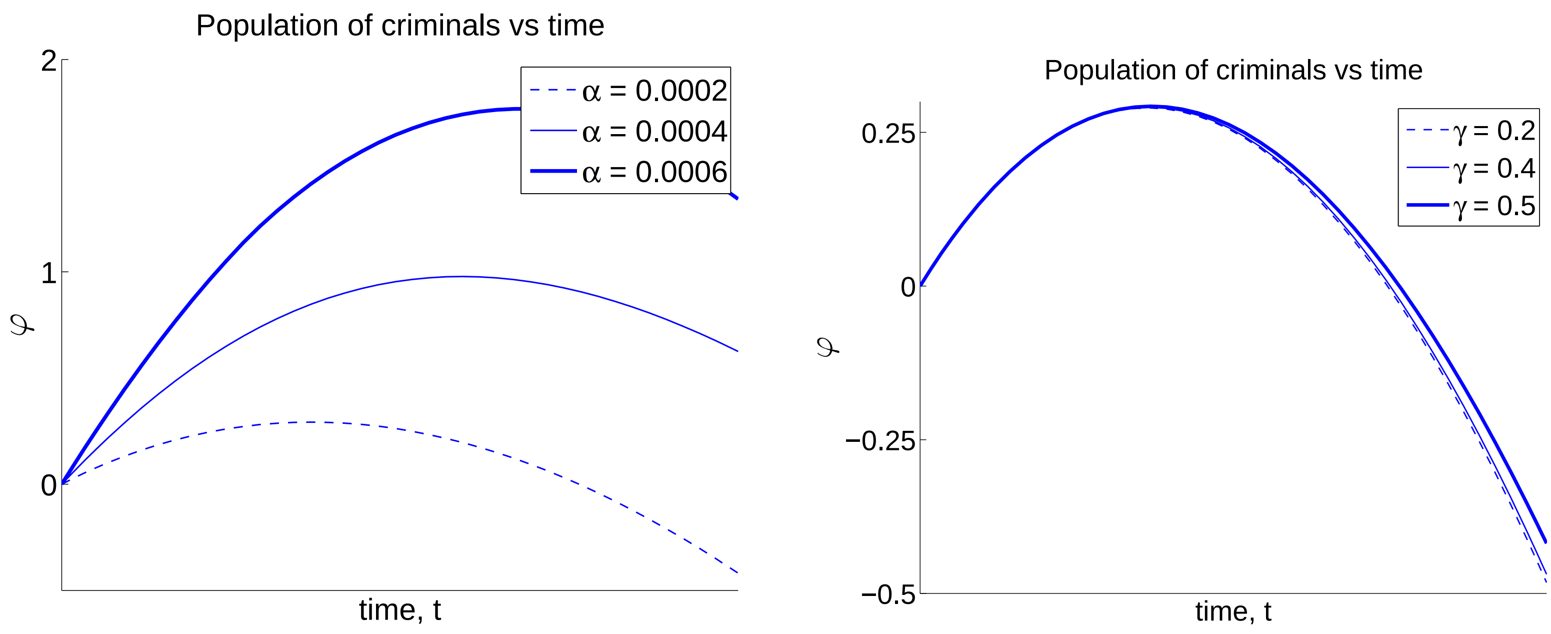
(c)



(d)

Figure 5: (c) Evolution of the size of functional subsystem 2 for different number of security agents per 100,000 citizens. (d) Evolution of the mean criminal ability, $\mathbb{E}_2(t)$, for different number of security agents per 100,000 citizens.

Simulations - Case 4



(a) (b)

Figure 6: Evolution of the size of functional subsystem 2, $n_2(t)$, for different values of (a) α_T and (b) γ.

Modeling on the Onset and Dynamics of Criminality

Looking ahead

- *Detailed analysis of bifurcation problems and asymptotic behaviors*
- *Interactions of different dynamics, e.g. welfare policy and criminality dynamics*
- *Predicting tipping points and the black swan*
- *Learning collective behaviors*
- *Dynamics on networks*
- *Control problems*
- *Looking ahead to a systems sociology approach*

THANK YOU FOR YOUR ATTENTION AND THANKS TO

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- **Juanjio Nieto** *Universidad de Granada*
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