Compactness and weak solutions to time fractional PDEs

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   - A special case of time fractional compressible Navier-Stokes equations
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Memory effects

- Memory effects are ubiquitous in physics and engineering (particles in heat bath; soft matter with viscoelasticity)
- One-sided convolution is usually used to model the memory effects.
- A typical example is the generalized Langevin equation (GLE) (Mori, Kubo, Zwangzig):

$$\begin{cases} \dot{x} = v, \\ m\dot{v} = -\nabla V(x) - \int_{t_0}^{t} \gamma(t-s)v(s)ds + R(t) \end{cases}$$

$R(t)$ is a random noise satisfying the so-called fluctuation-dissipation theorem:

$$\mathbb{E}(R(t_0)R(t_0+t)) = m\mathbb{E}(v(t_0)^2)\gamma(|t|) = kT\gamma(|t|).$$

GLE: Memory effects often lead to fractional calculus

- For absolutely continuous functions, the Caputo derivative is

$$D_c^{\alpha} x = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(s)}{(t-s)^{\alpha}} ds,$$

- Our recent work (Li, Liu and Lu) considers the over-damped regime of GLE driven by fractional noise, leading to the following fractional SDE model:

$$D_c^{2-2H} x = -\nabla V(x) + \dot{B}_H.$$

  - $D_c^{\alpha} x$ is the Caputo derivative, which corresponds to the $\int_{t_0}^{t} \gamma(t-s)v(s)ds$ term in the GLE model. $\dot{B}_H$ is the fractional noise.
  - For a physical system, the Caputo derivative and the fractional noise must appear in pairs.

Time FPDEs: Some probabilistic interpretations

- Hairer, Iyer et. al. recently showed that some intermediate time behaviors of celluclar flows can be described by time fractional diffusion equations with Caputo derivatives. (In other words, in certain scaling regime, the solutions to an advection-diffusion equations converge weakly to the solution of time fractional diffusion equations)
- Meerschaert and Scheffer noticed that the solution to the time fractional diffusion equation with order γ with initial data $u_0 = f(x)$ admits the following probabilistic representation:

$$u(x,t) = \mathbb{E}_x(f(W_{E_t}))$$

where W is a Brownian motion and E_t is an inverse γ-stable subordinator independent of β.

Fractional calculus

- Integrals of order $\gamma > 0$:

$$J^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds$$

The fractional integral is just the convolution between $\theta(t)f$ and $g_{\gamma}(t) = \theta(t)\frac{1}{\Gamma(\gamma)}t^{\gamma-1}$. $\theta(t)$ is the Heaviside step function.

  - Example: $J^2 f = \int_0^t \int_0^{\tau} f(s)ds\,d\tau = \int_0^t (t-s)f(s)\,ds$

- For derivatives, Riemann-Liouville and Caputo types are widely used: If $\gamma \in (n-1, n)$,

$$D_{rl}^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds,\ D_c^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}}ds$$

If $\gamma = n$, they are defined as the usual derivative $f^{(n)}(t)$.

Properties and issues for the traditional definition

- The integrals $\{J_\gamma\}$ form a semigroup. $D_{rl}^{\gamma} = D^n J_{\gamma-(n-1)}$ and $D_c^{\gamma} = J_{\gamma-(n-1)} D^n$.

- If $\alpha \ge \gamma > 0$:
$$D_{rl}^{\gamma} J_\alpha = D_c^{\gamma} J_\alpha = J_{\alpha-\gamma}.$$
The Riemann-Liouville operator and Caputo derivative are left inverse of integrals, but in general not the right inverse: $JDf = f(t) - f(0)$.

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    - $D_{rl}^{\gamma} 1 \neq 0$. This is due to a jump at $t = 0$ for causality.
    - $D_c^{\gamma} 1 = 0$. The Caputo derivative removes the singularity at $t = 0$. The dynamics is counted from $t = 0+$.

- The definition of Caputo derivative requires higher order derivatives. intuitively, if we want to define γ-th order derivative, we do not need it to be n-th order differentiable.

Some recent definitions to extend Caputo derivatives

- In the book by Kilbas et. al, the Caputo derivative with order $\gamma \in (0,1)$ was defined using Riemann-Liouville derivatives.
- In a recent work by Allen, Caffarelli and Vasseur, the Caputo derivative was defined by an integral form.
- In the work by Gorenflo, Luchko and Yamamoto, the Caputo derivative was extended to functions in some Sobolev spaces using functional analysis tools.
- Li and Liu extended the definition of Caputo derivatives to a certain class of locally integrable functions based on a convolution group. This definition reveals the underlying structures and is theoretically convenient .

Fractional differential equations

- Caputo derivatives count the dynamics from $t = 0^+$ and shares many similarities with usual derivatives, so more suited to initial value problems.

- Fractional ODE theory has been well established.
  - Books by Kilbas, and Diethelm for FODEs with Caputo derivatives.
  - Recently, Feng, Li, Liu and Xu studied 1D autonomous FODEs using a generalized definition. The monotonicity and blowup have been discussed thoroughly.

- Fractional SDEs: for physical systems, fractional noise must be paired with Caputo derivatives while for other models, they may not be paired.

- Very little rigorous study of FPDEs.
  - The time fractional diffusion equations have been studied by M. Taylor, and Caffarelli et. al.
  - Many papers used the old definition of Caputo derivatives to study other types of fractional PDEs, but are not mathematically rigorous.

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A convolution group and a generalized definition of Caputo derivatives

- In the book by Gel'fand and Shilov, integrals and derivatives of arbitrary order for a distribution supported on $[0,\infty)$ are defined to as

$$\varphi^{(\alpha)} = \varphi * g_\alpha, \quad g_\alpha := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \ \alpha \in \mathbb{C}.$$

  Here $t_+ = \max(t,0) = \theta(t)t$ and $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ must be understood as distributions for $\Re(\alpha) \le 0$.

- We use this group $\{g_\alpha : \alpha \in \mathbb{C}\}$ (with more convenient forms) in our work to define the modified Riemann-Liouville calculus. The derivatives are now inverse of integral operators.

A convolution group and a generalized definition of Caputo derivatives

The extended Caputo derivative:

Definition 1

Let $0<\gamma<1$. Consider $u\in L^1_{loc}(0,T;\mathbb{R})$ and there is $u_0\in\mathbb{R}$ such that $\lim_{t\to 0}\frac{1}{t}\int_0^t|u(s)-u_0|ds=0$. The γ-th order Caputo derivative of u is a distribution in $\mathscr{D}'(-\infty,T)$ with support in $[0,T)$, given by

$$D_c^\gamma u := g_{-\gamma} * \Big((u-u_0)\theta(t)\Big).$$

- For $\beta\in(-1,0)$, $g_\beta(t)=\frac{1}{\Gamma(1+\beta)}D(\theta(t)t^\beta)$, where D means the distributional derivative in $\mathscr{D}'$.

- One can check that this agrees with the traditional definition of Caputo derivatives if u is absolutely continuous.

Right derivatives

- Another group given by

$$\tilde{\mathscr{C}} := \{\tilde{g}_\alpha : \tilde{g}_\alpha(t) = g_\alpha(-t), \alpha \in \mathbb{R}\}.$$

  Clearly, *supp* $\tilde{g} \subset (-\infty, 0]$.

- Right Caputo derivatives:

Definition 2

Let $0 < \gamma < 1$. Consider $u \in L^1_{loc}(-\infty, T)$ such that T is a Lebesgue point. The γ-th order right Caputo derivative of u is a distribution in $\mathscr{D}'(\mathbb{R})$ with support in $(-\infty, T]$, given by

$$\tilde{D}^\gamma_{c;T} u := \tilde{g}_{-\gamma} * (\theta(T-t)(u(t) - u(T-))).$$

- If u is absolutely continuous on (a, T), $a < T$, then

$$\tilde{D}^\gamma_{c;T} u = -\frac{1}{\Gamma(1-\gamma)} \int_t^T (s-t)^{-\gamma} \dot{u}(s) ds, \ \forall t \in (a, T). \tag{1}$$

Integration by parts using right derivatives

Lemma 3

Let u, v be absolutely continuous on $(0, T)$, then we have the integration by parts formula for Caputo derivatives

$$\int_0^T (D_c^\gamma u)(v(t) - v(T-))dt = \int_0^T (u(t) - u(0+))(\tilde{D}_{c;T}^\gamma v)dt.$$

This relation also holds if $u \in L^1_{loc}(0, T)$ so that $u(0+)$ exists and $v \in C_c^\infty(-\infty, T)$.

Remark 1

If $\gamma \to 1$, it is not hard to see that $\tilde{D}_{c;T}^\gamma u \to -u'(t)$ weakly. Hence, the right derivatives carry a natural negative sign.

Remark 2

Assuming that u, v are smooth functions, this identity can be verified easily using traditional definitions of left and right Caputo derivatives.

Weak Caputo derivatives for functions valued in Banach spaces?

- Fix $T > 0$ and introduce the following sets:

$$\mathscr{D}' := \left\{ v \,\middle|\, v : C_c^\infty((-\infty, T); \mathbb{R}) \to B \text{ is a bounded linear operator} \right\}.$$

In other words, $\mathscr{D}'$ consists of functionals from $C_c^\infty((-\infty, T); \mathbb{R})$ to B.

- Definition of weak Caputo derivatives:

Definition 4

Let B be a Banach space and $u \in L^1_{loc}([0, T); B)$. Let $u_0 \in B$. We define the weak Caputo derivative of u associated with initial data u_0 to be $D_c^\gamma u \in \mathscr{D}'$ such that for any test function $\varphi \in C_c^\infty((-\infty, T); \mathbb{R})$,

$$\langle D_c^\gamma u, \varphi \rangle := \int_{-\infty}^{T} (u - u_0)\theta(t)(\tilde{D}_{c;T}^\gamma \varphi)\, dt = \int_0^T (u - u_0)\tilde{D}_{c;T}^\gamma \varphi\, dt.$$

Some basic properties of the weak Caputo derivatives

- supp $D_c^\gamma u \subset [0, T)$.
- If $B = \mathbb{R}^d$ and $u(0+) = u_0$, then the Caputo derivative is then given by
$$D_c^\gamma u = g_{-\gamma} * ((u - u_0)\theta(t)).$$
- Let $\gamma \in (0, 1)$. If $D_c^\gamma u \in L^1_{loc}([0, T); B)$, then
$$u(t) = J_\gamma(D_c^\gamma u) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} D_c^\gamma u \, ds, \ a.e. \text{ on } (0, T).$$
where the integral is understood as the Lebesgue integral.
- If u is absolutely continuous, then $D_c^\gamma u \in L^1_{loc}([0, T); B)$ for $\gamma \in (0, 1)$ and
$$D_c^\gamma u(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{u}(s)}{(t-s)^\gamma} ds, \ t \in [0, T). \tag{2}$$

An important lemma

Proposition 1

Let $\gamma \in (0,1)$. If $u : [0,T) \to B$ is $C^1((0,T);B) \cap C^0([0,T);B)$, and $u \mapsto E(u) \in \mathbb{R}$ is a C^1 convex functional on B, then

$$D_c^\gamma u(t) = \frac{1}{\Gamma(1-\gamma)} \left(\frac{u(t)-u(0)}{t^\gamma} + \gamma \int_0^t \frac{u(t)-u(s)}{(t-s)^{\gamma+1}} ds \right) \tag{3}$$

and

$$D_c^\gamma E(u(t)) \le \langle D_u E(u), D_c^\gamma u \rangle, \tag{4}$$

where $D_u E(\cdot) : B \to B'$ is the Fréchet differential and $\langle \cdot, \cdot \rangle$ is understood as the dual pairing between B' and B.

The proof is very straightforward by observing

$$E(u(t)) - E(b) \le D_u E(u(t)) \cdot (u(t) - b), \ \forall b \in B.$$

Regularity improvements by fractional integral

Proposition 2

Let B be a Banach space and $T > 0$ and $\gamma \in (0,1)$. Suppose $u \in L^1_{loc}((0,T);B)$ and $f = D_c^{\gamma} u$ with an assigned initial value $u_0 \in B$.

(i) *If $f \in L^{\infty}((0,T);B)$, then u is Hölder continuous with order $\gamma - \varepsilon$ for any $\varepsilon \in (0,\gamma)$. If f is continuous, then u is γ-th order Hölder continuous.*

(ii) *If further there exists $\delta > 0$, such that $f \in C^{m,\beta}([\delta/4, T];B)$, with $\beta \in [0,1]$, then*

$$u \in \begin{cases} C^{m,\beta+\gamma}([\delta,T];B), & \beta+\gamma < 1, \\ C^{m+1,\beta+\gamma-1}([\delta,T];B), & \beta+\gamma > 1, \\ C^{m,1;1}([\delta,T];B), & \beta+\gamma = 1. \end{cases}$$

(iii) *If there exists $\delta > 0$, such that $f \in H^s((\delta/4,T);B)$ (the Sobolev space $W^{1,s}((\delta/4,T);B)$), then $u \in H^{s+\gamma}((\delta,T);B)$*

The claims are not true in general if $\delta = 0$. For example, $f = 1$, then we have $u = u_0 + t^{\gamma}/\Gamma(1+\gamma)$. There is an intrinsic singularity at $t = 0$.

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The traditional Aubin-Lions Lemma and its variants

- The traditional Aubin-Lions-Simon lemma:
  Let X_0, X, X_1 be three Banach spaces such that X_0 is compactly embedded into X and X is continuously embedded into X_1. Let

$$W = \{u \in L^p([0,T];X_0) | u_t \in L^q([0,T];X_1)\}.$$

  (i) If $p < \infty$, W is relatively compact in $L^p([0,T];X)$.
  (ii) If $p = \infty$ and $q > 1$, W is relatively compact in $C([0,T];X)$.

- In the work by Chen, Jungel and Liu, several variants of Aubin-Lions lemma have been summarized and proved. The results here apply to many common nonlinear PDEs.

The role of Aubin-Lions Lemma in parabolic equations

The Aubin-Lions Lemma is a powerful tool for establishing the existence of weak solutions to nonlinear PDEs with diffusion.

As a trivial but illustrating example, consider

$$u_t = \Delta u,\ x \in \Omega;\ \ u = 0,\ x \in \partial\Omega.$$

Suppose $u|_{t=0} = u_0 \in L^2(\Omega)$. The proof of existence of weak solutions is as following (though I did not introduce the definition of weak solutions):

- Find a common basis to L^2 and H_0^1, $\{w_k\}$. Expand $u_0 = \sum_k \alpha^k w_k$.
- Let $u_m = \sum_{k=1}^m c_k(t) w_k$. Solve the ODE system

$$\langle \partial_t u_m, w_j \rangle = \langle \Delta u_m, w_j \rangle,\ j = 1, \ldots, m.$$

- We have $\frac{1}{2}\partial_t \|u_m\|_2^2 = -\|\nabla u_m\|_2^2$, which yields the uniform boundedness of u_m in $L^\infty(L^2)$ and $L^2(H_0^1)$.
- One can estimate $\|\partial_t u_m\|_{L^2(H^{-1})}$. Then, the Aubin-Lions lemma yields the relative compactness of $\{u_m\}$ in $L^2(L^2)$. The limit of a subsequence is then a weak solution.

FPDEs? Some a priori estimates and motivation

Consider the simple time fractional diffusion equation with Dirichlet condition:

$$D_c^{\gamma} u = \Delta u,\ x \in \Omega.$$

- Formula is available (See the notes by M. Taylor).
- Show the existence of solutions without using the formula? We do *a priori* estimates. $\int u\, D_c^{\gamma} u dx = -\int |\nabla u|^2 dx$. Using the important lemma about convex functional:

  $$\frac{1}{2} D_c^{\gamma} \|u\|_2^2 \le \int u\, D_c^{\gamma} u dx = -\int |\nabla u|^2 dx$$

  This implies that

  $$\|u\|_2^2(t) + \frac{2}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|\nabla u(s)\|_2^2\, ds \le \|u_0\|_2^2.$$

- Goal: use the boundedness of $\int_0^t (t-s)^{\gamma-1} \|\nabla u(s)\|_2^2\, ds$ and $L^{\infty}(0,\infty; L^2)$ to find a certain compactness criterion and then the existence of weak solution follows.

The first compactness criteria

We propose two compactness criteria, which could be useful at different situations (see the examples later).

Theorem 5

Let $T > 0, \gamma \in (0,1)$ and $p \in [1,\infty)$. Let M, B, Y be Banach spaces. $M \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Suppose $W \subset L^1_{loc}((0,T); M)$ satisfies:

(i) *There exists $C_1 > 0$ such that $\forall u \in W$,*

$$\sup_{t\in(0,T)} J_\gamma(\|u\|_M^p) = \sup_{t\in(0,T)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|u\|_M^p(s)ds \le C_1.$$

(ii) *There exist $r \in (\frac{p}{1+p\gamma}, \infty) \cap [1,\infty)$ and $C_3 > 0$ such that $\forall u \in W$, there is an assignment of initial value u_0 for u so that the weak Caputo derivative satisfies:*

$$\|D_c^\gamma u\|_{L^r((0,T);Y)} \le C_3.$$

Then, W is relatively compact in $L^p((0,T); B)$.

The second compactness criteria

Theorem 6

Let $T > 0$,$\gamma \in (0,1)$ and $p \in [1,\infty)$. Let M, B, Y be Banach spaces. $M \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Suppose $W \subset L^1_{loc}((0,T);M)$ satisfies:
(i). There exists $r_1 \in [1,\infty)$ and $C_1 > 0$ such that $\forall u \in W$,

$$\sup_{t\in(0,T)} J_\gamma(\|u\|_M^{r_1}) = \sup_{t\in(0,T)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|u\|_M^{r_1}(s)ds \le C_1.$$

(ii). There exists $p_1 \in (p,\infty]$, W is bounded in $L^{p_1}((0,T);B)$.
(iii). There exist $r_2 \in [1,\infty)$ and $C_2 > 0$ such that $\forall u \in W$, there is an assignment of initial value u_0 for u so that the weak Caputo derivative satisfies:

$$\|D_c^\gamma u\|_{L^{r_2}((0,T);Y)} \le C_2,$$

Then, W is relatively compact in $L^p((0,T);B)$.

Sketch of the proof 1: The implication of boundness fractional integral

- Let $\gamma \in (0,1)$. If $\sup_{t\in(0,T)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}\|f(s)\|_M^p ds < \infty$, then $f \in L^p((0,T);M)$.
  - Proof: $\int_0^T \|f\|_M^p(s)ds \le T^{1-\gamma}\int_0^T (T-s)^{\gamma-1}\|f\|_M^p ds$.
- The results can not essentially be improved: Suppose μ is the middle $1/3$ Cantor measure that is Ahlfors-regular of degree (or dimension) $\alpha = \ln 2/\ln 3$. Then, if $\gamma > 1-\alpha$,

$$\sup_{t\in[0,1]} \int_0^1 |t-s|^{\gamma-1} d\mu(s) < \infty.$$

Sketch of the proof 2: Time shift estimates

Proposition 3

Fix $T > 0$. Let B be a Banach space and $\gamma \in (0,1)$. Suppose $u \in L^1_{loc}((0,T);B)$ has a weak Caputo derivative $D_c^\gamma u \in L^p((0,T);B)$ associated with initial value $u_0 \in B$. If $p\gamma \geq 1$, we set $r_0 = \infty$ and if $p\gamma < 1$, we set $r_0 = p/(1-p\gamma)$. Then, there exists $C > 0$ independent of h and u such that

$$\|\tau_h u - u\|_{L^r((0,T-h);B)} \leq \begin{cases} Ch^{\gamma+\frac{1}{r}-\frac{1}{p}}\|D_c^\gamma u\|_{L^p((0,T);B)}, & r \in [p, r_0), \\ Ch^{\gamma}\|D_c^\gamma u\|_{L^p((0,T);B)}, & r \in [1, p]. \end{cases} \tag{5}$$

Sketch of the proof 3: Some key lemmas

- The first:

Lemma 7

Suppose M, B, Y are three Banach spaces. $M \hookrightarrow B \hookrightarrow Y$ with the embedding $M \to B$ be compact. $1 \le p < \infty$ and
(i). W is bounded in $L^p((0,T);M)$;
(ii). $\|\tau_h f - f\|_{L^p((0,T-h);Y)} \to 0$ uniformly as $h \to 0$.
Then, W is relatively compact in $L^p((0,T);B)$.

- The second:

Lemma 8

Let $1 < p_1 \le \infty$. If W is a bounded set in $L^{p_1}((0,T);B)$ and relatively compact in $L^1_{loc}((0,T);B)$, then it is relatively compact in $L^p((0,T);B)$ for all $1 \le p < p_1$.

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Time fractional compressible Navier-Stokes equations

The first toy problem is:

$$\begin{cases} D_c^\gamma u + \nabla\cdot(uu) + \dfrac{1}{2}\nabla(|u|^2) = \Delta u, \quad x\in\Omega, \\ u|_{\partial\Omega} = 0, \ x\in\partial\Omega. \end{cases}$$

- $\Omega\subset\mathbb{R}^d (d=2,3)$ is an open bounded domain with smooth boundary.
- The nonconservative form $D_c^\gamma u + u\cdot\nabla u + (\nabla u)\cdot u + (\nabla\cdot u)u = \Delta u$ is the time fractional Burgers equation.
- For the usual time derivative, it is also called the Euler-Poincare equation.

Weak formulation

Using the definition of weak Caputo derivative, we can formulate the definition of weak solutions as following

Definition 9

Let $\gamma \in (0,1)$. We say $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H_0^1(\Omega))$ with

$$D_c^\gamma u \in L^{q_1}((0,T);H^{-1}(\Omega)),\ q_1 = \min(2, 4/d),$$

is a weak solution with initial data $u_0 \in L^2(\Omega)$, if

$$\begin{aligned}\left\langle u(x,s) - u_0, \tilde{D}_{c;T}^\gamma \varphi \right\rangle - \int_0^T \int_\Omega \nabla\varphi : u \otimes u\, dxdt \\ - \frac{1}{2}\int_0^T \int_\Omega \nabla\cdot\varphi |u|^2\, dxdt = \int_0^T \int_\Omega u \cdot \Delta\varphi\, dxdt, \quad (6)\end{aligned}$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^d)$. We say a weak solution is a regular weak solution if $u(0+) = u_0$.

Existence of weak solutions

- The idea is to use Galerkin's method to form an approximating function sequence and then apply our first compactness criteria.

- Pick a common basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ and $H_0^1(\Omega)$, orthonormal in L^2. By the uniform boundedness principle,

$$P_m : v \mapsto \sum_{k=1}^{m} \alpha_k w_k$$

  is uniformly bounded in both H_0^1 and L^2.

- Decompose $u_0 = \sum_{k=1}^{\infty} \alpha^k w_k(x)$ in $L^2(\Omega)$. Consider $u_m(t) = \sum_{k=1}^{m} c_m^k(t) w_k$ satisfying

$$\begin{aligned} &\langle D_c^{\gamma} u_m, w_j\rangle + \langle \nabla\cdot(u_m \otimes u_m), w_j\rangle + \frac{1}{2}\langle \nabla |u_m|^2, w_j\rangle = \langle \Delta u_m, w_j\rangle, \\ &u_m(0) = \sum_{k=1}^{m} c_m^k(0) w_k = \sum_{k=1}^{m} \alpha^k w_k. \end{aligned} \tag{7}$$

Existence of weak solutions (continued)

- The equation for u_m is reduced to an FODE for the coefficient c_m. By the result in Feng, Li, Liu and Xu:

$$u_m \in C^1((0,\infty); H_0^1(\Omega)) \cap C^0([0,\infty); H_0^1(\Omega)).$$

Note that $u_m(0) \in H_0^1(\Omega)$ though u_0 is not necessarily in H_0^1.

- Using the important lemma about convex functional, we have the energy estimates:

$$\|u_m\|_{L^\infty((0,\infty);L^2(\Omega))} \le \|u_0\|_2, \ \sup_{0\le t<\infty} \int_0^t (t-s)^{\gamma-1} \|\nabla u_m\|_2^2 ds \le \frac{1}{2}\Gamma(\gamma)\|u_0\|_2.$$

- Using the fact that P_m is uniformly bounded, we find

$$\|D_c^\gamma u_m\|_{L^{q_1}(0,T;H^{-1})} \le C, \ q_1 = \min\left(2, \frac{4}{d}\right).$$

- The second compactness criterion then yields a convergent subsequence in $L^2(0,T;L^2)$, and the limit is a weak solution.

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2D time fractional Keller-Segel equations

As another toy problem, we consider

$$\begin{cases} D_c^\gamma \rho + \nabla\cdot(\rho\nabla c) = \Delta\rho,\ x\in\mathbb{R}^2, \\ -\Delta c = \rho,\ x\in\mathbb{R}^2. \end{cases}$$

Initial condition is $\rho(x,0) = \rho_0 \geq 0$.

Definition 10

We say $\rho \in L^\infty(0,T;L^1(\mathbb{R}^2))\cap L^\infty(0,T;L^2(\mathbb{R}^2))\cap L^2(0,T;H^1(\mathbb{R}^2))$ is a weak solution with initial data $\rho_0 \geq 0$ and $\rho_0 \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, if
(i). $\rho(x,t)\geq 0$. (ii). There exists $q\in(1,2)$ such that $D_c^\gamma u \in L^{q_1}((0,T);W^{-2,q}(\mathbb{R}^2))$ for any $q_1\in(1,\infty)$. (iii). For any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2)$,

$$\left\langle u(x,s)-u_0, \tilde{D}_{c;T}^\gamma \varphi\right\rangle - \int_0^T\int_{\mathbb{R}^2} \nabla\varphi\cdot(\nabla(-\Delta)^{-1}\rho)\rho\,dxdt = \langle u,\Delta\varphi\rangle.$$

We say a weak solution is a regular weak solution if $u(0+) = u_0$.

The regularized system

To prove the existence of weak solutions, we construct an approximating sequence. One standard idea is to consider a regularized system.

- $J(x) \in C_c^\infty(\mathbb{R}^2)$, $J(x) \geq 0$ and $\int_{\mathbb{R}^2} J(x)dx = 1$, $J_\varepsilon = \frac{1}{\varepsilon^2} J\left(\frac{x}{\varepsilon}\right)$.
-
$$\begin{cases} D_c^\gamma \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon \nabla c^\varepsilon) = \Delta \rho^\varepsilon, \\ -\Delta c^\varepsilon = \rho^\varepsilon * J_\varepsilon, \end{cases}$$

  Initial data $\rho_0^\varepsilon = \rho_0 * J_\varepsilon$, which has the same L^1 norm as ρ_0.
- The question is that how we can show the existence of strong solutions of the regularized system.
  - The answer is to show the existence of mild solution and then show the mild solution is a strong solution with the regularity proved.

Linear time fractional advection diffusion equations

$$D_c^\gamma \rho + \nabla\cdot(\rho a(x,t)) = \Delta\rho,$$

Initila data $\rho(x,0)=\rho_0$.

- By the Laplace transform, we have ($A=-\Delta$):

$$\rho(x,t) = E_\gamma(-t^\gamma A)\rho_0 + \gamma\int_0^t \tau^{\gamma-1}E_\gamma'(-\tau^\gamma A)(-\nabla\cdot(\rho a)|_{t-\tau})d\tau.$$

- An important observation is

$$\begin{aligned}\|E_\gamma'(-\tau^\gamma A)\nabla f\|_{H^\alpha}^2 &\le C\int_{\mathbb{R}^2} E_\gamma'(-\tau^\gamma |k|^2)^2|k|^2|\hat{f}_k|^2(1+|k|^{2\alpha})dk \\ &\le C\tau^{-\gamma}\int_{\mathbb{R}^2}|\hat{f}_k|^2(1+|k|^{2\alpha})dk = C\tau^{-\gamma}\|f\|_{H^\alpha}^2.\end{aligned}$$

Linear time fractional advection diffusion equations

Lemma 11

Suppose $a(x,t)$ is smooth and all the derivatives are bounded. Then:

(i) *If $\rho_0 \in L^1(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$, then $\forall T > 0$, the equation has a unique mild solution in $C([0,T]; H^\alpha(\mathbb{R}^2))$.*

(ii) *For the unique mild solution in (i), $\forall T > 0$,*

$$\rho \in C^{0,\gamma}([0,T]; H^\alpha(\mathbb{R}^2)) \cap C^\infty((0,T); H^\alpha(\mathbb{R}^2)).$$

The mild solution is a strong solution in $C([0,T]; H^{\alpha-2})$ so that:

$$D_c^\gamma \rho = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{\rho}(s)}{(t-s)^\gamma} ds = -\nabla \cdot (\rho a(x,t)) + \Delta \rho.$$

(iii) *If $\rho_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $\rho_0 \geq 0$, then $\rho(x,t) \geq 0$, and*

$$\int_{\mathbb{R}^2} \rho \, dx = \int_{\mathbb{R}^2} \rho_0 \, dx.$$

Strong solutions of the regularized system

$$
\begin{cases}
D_c^\gamma \rho^\varepsilon + \nabla\cdot(\rho^\varepsilon \nabla c^\varepsilon) = \Delta \rho^\varepsilon, \\
-\Delta c^\varepsilon = \rho^\varepsilon * J_\varepsilon,
\end{cases}
$$

Initial data $\rho_0^\varepsilon = \rho_0 * J_\varepsilon$.

- Step 1: Existence and uniqueness of mild solutions. This is similar as what we do for the linear equations.
- Step 2: Let $c^\varepsilon = (-\Delta)^{-1}\rho^\varepsilon$ and consider the linear problem:

  $$D_c^\gamma v + \nabla\cdot(v\nabla c^\varepsilon) = \Delta v.$$

  By the results for linear advection-diffusion equation, we know this has strong solutions, which then must be the mild solution ρ^ε. Also, it is nonnegative and preserves L^1 norm.

Energy estimates and compactness

- By the standard Sobolev embeddings and the important lemma about convex energy functional: $\frac{1}{q}D_c^{\gamma}\|\rho^{\varepsilon}\|_q^q \le \langle(\rho^{\varepsilon})^{q-1}, D_c^{\gamma}\rho^{\varepsilon}\rangle$, we have

Lemma 12

Suppose $\rho_0 \ge 0$ satisfies that $\rho_0 \in L^1 \cap L^2$ and $M_0 = \|\rho_0\|_1$ is sufficiently small. Then, $\rho^{\varepsilon} \ge 0$ and for any fixed $T > 0$,

$$\|\rho^{\varepsilon}\|_{L^{\infty}(0,T;L^q)} \le C(q,T), \forall q \in [1,2],$$

$$\sup_{0\le t\le T}\int_0^t (t-s)^{\gamma-1}\|\nabla\rho^{\varepsilon}\|_2^2 ds \le C(T).$$

Further, there exists $q \in (1,2)$ such that $D_c^{\gamma}\rho^{\varepsilon}$ is uniformly bounded in $L^{q_1}(0,T;W^{-2,q}(\mathbb{R}^2))$ for any $q_1 \in (1,\infty)$.

- The first compactness criterion then yields a convergent subsequence, which turns out to be a weak solution.

- This is a joint work with Prof. Jian-Guo Liu:

  L. Li and J. Liu *Some compactness criteria for weak solutions of time fractional PDEs*. Preprint.

- Other references
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Thank you for your attention!