Multi-agent System Learning and Control

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“Collective dynamics, control and imaging”
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Joint work with M. Bongini, M. Caponigro, M. Hansen, D. Kalise, M. Maggioni, B. Piccoli, F. Rossi, F. Solombrino, E. Trelat
What is a self-organizing system?
**Social dynamics**

We consider large particle systems of form:

\[
\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= (S + K * \mu_N)(x_i, v_i), \\
i &= 1, \ldots, N, \quad t \in [0, T], \\
\text{where,} \quad \mu_N &= \frac{1}{N} \sum_{j=1}^{N} \delta(x_j, v_j),
\end{aligned}
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“Social forces” encoded in $S$ and $K$:

- Repulsion-attraction
- Alignment
- Self-propulsion/friction
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Understanding how superposition of re-iterated binary “social forces” yields global self-organization.
For $S \equiv 0$ and $K(x, v) = a(\|x\|)(-v)$ we get the Cucker-Smale model

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\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i), \quad i = 1, \ldots, N,
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Usually, $a(r) = \frac{1}{(1+r^2)^\beta}$, where $\beta \in [0, +\infty]$.

If $\beta < \frac{1}{2}$, alignment occurs everytime

If $\beta \geq \frac{1}{2}$, alignment occurs only for certain initial data
Models for social dynamics - Cucker-Dong

For $S(x, v) = -bv$ and $K(x, v) = a(\|x\|^2)(-x) - f(\|x\|^2)(-x)$ we get the Cucker-Dong model

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= -bv_i + \frac{1}{N} \sum_{j=1}^{N} (a - f)(\|x_i - x_j\|^2)(x_j - x_i), \quad i = 1, \ldots, N,
\end{align*}
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Models for social dynamics - Cucker-Dong

For $S(x, \nu) = -bv$ and $K(x, \nu) = a(\|x\|^2)(-x) - f(\|x\|^2)(-x)$ we get the Cucker-Dong model

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$$

Usually, $a(r) = \frac{1}{(1+r^2)^\beta}$ and $f(r) = \frac{1}{r^\delta}$. A sufficient condition for self-organization is given by the quantities

$$
E(t) = \sum_{i=1}^{N} \|\nu_i(t)\|^2 + \frac{1}{2N} \sum_{i<j}^{N} \left( \int_{0}^{\|x_i(t)-x_j(t)\|^2} a(r) dr + \int_{\|x_i(t)-x_j(t)\|^2}^{+\infty} f(r) dr \right)
$$

$$
\theta = \frac{N - 1}{2} \int_{0}^{+\infty} a(r) dr
$$
If $E(0) \leq \theta$, cohesiveness occurs every-time.

If $E(0) > \theta$, cohesiveness occurs only for certain initial data.
For \( S(x, v) = (\alpha - \beta\|v\|^2)v \) and \( K(x, v) = -\nabla U(\|x\|) \frac{x}{\|x\|} \) we get the D’Orsogna-Bertozzi et al. model

\[
\begin{cases}
\dot{x}_i = v_i, \\
\dot{v}_i = (\alpha - \beta\|v_i\|^2)v_i - \frac{1}{N} \sum_{j=1}^{N} \nabla U(\|x_i - x_j\|) \frac{x_i - x_j}{\|x_i - x_j\|}, 
\end{cases} \quad i = 1, \ldots, N,
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Usually, $U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}$.

If $\frac{C_R}{C_A} \left( \frac{\ell_R}{\ell_A} \right)^d < 1$, crystalline structures appear

If $\frac{C_R}{C_A} \left( \frac{\ell_R}{\ell_A} \right)^d \geq 1$, mill patterns arise
A society is said to be homophilious whenever its agents are more influenced by near agents than far ones;
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However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments.
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**Question:** can a government endowed with limited resources rescue/stabilize a society by minimal interventions? Which ones?
A parametric model of homo-to-hetero-philia

The Cucker-Smale model:

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\begin{align*}
\dot{x}_i &= v_i \in \mathbb{R}^d \\
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\end{align*}
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where \( a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta} \), \( \beta > 0 \) governs the rate of communication.

\(^1\)The Laplacian \( L \) of \( A \) is given by \( L = D - A \), with \( D = \text{diag}(d_1, \ldots, d_N) \) and \( d_k = \sum_{j=1}^{N} a_{kj} \)}
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\begin{align*}
\dot{x} &= v \\
\dot{v} &= -L_x v
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where \( L_x \) is the Laplacian of the matrix\(^1\) \( (a(||x_j - x_i||)/N)_{i,j=1}^{N} \) and depends on \( x \).

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Conditional consensus emergence for a generic communication rate $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$
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**Theorem (Ha-Ha-Kim)**

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\gamma(X_0) := \sqrt{N} \int_{\sqrt{NX_0}}^{\infty} a(\sqrt{2r})dr > \sqrt{V_0}.$$

Then the solution with initial data $(x_0, v_0)$ tends to consensus.
Consider $\beta = 1$ and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative pos. and vel. of two agents on the line:

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\frac{v}{1 + x^2}
\end{align*}
\]

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$. 
Non-consensus events

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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$ 

Hence, if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$
Self-organization Vs organization by intervention

We introduce the notion of organization via intervention.
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We introduce the notion of organization via intervention. Admissible controls: measurable functions $u = (u_1, \ldots, u_N) : [0, +\infty) \to \mathbb{R}^N$ such that $\sum_{i=1}^{N} \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant $M$:

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\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) + u_i
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for $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$. 

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for \( i = 1, \ldots, N \), and \( x_i \in \mathbb{R}^d \), \( v_i \in \mathbb{R}^d \).

Our aim is then to find admissible controls steering the system to the consensus region.
Total control

Proposition (Caponigro-F.-Piccoli-Trélat)

For every initial condition \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) and \(M > 0\) there exist \(T > 0\) and \(u : [0, T] \rightarrow (\mathbb{R}^d)^N\), with \(\sum_{i=1}^{N} \|u_i(t)\| \leq M\) for every \(t \in [0, T]\) such that the associated solution reaches the consensus region.
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Proof.

Consider a solution of system with initial data \((x_0, v_0)\) associated with a feedback control \(u = -\alpha(v - \bar{v})\), with \(0 < \alpha \leq M/(N\sqrt{B(v_0, v_0)})\).
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Then

\[
\frac{d}{dt} V(t) = \frac{d}{dt} B(v(t), v(t))
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\[
= -2B(L_x v(t), v(t)) + 2B(u(t), v(t))
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\leq 2B(u(t), v(t)) = -2\alpha B(v - \bar{v}, v - \bar{v}) = -2\alpha V(t).
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\]

Therefore \(V(t) \leq e^{-2\alpha t} V(0)\) and \(V(t)\) tends to 0 exponentially fast as \(t \to \infty\). Moreover \(\sum_{i=1}^N \|u_i\| \leq M\). \(\square\)
More economical choices?

We wish to make

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the smallest possible and use the minimal amount of intervention:
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the smallest possible and use the minimal amount of intervention: minimize \( B(u(t), v(t)) \) with additional sparsity constraints.
Greedy sparse control

**Theorem (Caponigro-F.-Piccoli-Trélat)**

*For every initial condition \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) and \(M > 0\) there exist \(T > 0\) and a sparse control \(u : [0, T] \to (\mathbb{R}^d)^N\), with \(\sum_{i=1}^N \|u_i(t)\| \leq M\) for every \(t \in [0, T]\) such that the associated AC solution reaches the consensus region.*
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\[
\min B(v, u) + \frac{\gamma(x)}{N} \sum_{i=1}^{N} \|u_i\| \quad \text{subject to} \quad \sum_{i=1}^{N} \|u_i\| \leq M,
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where \(\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2r})dr\) threshold by Ha-Ha-Kim.
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For every initial condition \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) and \(M > 0\) there exist \(T > 0\) and a \textit{sparse} control \(u : [0, T] \to (\mathbb{R}^d)^N\), with \(\sum_{i=1}^{N} \|u_i(t)\| \leq M\) for every \(t \in [0, T]\) such that the associated AC solution reaches the consensus region. More precisely, we can choose adaptively the control law explicitly as one of the solutions of the variational problem

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where \(\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2r})dr\) threshold by Ha-Ha-Kim. This choice of the control makes \(V(t) = B(v(t), v(t))\) vanishing in finite time, hence there exists \(T\) such that \(B(v(t), v(t)) \leq \gamma(x)^2\), \(t \geq T\).
Geometrical interpretation in the scalar case

For $|v| \leq \gamma$ the minimal solution $u \in [-M, M]$ is zero.

For $|v| > \gamma$ the minimal solution $u \in [-M, M]$ is $|u| = M$. 
Explicit sparse control

Denote $\nu_\perp = \nu - \bar{\nu}$. We construct the control law from the variational problem.
Explicit sparse control

Denote $v_\perp = v - \bar{v}$. We construct the control law from the variational problem.

If $\|v_\perp_i\| \leq \gamma(x)$ for every $i = 1, \ldots, N$, then

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Otherwise there exists a “best index” $i \in \{1, \ldots, N\}$ such that

$$\|v_\perp_i\| > \gamma(x) \quad \text{and} \quad \|v_\perp_i\| \geq \|v_\perp_j\| \quad \text{for every } j = 1, \ldots, N.$$

Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

$$u_i = -M \frac{v_\perp_i}{\|v_\perp_i\|}, \quad \text{and} \quad u_j = 0, \quad \text{for every } j \neq i.$$
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Hence the control acts on the most “stubborn”. We may call this control the “shepherd dog strategy”.

![Sheep and Shepherd Dog Image](image)
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Instantaneous optimality of the greedy strategy

Consider generic control $u$ (solution of the variation problem) of components

$$u_i(x, v) = \begin{cases} 
0 & \text{if } v_{\perp i} = 0 \\
-\alpha_i \frac{v_{\perp i}}{\|v_{\perp i}\|} & \text{if } v_{\perp i} \neq 0
\end{cases}$$

where $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i \leq M$. 
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where $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i \leq M$.

Theorem (Caponigro-F.-Piccoli-Trélat)

* A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.

* Homophilious society can be stabilized by parsiminiuous interventions!

The 1-sparse control is the minimizer of

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the control of the previous form.
Homophilious societies are sparsely stabilizable – 1

If we allow external intervention, the CS system in the homophilious regime ($\beta > \frac{1}{2}$)

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} a(|x_i - x_j|) (x_j - x_i) + u_i
\end{align*}
\]

can be stabilized for any initial condition by using only sparse controls, i.e., zero for almost every agent \(^2\).

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can be stabilized for any initial condition by using only sparse controls, i.e., zero for almost every agent. The sparse control acts on the most “stubborn” agent at every time, like the “shepherd dog strategy”.

---

Homophilious societies are sparsely stabilizable – 2

For the CD system in the homophilious regime (that is \( E(0) \geq \theta \)), the shepard dog strategy stabilization of

\[
\begin{align*}
\dot{x}_i & = v_i, \\
\dot{v}_i & = -bv_i + \frac{1}{N} \sum_{j=1}^{N} (a - f)(|x_i - x_j|)(x_j - x_i) + u_i,
\end{align*}
\]

with \( b \equiv 0 \) occurs under the additional hypothesis

\[
\theta > E(0) \exp \left( -\frac{2\sqrt{3}}{9} \frac{N\|\bar{v}(0)\|^3}{E(0)^{3/2}} \right)
\]
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$$\theta > E(0) \exp \left( -\frac{2\sqrt{3}}{9} \frac{N\|\vec{v}(0)\|^3}{E(0)^{3/2}} \right)$$

The shepherd dog strategy does not work for every initial condition!
Observing the future: sparse optimal control

The problem is to minimize, for a given $\gamma > 0$

$$\mathcal{J}(u) = \int_0^T \frac{1}{N} \sum_{i=1}^N \left( \left( v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \right)^2 + \gamma \| u_i(t) \| \right) dt,$$

s.t.

$$\sum_{i=1}^N \| u_i \| \leq M$$

where the state is a trajectory of the control system

$$\begin{cases}
\dot{x}_i = v_i \\
\dot{v}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i
\end{cases}$$

with initial constraint

$$(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N.$$
Beyond a greedy approach: sparse optimal control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every \((x_0, v_0)\) in \((\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\), for every \(M > 0\), and for every \(\gamma > 0\) the optimal control problem has an optimal solution. The optimal control \(u(t)\) is “usually” instantaneously a vector with at most one nonzero coordinate.
Beyond a greedy approach: sparse optimal control

**Theorem (Caponigro-F.-Piccoli-Trélat)**

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The PMP ensures the existence of \(\lambda \geq 0\) and of a nontrivial covector \((p_x, p_v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) satisfying the adjoint equations, for \(i = 1, \ldots, N\),

\[
\begin{aligned}
\dot{p}_{x_i} &= \frac{1}{N} \sum_{j=1}^{N} \frac{a(\|x_j - x_i\|)}{\|x_j - x_i\|} \langle x_j - x_i, v_j - v_i \rangle (p_{v_j} - p_{v_i}) \\
\dot{p}_{v_i} &= -p_{x_i} - \frac{1}{N} \sum_{j \neq i} a(\|x_j - x_i\|) (p_{v_j} - p_{v_i}) - 2\lambda v_i + \frac{2\lambda}{N} \sum_{j=1}^{N} v_j.
\end{aligned}
\]

The application of the PMP leads to minimize

\[
\min \sum_{i=1}^{N} \langle p_{v_i}, u_i \rangle + \lambda \gamma \sum_{i=1}^{N} \|u_i\|, \quad \text{subject to} \quad \sum_{i=1}^{N} \|u_i\| \leq M.
\]
What if the population is too large $N \approx \infty$?
Mean-field (sparse) optimal control?

What if $N \to \infty$?
Mean-field (sparse) optimal control?

What if \( N \to \infty \)? Did you know that the term “curse of dimensionality” was first introduced by Richard E. Bellman precisely for high-dimensional optimal control problems?
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What if $N \to \infty$? Did you know that the term “curse of dimensionality” was first introduced by Richard E. Bellman precisely for high-dimensional optimal control problems? How can we define an approximating “infinite dimensional” sparse optimal control?

We consider a perhaps natural control problem:

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\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= (H \ast \mu_N)(x_i, v_i) + u_i, \quad i = 1, \ldots, N, \quad t \in [0, T],
\end{aligned}
\]

where \( \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_j, v_j) \),

controlled by the minimizer of the cost functional

\[
\mathcal{J}(u) := \int_0^T \int_{\mathbb{R}^{2d}} \left( L(x, v, \mu_N) d\mu_N(t, x, v) + \frac{1}{N} \sum_{i=1}^{N} \|u_i\| \right) dt,
\]
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\]

Which topology on \(\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_j, v_j)\)? Which topology on \(\nu_N = \frac{1}{N} \sum_{j=1}^{N} u_i \delta(x_j, v_j)\)?
Too weak convergence

The compactness of the problem is way too weak

\[ \nu_N = \frac{1}{N} \sum_{j=1}^{N} u_i \delta(x_j, v_j) \rightharpoonup \nu, \quad \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_j, v_j) \rightharpoonup \mu, \]

as it can happen that \( \nu \perp \mu \).
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as it can happen that $\nu \perp \mu$. Natural limit equation

$$\frac{\partial \mu}{\partial t} + \nu \cdot \nabla_x \mu = \nabla \nu \cdot [(H \ast \mu) \mu + \nu].$$
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$$\nu_N = \frac{1}{N} \sum_{j=1}^{N} u_i \delta(x_j,v_j) \rightarrow \nu, \quad \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x_j,v_j) \rightarrow \mu,$$

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$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \ast \mu) \mu + \nu].$$

Steering the cloud $\mu$ by means of the toothpicks $\nu$!!
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$$\frac{\partial \mu}{\partial t} + \nu \cdot \nabla x \mu = \nabla \nu \cdot [(H \ast \mu + f)\mu],$$
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as it can happen that $\nu \perp \mu$. Natural limit equation

$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot [(H * \mu)\mu + \nu].$$

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but $f \in L^1_\mu(\mathbb{R}^{2d}, \mathbb{R}^d)$ only and no well-posedness can be expected!
Mean-field Sparse Optimal Control?

“Ultimately it would be good to have a theory that combined both the collective behaviour of a large number of “ordinary” agents with the decisions of a few key players of unusually large (relative) influence – some complicated combination of PDE and game theory, presumably – but our current mathematical technology is definitely insufficient for even a zeroth approximation to this task”.

– Terry Tao, January 7, 2010

https://terrytao.wordpress.com/2010/01/07/mean-field-equations
A natural relaxation: smoother controls

Definition
For a horizon time $T > 0$, and an exponent $1 \leq q < +\infty$ we fix a control bound function $\ell \in L^q(0, T)$. The class of admissible control functions $\mathcal{F}_\ell([0, T])$ is so defined: $f \in \mathcal{F}_\ell([0, T])$ if and only if

(i) $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a Carathéodory function,
(ii) $f(t, \cdot) \in W^{1,\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^d)$ for almost every $t \in [0, T]$, and
(iii) $\|f(t, 0)\| + \text{Lip}(f(t, \cdot), \mathbb{R}^d) \leq \ell(t)$ for almost every $t \in [0, T]$. 
Mean-field optimal control

Theorem (F. and Solombrino)

Assume that we are given maps $H$, $L$, and $\psi$ as in assumptions (H), (L), and (Ψ). For $N \in \mathbb{N}$ and an initial datum 
$((x_N^0)_1, \ldots, (x_N^0)_N, (v_N^0)_1, \ldots, (v_N^0)_N) \in B(0, R_0) \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for $R_0 > 0$ independent of $N$, we consider

$$\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} [L(x, v, \mu_N(t, x, v)) + \psi(f(t, x, v))] \, d\mu_N(t, x, v) \, dt,$$

where $\mu_N(t, x, v) = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j(t), v_j(t))}(x, v)$, constrained by being the solution of

$$\begin{cases} 
\dot{x}_i = v_i, \\
\dot{v}_i = (H \ast \mu_N)(x_i, v_i) + f(t, x_i, v_i), & i = 1, \ldots N, \quad t \in [0, T],
\end{cases}$$

with initial datum $(x(0), v(0)) = (x_N^0, v_N^0)$ and, for consistency, we set

$$\mu_N^0 = \frac{1}{N} \sum_{i=1}^M \delta_{((x_N^0)_i, (v_N^0)_i)}(x, v).$$

For all $N \in \mathbb{N}$ let us denote the function $f_N \in \mathcal{F}_\ell$ as a solution of the finite dimensional optimal control problem.
Mean-field optimal control

If there exists a compactly supported $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ such that
\[
\lim_{N \to \infty} W_1(\mu_N^0, \mu^0) = 0,
\]
then there exists a subsequence $(f_{N_k})_{k \in \mathbb{N}}$ and a function $f_\infty \in \mathcal{F}_\ell$ such that $f_{N_k}$ converges to $f_\infty$ and $f_\infty$ is a solution of the infinite dimensional optimal control problem

\[
\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} \left[ L(x, v, \mu(t, x, v)) + \psi(f(t, x, v)) \right] d\mu(t, x, v) dt,
\]

where $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ is the unique weak solution of

\[
\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot \left[ (H \ast \mu + f) \mu \right],
\]

with initial datum $\mu(0) := \mu^0$ and forcing term $f$. 
Mean-field optimal control

If there exists a compactly supported $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ such that
$$\lim_{N \to \infty} \mathcal{W}_1(\mu^0_{N}, \mu^0) = 0,$$
then there exists a subsequence $(f_{N_k})_{k \in \mathbb{N}}$ and a function $f_{\infty} \in \mathcal{F}_\ell$ such that $f_{N_k}$ converges to $f_{\infty}$ and $f_{\infty}$ is a solution of the infinite dimensional optimal control problem

$$\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} \left[ L(x, v, \mu(t, x, v)) + \psi(f(t, x, v)) \right] d\mu(t, x, v) dt,$$

where $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ is the unique weak solution of

$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot \left[ (H \ast \mu + f) \mu \right],$$

with initial datum $\mu(0) := \mu^0$ and forcing term $f$.

The proof is based on the simultaneous development of the mean-field limit for the equation and the $\Gamma$-limit for the optimization of the control.
Let us now consider a controlled system with \( m \) leaders for \( m \gg N \)

\[
\begin{align*}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H * \mu_N(y_k, w_k) + H * \mu_m(y_k, w_k) + u_k \quad k = 1, \ldots m, \\
\dot{x}_i &= v_i, \\
\dot{v}_i &= H * \mu_N(x_i, v_i) + H * \mu_m(x_i, v_i) \quad i = 1, \ldots N,
\end{align*}
\]
Sparse optimal control? Mixing diffuse and granular

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\dot{v}_i &= H \ast \mu_N(x_i, v_i) + H \ast \mu_m(x_i, v_i) \quad i = 1, \ldots, N,
\end{align*}
\]

For \( N \to \infty \) the limit dynamics is

\[
\begin{align*}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots, m, \\
\partial_t \mu + v \cdot \nabla_x \mu &= \nabla_v \cdot [(H \ast (\mu + \mu_m)) \mu],
\end{align*}
\]

where the weak solutions of the equations have to be interpreted in the Carathéodory sense.
Mixing diffuse and granular

Figure: A mixed granular-diffuse crowd leaving a room through a door. This figure was kindly provided by the authors. Copyright ©2011 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.
Main result on mean-field sparse control, I

Denote $\mathcal{X} := \mathbb{R}^{2d \times m} \times \mathcal{P}(\mathbb{R}^{2d})$.

Theorem (F., Piccoli, and Rossi)

Let $H$ and $L$ be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with $\mu^0$ compactly supported, $\text{supp}(\mu^0) \subset B(0, R)$, $R > 0$, the optimal control problem

$$
\min_{u \in L^1([0, T], \mathcal{U})} \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m} \sum_{k=1}^m \|u_k(t)\| \right\} dt,
$$

has solutions, where the triplet $(y, w, \mu)$ defines the unique solution of

$$
\begin{aligned}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast (\mu + \mu_m)(y_k, w_k) + u_k, \\
\partial_t \mu + v \cdot \nabla x \mu &= \nabla v \cdot [(H \ast (\mu + \mu_m)) \mu],
\end{aligned}
\quad k = 1, \ldots, m, \quad t \in [0, T]
$$

with initial datum $(y^0, w^0, \mu^0)$ and control $u$, and

$$
\mu_m(t) = \frac{1}{m} \sum_{k=1}^n \delta(y_k(t), w_k(t)).
$$
Main result on mean-field sparse control, II

Moreover, solutions to the problem can be constructed as weak limits $u^*$ of sequences of optimal controls $u^*_N$ of the finite dimensional problems

$$
\min_{u \in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^m \|u_k(t)\| \right\} dt,
$$

where $\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{(x_{i,N}(t),v_{i,N}(t))}$ and $\mu_{m,N}(t) = \frac{1}{m} \sum_{k=1}^m \delta_{(y_{k,N}(t),w_{k,N}(t))}$ are the time-dependent atomic measures supported on the trajectories defining the solution of the system

$$
\begin{align*}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \ast \mu_N(y_k, w_k) + H \ast \mu_{m,M}(y_k, w_k) + u_k & k = 1, \ldots, m, & t \in [0, T], \\
\dot{x}_i &= v_i, \\
\dot{v}_i &= H \ast \mu_N(x_i, v_i) + H \ast \mu_{m,M}(x_i, v_i) & i = 1, \ldots, N, & t \in [0, T],
\end{align*}
$$

with initial datum $(y^0, w^0, x_{i,0}, v_{i,0})$, control $u$, and $\mu_{N,0} = \frac{1}{N} \sum_{i=1}^N \delta_{(x^0_{i},v^0_{i})}$ is such that $\mathcal{W}_1(\mu_{N,0}, \mu^0) \to 0$ for $N \to +\infty$. 
Evacuating an unknown environment
Simulations I
Simulations II
Consider the dynamics

\[ \dot{x}_i(t) = \frac{1}{N} \sum_{j \neq i} a(\|x_i - x_j\|)(x_j - x_i), \quad i = 1, \ldots, N. \]

with \( a \in X = \{ b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L^\infty(\mathbb{R}_+) \cap W^{1,\infty}_{loc}(\mathbb{R}_+) \} \).
Learning the dynamics

Consider the dynamics

$$\dot{x}_i(t) = \frac{1}{N} \sum_{j \neq i} a(\|x_i - x_j\|)(x_j - x_i), \quad i = 1, \ldots, N.$$  

with $a \in X = \{ b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L^\infty(\mathbb{R}_+) \cap W_{loc}^{1,\infty}(\mathbb{R}_+) \}$. Can we ”learn” the interaction function $a$ from observations of the dynamics?
A least square functional

As an approximation to $a$ we seek for a minimizer of the following *discrete error functional*

$$
\mathcal{E}_N(\hat{a}) = \frac{1}{T} \int_0^T \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{N} \sum_{j=1}^N (\hat{a}(\|x_i(t) - x_j(t)\|) (x_i(t) - x_j(t)) - \dot{x}_i(t)) \right\|^2 dt,
$$

among all functions $\hat{a} \in X$. 
A least square functional

As an approximation to \( a \) we seek for a minimizer of the following "discrete error functional"

\[
\mathcal{E}_N(\hat{a}) = \frac{1}{T} \int_0^T \frac{1}{N} \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \left( \hat{a}(\|x_i(t) - x_j(t)\|)(x_i(t) - x_j(t)) - \dot{x}_i(t) \right)^2 \, dt,
\]

among all functions \( \hat{a} \in X \). In particular, given a finite dimensional space \( V \subset X \), we consider the minimizer:

\[
\hat{a}_{N,V} = \arg\min_{\hat{a} \in V} \mathcal{E}_N(\hat{a}).
\]
A least square functional

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$$

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$$
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$$

The fundamental question is

(Q) For which choice of the approximating spaces $V \in \Lambda$ (we assume here that $\Lambda$ is a countable family of invading subspaces of $X$) does $\hat{a}_{N,V} \rightarrow a$ for $N \rightarrow \infty$ and $V \rightarrow X$ and in which topology should this convergence hold?
Mean-field limit

The empirical measure \( \mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(t) \) weakly converges for \( N \to \infty \) to the probability measure valued trajectory \( t \to \mu(t) \) satisfying weakly the equation

\[
\partial_t \mu(t) = -\nabla \cdot \left( \left( H[a] * \mu(t) \right) \mu(t) \right), \quad \mu(0) = \mu^0.
\]

where \( H[a](x) = -a(\|x\|)x, \) for \( x \in \mathbb{R}^d. \)
The empirical measure $\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}$ weakly converges for $N \to \infty$ to the probability measure valued trajectory $t \to \mu(t)$ satisfying weakly the equation

$$\partial_t \mu(t) = -\nabla \cdot \left( (H[a] \ast \mu(t)) \mu(t) \right), \quad \mu(0) = \mu^0.$$  

where $H[a](x) = -a(\|x\|)x$, for $x \in \mathbb{R}^d$. We define

$$\mathcal{E}(\hat{a}) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \left\| (H[\hat{a}] - H[a]) \ast \mu(t) \right\|^2 \mu(t)(x) dt,$$
Coercivity property I

By Jensen inequality

$$\mathcal{E}(\hat{a}) \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{a}(\|x - y\|) - a(\|x - y\|)|^2 \|x - y\|^2 d\mu_t(x)d\mu_t(y)dt$$

$$= \frac{1}{T} \int_0^T \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 s^2 d\varrho(t)(s)dt$$

(1)

where $\varrho(t) = (\|x - y\| \# \mu_x(t) \otimes \mu_y(t))$. 
Coercivity property I

By Jensen inequality

\[ E(\hat{a}) \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{a}(\|x - y\|) - a(\|x - y\|)|^2 \|x - y\|^2 d\mu(t)(x)d\mu(t)(y) dt \]

\[ = \frac{1}{T} \int_0^T \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 s^2 d\varrho(t)(s) dt \]  

(1)

where \( \varrho(t) = (\|x - y\| \# \mu_x(t) \otimes \mu_y(t)) \). We define the prob. measure

\[ \tilde{\rho} := \frac{1}{T} \int_0^T \varrho(t) dt. \]
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By Jensen inequality
\[
\mathcal{E}(\hat{a}) \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{a}(\|x - y\|) - a(\|x - y\|)|^2 \|x - y\|^2 d\mu(t)(x) d\mu(t)(y) dt
\]
\[
= \frac{1}{T} \int_0^T \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 s^2 d\varrho(t)(s) dt
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where \( \varrho(t) = (\|x - y\| \# \mu_x(t) \otimes \mu_y(t)) \). We define the prob. measure
\[
\bar{\rho} := \frac{1}{T} \int_0^T \varrho(t) dt.
\]

Finally we define the weighted measure
\[
\rho(A) := \int_A s^2 d\bar{\rho}(s),
\]
Coercivity property I

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Then one can reformulate (1) in a very compact form as follows

\[ \mathcal{E}(\hat{a}) \leq \int_{\mathbb{R}_+} |\hat{a}(s) - a(s)|^2 d\rho(s) = \|\hat{a} - a\|^2_{L^2(\mathbb{R}_+,\rho)}. \]
Coercivity property II

To establish coercivity of the learning problem it is essential to assume that there exists $c_T > 0$ such that also the following additional lower bound holds

$$c_T \| \hat{a} - a \|_{L^2(\mathbb{R}_+, \rho)}^2 \leq \mathcal{E}(\hat{a}),$$

for all relevant $\hat{a} \in X \cap L^2(\mathbb{R}_+, \rho)$. This crucial assumption eventually determines also the natural space $X \cap L^2(\mathbb{R}_+, \rho)$ for the solutions.
Uniform approximation property

For $M > 0$ and an interval $K = [0, 2R]$ define the set

$$X_{M,K} = \{ b \in W^{1,\infty}(K) : \|b\|_{L^\infty(K)} + \|b'\|_{L^\infty(K)} \leq M \}.$$

Additionally for every $N \in \mathbb{N}$, let $V_N$ be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence on $K$ with the following uniform approximation property: for all $b \in X_{M,K}$ there exists a sequence $(b_N)_{N \in \mathbb{N}}$ converging uniformly to $b$ on $K$ and such that $b_N \in V_N$ for every $N \in \mathbb{N}$. 
Main learnability result I

Theorem (Bongini, F., Hansen, Maggioni, 2015)

Fix $M \geq \|a\|_{L^\infty(K)} + \|a'\|_{L^\infty(K)}$ for $K = [0, 2R]$, for $R > 0$ large enough. For every $N \in \mathbb{N}$, let $V_N$ be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence on $K$ with the uniform approximation property. Then the minimizers

$$\hat{a}_N \in \arg \min_{\hat{a} \in V_N} \mathcal{E}_N(\hat{a}).$$

converge uniformly up to subsequences for $N \to \infty$ to a continuous function $\hat{a} \in X_{M,K}$ such that $\mathcal{E}(\hat{a}) = 0$. 
Main learnability result II

If we additionally assume the coercivity condition, then $\hat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$. Moreover, in this latter case, if there exist rates $\alpha, \beta > 0$, constants $C_1, C_2 > 0$, and a sequence $(a_N)_{N \in \mathbb{N}}$ of elements $a_N \in V_N$ such that

$$\|a - a_N\|_{L_\infty(K)} \leq C_1 N^{-\alpha},$$

and

$$W_1(\mu_0^N, \mu_0) \leq C_2 N^{-\beta},$$

then there exists a constant $C_3 > 0$ such that

$$\|a - \hat{a}_N\|_{L_2(\mathbb{R}_+, \rho)}^2 \leq C_3 N^{-\min\{\alpha, \beta\}},$$

for all $N \in \mathbb{N}$. In particular, in this case, it is the entire sequence $(\hat{a}_N)_{N \in \mathbb{N}}$ (and not only subsequences) to converge to $a$ in $L^2(\mathbb{R}_+, \rho)$. 

Numerical experiments
Conclusion

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- We presented dynamical systems with self-organization features ⇒ organization by external intervention.
- We proved that the most effective greedy strategy is by instantaneous 1-sparse controls.
- We presented relaxations of the (sparse) finite dimensional optimal control problems and a general technique to derive their mean-field limits.
- We showed recent results on learnability of systems modeling social dynamics.
A few info

▶ **WWW:** http://www-m15.ma.tum.de/

▶ **References:**
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