Exponential tail behavior for solutions to the homogeneous Boltzmann equation

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Introduction

The Boltzmann equation (late 1860s and 1870s, Maxwell and Boltzmann) describes evolution of the probability density $f(t, x, v)$ of gas particles in a rarefied gas, for the time $t \in \mathbb{R}^+$ velocity $v \in \mathbb{R}^d$ and position $x \in \mathbb{R}^d$. It reads

$$\boxed{\partial_t f + v \cdot \nabla_x f = Q(f, f)}, \tag{1.1}$$

where $Q(f, f)$ is a **quadratic integral operator**. Before we write the formula for the collision operator, we recall notation associated to a collision of a pair of particles.

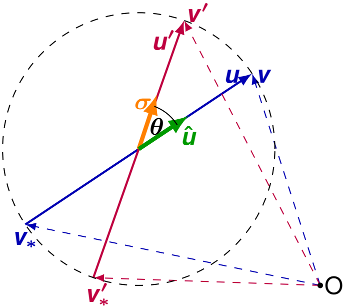
Pre-post collisional velocities:

$$v' = v + \frac{1}{2}(|u|\sigma - u)$$
$$v'_* = v_* + \frac{1}{2}(|u|\sigma - u), \quad \sigma \in S^{d-1}.$$

Notation:

$u = v - v_*, \quad u' = v' - v'_*$

$f' = f(v'), f_* = f(v_*)$ etc.



The collisional operator

The strong form of the collisional operator is the following:

$$Q(f,f)(t,x,v) \;=\; \int_{\mathbb{R}^d} \int_{S^{d-1}} \left(f' f'_* - f f_*\right) \, B(|u|, \hat{u}\cdot\sigma) \, d\sigma \, dv_*$$

The collision kernel $B(|u|, \hat{u}\cdot\sigma)$ is assumed to have the factorized form:

$$B(|u|, \hat{u}\cdot\sigma) \;=\; |u|^{\gamma} \; b(\cos\theta).$$

In physically motivated models, $\gamma \in (-d, 1]$, with the following special cases

- $\gamma \in (0, 1]$: **hard potentials**; $\gamma = 1$: hard spheres,
- $\gamma = 0$: Maxwell molecules,
- $\gamma \in (-d, 0)$: soft potentials.

The angular cross-section

The angular kernel $b(\cos\theta)$ has a singularity at $\theta = 0$, which makes it non-integrable over the sphere. Traditionally, however, integrability was assumed

- **Grad's cutoff:** $\int_0^\pi b(\cos\theta)\ \sin^{d-2}\theta\, d\theta < \infty.$

This simplifies the analysis of the collision operator as it can then be split in the so called gain and loss terms $Q(f,f) = Q^+(f,f) - Q^-(f,f)$. For a long time, it was believed that this removal of the singularity does not influence the equation significantly. However, recently it has been observed that singularity carries regularizing effect.

We work in the non-cutoff regime, which means that the integral above is infinite, while its weighted version is finite. More precisely:

- **Non-cutoff:**

$$\int_0^\pi b(\cos\theta)\ \sin^{d-2}\theta\, d\theta = \infty,$$

$$\int_0^\pi b(\cos\theta)\ \sin^\beta\theta\ \sin^{d-2}\theta\, d\theta < \infty, \qquad \beta \in (0,2].$$

Another simplification of the Boltzmann equation

Another major simplification of the Boltzmann equation is removal of the dependence on the space variable x. This leads to the so called **spatially homogeneous Boltzmann equation**:

$$\boxed{\partial_t f = Q(f, f).} \tag{1.2}$$

We study this spatially homogeneous Boltzmann equation, in non-cutoff regime, for hard potentials.

Goal

We study **Exponential Tail Behavior** of a solution in L^1 and L^∞ sense. We say f has exponential tail behavior in L^1 or in L^∞ sense is for some $\alpha > 0$ and some $s > 0$, the following norms are finite, respectively:

$$\|f\|_{L^1_{\exp(\alpha\,\langle v\rangle^s)}} := \int_{\mathbb{R}^d} f(t,v)\, e^{\alpha\,\langle v\rangle^s}\, dv < \infty,$$

$$\|f\|_{L^\infty_{\exp(\alpha\,\langle v\rangle^s)}} := \sup_v\, f(t,v)\, e^{\alpha\langle v\rangle^s} < \infty.$$

- ***L^1 exponential tail behavior in time***:
  - **Propagation:** If initial data $f(0,v)$ has $\exp(\alpha_0\,\langle v\rangle^s)$ tail in L^1 sense, does $f(t,v)$ has $\exp(\alpha\,\langle v\rangle^s)$ tail in L^1 sense, uniformly in time, for some $\alpha < \alpha_0$?
  - **Generation:** If initial data has finite first two moments, does $f(t,v)$ generates exponential tail in L^1 sense? If so, for what order s?
- ***L^∞ exponential tail behavior in time***:
  - **Propagation:** If $f(0,v) \le e^{-\alpha_0\,\langle v\rangle^s}$, is it true that then $f(t,v) \le e^{\alpha\,\langle v\rangle^s}$ holds uniformly in time, with potentially smaller α?
  - **Generation:** If initial data has finite finite first few moments, is it possible to generate pointwise exponential tail $f(t,v) \le e^{\alpha\,\langle v\rangle^s}$ for $t > 0$? For what s?

L^1 exponential tails

- **First results were on polynomial moments**, defined as $m_p(t) = \int f(t,v)\langle v\rangle^p dv$. It has been shown (*Arkeryd '72, Elmroth '83, Desvilletes '93, Wennberg '97, Mischler-Wennberg '99*) that as soon as the initial energy (second moment) is finite, all higher moments are generated and remain bounded uniformly in time.
- Exponential tail behavior in L^1 sense was first studied under the Grad's cutoff by *Bobylev* (Maxwell molecules in 1984, hard spheres in 1997). In these seminal works, by Taylor expanding exponential function the question is reformulated (formally) to showing summability of polynomial moments renormalized by Gamma functions:

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha\,\langle v\rangle^s}\, dv = \int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{\langle v\rangle^{qs}\,\alpha^q}{q!} = \sum_{q=0}^{\infty} \frac{m_{qs}(t)\,\alpha^q}{q!}.$$

  To estimate this sum, ordinary differential inequalities for polynomial moments are developed, also used in the study of polynomial moments, but now because of the required summability, constants need to be shaper.

The weak form of the collision operator:

$$\int_{\mathbb{R}^d} Q(f,f)\,\phi\,dv = \frac{1}{2}\int_{\mathbb{R}^{2d}} f\,f_* \left(\int_{S^{d-1}} (\phi' + \phi'_* - \phi - \phi_*)\,B(|u|, \hat{u}\cdot\sigma)\,d\sigma\right) dv_*\,dv$$

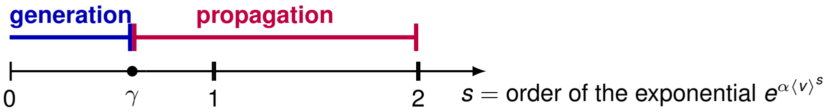
Hence, if the Boltzmann equation is multiplied by $\langle v\rangle^{2k}$, i.e. $\partial_t f\,\langle v\rangle^{2k} = Q(f,f)\,\langle v\rangle^{2k}$, integration in velocity yields the first step toward an ordinary differential inequality of moments:

$$\begin{aligned} m'_{2k}(t) &= \int_{\mathbb{R}^d} Q(f,f)\,\phi\,dv \\ &= \frac{1}{2}\int_{\mathbb{R}^{2d}} f\,f_* \left(\int_{S^{d-1}} \left(\langle v'\rangle^{2k} + \langle v'_*\rangle^{2k} - \langle v\rangle^{2k} - \langle v_*\rangle^{2k}\right) B(|u|, \hat{u}\cdot\sigma)\,d\sigma\right) dv dv_* \end{aligned}$$

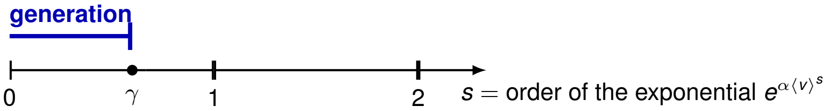
In the Grad's cutoff case, the positive and negative parts of the above integral can be bounded separately. That leads to an ODI for polynomial moments, from which one can obtain bounds of the infinite sum representation of exponential moments.

Summary of previous $L^1_{\exp}$ results

Grad's cut-off case:
Term-by-term method: *Bobylev 97*, *Bobylev-Gamba-Panferov 04*, *Gamba-Panferov-Villani 09*, *Mouhot 06*
Partial sum method: *Alonso-Canižo-Gamba-Mouhot 2013*



Non-cutoff case: Prior to our work, there was one result on exponential tail behavior in non-cutoff. Namely, *Lu and Mouhot in 2012* adapted the term-by term technique to the non-cutoff case to show generation of exponential tails of order up to $s = \gamma$.

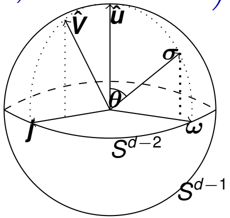


How do we approach the problem in the non-cutoff?

We adapt the partial sum technique to the non-cutoff case. The story begins in the same way as before

$$m'_{2q}(t) = \int_{\mathbb{R}^d} Q(f,f)\, \langle v\rangle^{2q}\, dv$$
$$= \frac{1}{2}\int_{\mathbb{R}^{2d}} ff_* \left(\int_{S^{d-1}} \left(\langle v'\rangle^{2q} + \langle v'_*\rangle^{2q} - \langle v\rangle^{2q} - \langle v_*\rangle^{2q} \right) B(|u|, \hat{u}\cdot\sigma)\, d\sigma \right) dv dv_*.$$

One of the challenges compared to Grad's cutoff, is the angular singularity. To overcome it, we exploit certain **cancellation properties** that become visible after an application of the Taylor expansion to the test functions in the weak formulation above.



This leads to an ODI for polynomial moments, which compared to the Grad's cutoff case has two extra powers of q in the last term of the following inequality:

$$m'_{2q} \leq -K_1\, m_{2q+\gamma} + K_2\, m_{2q}$$
$$+ K_3\, \varepsilon_q\, q\,(q-1) \sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma} \right)$$

To reduce the quadratic power of $q(q-1)$, we renormalize moments in the last term, and exploit the following properties of Gamma and Beta function, as well as a combinatorial sum of Beta functions:

$$\Gamma(x)\,\Gamma(y) = \Gamma(x+y)\,B(x,y)$$

$$\sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \leq C_a \frac{1}{q^{a+1}}$$

The last inequality holds only if $a > 1$. This leads us to study partial sums of the form

$$\mathcal{E}^n(t) := \sum_{q=0}^{n} \frac{m_{2q}(t)\,\alpha^{aq}}{\Gamma(aq+1)},$$

which differ from partial sums of exponential moments

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha\,\langle v \rangle^s}\, dv \;=\; \sum_{q=0}^{\infty} \frac{m_{qs}(t)\,\alpha^q}{q!}.$$

The appearance of $\Gamma(aq+1)$, with a non-integer coefficient a, inspired us to use **Mittag-Leffler functions** defined by:

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}$$

instead of the classical exponential functions. Mittag-Leffler functions generalize exponentials and are known to **asymptotically behave like exponentials**:

$$\mathcal{E}_a(x) \sim e^{x^{1/a}}, \qquad \text{for } x \gg 1.$$

Definition (Mittag-Leffler moment)

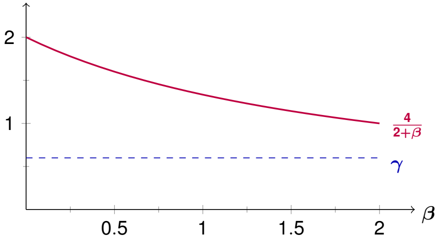
The **Mittag-Leffler moment** of f of order s and rate $\alpha > 0$ is introduced via:

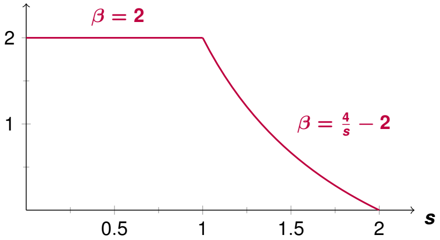
$$ML^{(\alpha,s)} = \sum_{q=0}^{\infty} \frac{m_{2q}\, \alpha^{2q/s}}{\Gamma(\frac{2}{s}\, q+1)}.$$

"Statement" of our $L^1_{\exp}$ result:

- **Generation of tails of order $s \leq \gamma$:**
  We provide a new proof of the generation of exponential tails $e^{-\alpha\langle v\rangle^s}$ of order up to $s = \gamma$ for the most singular kernel $\int_0^\pi b(\cos\theta)\,\sin^2\theta\;\sin^{d-2}\theta\,d\theta < \infty$.

- **Propagation of higher-order tails:**
  This result depends on the strength of the angular singularity. If the angular kernel satisfies $\int_0^\pi b(\cos\theta)\,\sin^\beta\theta\;\sin^{d-2}\theta\,d\theta < \infty$, with $\beta \in (0,2]$, we establish **propagation of Mittag-Leffler moments of order $s < \frac{4}{2+\beta}$.**





Application to the inverse-power law model in 3D

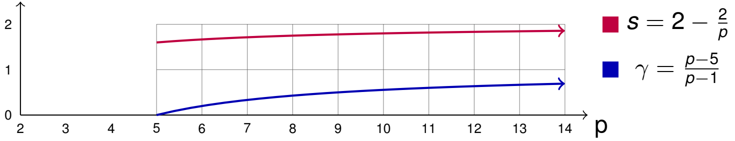
If the intermolecular force is given by $\boldsymbol{U(r) = r^{-(p-1)}}$, where r is the distance between interacting particles, then $B = |u|^\gamma b(\cos\theta)$ with

- $\gamma = \frac{p-5}{p-1}$,
- $\boldsymbol{b(\cos\theta) \sim K\theta^{-2-\frac{2}{p-1}}}$ as $\theta \to 0$.

Hard potentials correspond to $\boldsymbol{p > 5}$. In this case, we have that:

1. Exponential moments of order $\boldsymbol{s \in (0, \gamma]}$ are generated.
2. Mittag-Leffler moments of order $\boldsymbol{s \in (\gamma, 2 - \frac{2}{p})}$ are propagated.

$s =$ order of the exponential $e^{\alpha \langle v \rangle^s}$



L^∞ exponential tails

Previous $L^\infty_{\exp}$ result

Grad's cut-off case: *Gamba, Panferov, Villani 2009* established propagation of pointwise Gaussian tails. In other words:

$$f_0(v) \le c\, e^{-\alpha |v|^2} \quad \Rightarrow \quad f(t,v) \le c_1\, e^{-\alpha_1 |v|^2}.$$

For this purpose, they established comparison principle for the Boltzmann equation. Formally, it says that if

$$\begin{aligned} \partial_t f &= Q(f,f), \quad f(0,v) = f_0 \\ \partial_t g &\ge Q(f,g), \quad g(0,v) = g_0, \end{aligned}$$

then $f_0 \le g_0$ implies $f \le g$. So to establish propagation of pointwise Gaussian tails, all they need to show is that $Q(f,M) < 0$ (for large velocities). This last bound heavily relies on the possibility to split gain and loss, which is not possible in the non-cutoff. But, another beautiful aspect of their calculation is that the bound of $Q(f,M)$ will use L^1 exponentially weighted norm. In that sense, their result enhances L^1 exponentially weighted bounds to L^∞ exponentially weighted bounds. Moreover, they show

$$\|f\|_{L^\infty_{\exp(\alpha|v|^2)}} \le F(\|f\|_{L^1_{\exp(\alpha_1|v|^2)}}).$$

Previous L^∞ result without weights

Very recently, *Silvestre 2014* applied general techniques he developed with Schwab for non-local equations to the Boltzmann equation. Along the way of proving Hölder continuity of solutions to the Boltzmann equation, he establishes the lifting of from L^1 to L^∞ (notice there are no weights). This is done via a smart contradiction argument that enables the author to extract a **negative contribution out of the collision operator in the strong form**.

Our goal:

- **Adapt the contradiction argument of Silvestre to work with weighted spaces**
- **Use L^1 exponential bounds we established before** to conclude propagation of L^∞ exponentially weighted bound.

"Statement" of our L^∞_{exp} result

Suppose:

- Function f is a classical solution to the Boltzmann equation.
- The angular kernel satisfies:

$$b(\cos\theta)(\sin\theta)^{d-2} \sim (\sin\theta)^{-1-\nu}, \qquad \nu \in (0,1].$$

- Exponential tails of order s propagate in time (by our L^1_{exp} result, $s \in [0, \frac{4}{2+\nu})$).

Then for every α_0 there exists $\alpha_1 < \alpha_0$ and a uniform in time constant C so that

$$\|f_0\|_{L^\infty_{\exp(\alpha_0 \langle v \rangle^s)}} < \infty \quad \Rightarrow \quad \|f(t,v)\|_{L^\infty_{\exp(\alpha_1 \langle v \rangle^s)}} \leq \|f(t,v)\|_{L^1_{\exp(\alpha \langle v \rangle^s)}} \leq C.$$

Few words on the proof: contradiction argument

Inspired by the contradiction argument of Silvestre, set as a goal the following estimate:

$$\boldsymbol{m(t) := \|f\|_{L^\infty,(\alpha,s)} = \left\| \frac{f(t,v)}{M(v)} \right\|_{L^\infty} < a + bt^{-d/\nu}},$$

where $M(v) = e^{-\alpha\langle v\rangle^s}$ and where constants $a, b > 0$ will be determined later (they will be multiples of $\|f/M\|_{L^1_v}$). Suppose t_0 **is the first time the inequality fails**, i.e. $m(t_0) = a + bt_0^{-d/\nu}$, and let v_0 the the corresponding velocity where the maximum is achieved. Then

$$\boldsymbol{\partial_t \left(\frac{f}{M}\right) \geq \frac{d}{dt}\left(a + bt^{-d/\nu}\right)},$$

which, after an algebraic manipulation, leads to a lower bound on $\partial_t f$

$$\boldsymbol{\partial_t f(t_0, v_0) \geq -\frac{d}{\nu}\, b^{-\nu/d}\, M(v_0)\, (m(t_0) - a)^{1+d/\nu}}.$$

Goal: find an upper bound on $Q(f,f)(t_0,v_0)$ that will contradict the last inequality.

Few words on the proof: splitting and the negative contribution

The strong form of $Q(f,f)$ in the non-cutoff is usually split into two parts that are finite thanks to the Cancellation lemma (e.g. *Alexandre-Desvillettes-Villani-Wennberg*):

$$Q(f,f) = Q_1 + Q_2 = \int_{\mathbb{R}^N}\int_{\mathcal{S}^{N-1}} (f' - f)\, f'_*\, B\, d\sigma dv_* + f(v) \int_{\mathbb{R}^N}\int_{\mathcal{S}^{N-1}} (f'_* - f_*)\, B\, d\sigma dv_*$$

We further split the first term $Q_1(f,f)$ in a way that is more suitable to deal with f/M functions, so now $\boldsymbol{Q = Q_{1,1} + Q_{1,2} + Q_2}$, where:

$$\begin{aligned} \boldsymbol{Q_{1,1}} &= \boldsymbol{M(v) \int_{\mathbb{R}^N}\int_{\mathcal{S}^{N-1}} \left(\frac{f'}{M'} - \frac{f}{M}\right) f'_*\, B\, d\sigma dv_*} \\ \boldsymbol{Q_{1,2}} &= \boldsymbol{\int_{\mathbb{R}^N}\int_{\mathcal{S}^{N-1}} \frac{f'}{M'} (M' - M)\, f'_*\, B\, d\sigma dv_*} \\ \boldsymbol{Q_2} &= \boldsymbol{f(v) \int_{\mathbb{R}^N}\int_{\mathcal{S}^{N-1}} (f'_* - f_*)\, B\, d\sigma dv_*}. \end{aligned}$$

At the point (t_0, v_0), the function f/M achieves its maximum. Hence,

$$\begin{aligned} Q_{1,1}(t_0, v_0) &= M(v_0) \int_{\mathbb{R}^N} \int_{S^{N-1}} \left(\frac{f'}{M'} - \frac{f(t_0, v_0)}{M(v_0)} \right) f'_* \, B \, d\sigma \, dv_* \\ &= -M(v_0) \int_{\mathbb{R}^N} \int_{S^{N-1}} \left(m(t_0) - \frac{f'}{M'} \right) f'_* \, B \, d\sigma \, dv_*, \end{aligned}$$

so **the term $Q_{1,1}$ is negative** at the point (t_0, v_0).

At the end of the day, one gets:

$$\begin{aligned} Q_{1,1}(t_0, v_0) &\leq -C \, m(t_0)^{1+\nu/d} \, \langle v_0 \rangle^{1+\gamma+\nu} \\ Q_{1,2}(t_0, v_0) &\leq C \, m(t_0) \, \langle v_0 \rangle^{1+\gamma+\nu} \\ Q_2(t_0, v_0) &\leq C \, m(t_0) \, \langle v_0 \rangle^{\gamma}. \end{aligned}$$

Note that the negative term is dominating...

Possible further questions

1. **Soft potentials:** tail behavior for $\gamma < 0$?
2. **Tail behavior in L^∞ sense:** remove the assumption of classical solutions?
3. Can one use exp moments to get some results related to convergence to the equilibrium?

Possible further questions

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Thank you!!!