The multivariate Hermite–Laguerre connection

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joint work with Stephanie Troppmann (arXiv:1303.5192)

Chicago, March 29, 2014

Let $\varepsilon > 0$,

$$\varphi_0(x) = \pi^{-1/4} \exp(-\tfrac{1}{2\varepsilon}x^2), \qquad x \in \mathbb{R},$$

and $a = \frac{1}{\sqrt{2\varepsilon}}(x + \nabla_x)$, $a^\dagger = \frac{1}{\sqrt{2\varepsilon}}(x - \nabla_x)$.

The eigenfunctions of $\frac{1}{2}(aa^\dagger + a^\dagger a) = \frac{1}{2}(-\varepsilon^2 \Delta_x + x^2)$ are

$$\varphi_{k+1} = \frac{1}{\sqrt{k+1}} a^\dagger \varphi_k, \qquad k \in \mathbb{N}.$$

$$\varphi_k = \frac{1}{\sqrt{2^k k!}} h_k \varphi_0,$$

where (h_k) is defined by the 3-term recurrence

$$h_{k+1} = \frac{2}{\sqrt{\varepsilon}} x h_k - 2k h_{k-1}$$

or the Rodriguez formula

$$h_k(x) = \exp(\tfrac{1}{\varepsilon} x^2)(-\nabla_x)^k \exp(-\tfrac{1}{\varepsilon} x^2).$$

Let $\varepsilon > 0$ and $\varphi : \mathbb{R}^d \to \mathbb{C}$. Define

$$\mathcal{W}^\varepsilon(\varphi)(x,\xi) = (2\pi\varepsilon)^{-d} \int \overline{\varphi(x + \tfrac{1}{2}y)} \varphi(x - \tfrac{1}{2}y) \mathrm{e}^{iy\cdot\xi/\varepsilon} dy$$

for $(x,\xi) \in \mathbb{R}^{2d}$. Then,

$$\int \mathcal{W}^\varepsilon(\varphi)(x,\xi) d\xi = |\varphi(x)|^2, \qquad \int \mathcal{W}^\varepsilon(\varphi)(x,\xi) dx = |\mathcal{F}^\varepsilon(\xi)|^2.$$

Hudson ’74, Soto/Claverie ’83:

$$\mathcal{W}^\varepsilon(\varphi) \geq 0 \quad \Longleftrightarrow \quad \varphi(x) = \exp(-\tfrac{1}{2\varepsilon}(x \cdot Cx + 2b \cdot x + \gamma)),$$

for $b \in \mathbb{C}^d$, $\gamma \in \mathbb{C}$, and $C \in \Sigma_d$ with

$$\Sigma_d = \left\{ C \in \mathbb{C}^{d \times d} \mid C = C^T, \, \mathrm{Re}(C) > 0 \right\}.$$

Groenewold '46:

$$\mathcal{W}^\varepsilon(\varphi_k)(x,\xi) = \frac{(-1)^k}{\pi\varepsilon} L_k(\tfrac{2}{\varepsilon}|z|^2)\exp(-|z|^2/\varepsilon)$$

with $z = x + i\xi$ for $x, \xi \in \mathbb{R}$ and

$$L_k(x) = \frac{\mathrm{e}^x}{n!}\nabla_x^k\left(\mathrm{e}^{-x}x^k\right)$$

the kth Laguerre polynomial.

$$\varphi_0(x) = (\pi\varepsilon)^{-1/4} \det(Q)^{-1/2} \exp\left(\tfrac{i}{2\varepsilon}(x-q)\cdot PQ^{-1}(x-q) + \tfrac{i}{\varepsilon} p \cdot (x-q)\right)$$

Let

$$A^\dagger = \tfrac{i}{\sqrt{2\varepsilon}}\left(P^*(x-q) - Q^*(-i\varepsilon\nabla_x - p)\right).$$

The eigenfunctions of $\frac{1}{2}(A\cdot A^\dagger + A^\dagger \cdot A)$ are

$$\varphi_{k+e_j} = \tfrac{1}{\sqrt{k_j+1}} A_j^\dagger \varphi_k, \qquad k \in \mathbb{N}^d.$$

$Q, P \in \mathbb{C}^{d\times d}$ satisfy

$$Q^T P - P^T Q = 0, \qquad Q^* P - P^* Q = 2i\mathrm{Id}.$$

That is,

$$F = \begin{pmatrix} \mathrm{Re}(Q) & \mathrm{Im}(Q) \\ \mathrm{Re}(P) & \mathrm{Im}(P) \end{pmatrix}$$

satisfies $F^T J F = J$ with

$$J = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

$\Rightarrow$: For all symplectic pairs $P, Q \in \mathbb{C}^{d\times d}$, $C = PQ^{-1} \in \Sigma_d$.

$\Leftarrow$: For all $C \in \Sigma_d$ there exists a symplectic pair $P, Q \in \mathbb{C}^{d\times d}$ with

$$C = PQ^{-1}.$$

$$\varphi_k = \frac{1}{\sqrt{2^{|k|}k!}} p_k \varphi_0,$$

where (p_k) is defined by the 3-term recurrence

$$\left(p_{k+e_j}\right)_{j=1}^d = \sqrt{\tfrac{2}{\varepsilon}} Q^{-1}(x-q)p_k - 2Q^{-1}\overline{Q}\left(k_j p_{k-e_j}\right)_{j=1}^d$$

or the new Rodriguez formula

$$p_k(x) = |\varphi_0(x)|^{-2}(-\sqrt{\varepsilon}Q^*\nabla_x)^k |\varphi_0(x)|^2.$$

If $Q = Q^T \in \mathbb{R}^{d\times d}$, then

$$p_k(x) = \prod_{j=1}^{d} h_{k_j}\left(\frac{1}{\sqrt{\varepsilon}}(Q^{-1}(x-q))_j\right), \qquad x \in \mathbb{R}^d,$$

see Hagedorn '85.

Intriguing thought:
Given $C = PQ^{-1}$. Why not choose Q real symmetric?

$$\mathrm{e}^{-i(-\frac{\varepsilon^2}{2}\Delta_x+V)t/\varepsilon}\varphi_k[q_0,p_0,Q_0,P_0] \;\approx\; \mathrm{e}^{iS_t/\varepsilon}\varphi_k[q_t,p_t,Q_t,P_t]$$

as $\varepsilon \to 0$, where

$$\begin{aligned} \dot{q}_t &= p_t, \qquad & \dot{p}_t &= -\nabla V(q_t), \\ \dot{Q}_t &= P_t, \qquad & \dot{P}_t &= -D^2V(q_t)Q_t, \end{aligned}$$

see Hagedorn '81, '85, '98, ...

L./Troppmann '13:

$$\mathcal{W}^\varepsilon(\varphi_k)(x,\xi) = \frac{(-1)^{|k|}}{(\pi\varepsilon)^d} \prod_{j=1}^{d} L_{k_j}(\tfrac{2}{\varepsilon}|z_j|^2)\, \exp(-|z|^2/\varepsilon)$$

with $z = -i(P^T(x-q) - Q^T(\xi - p))$ for $x, \xi \in \mathbb{R}^d$.

Note that

$$\begin{pmatrix} \mathrm{Re}(z) \\ \mathrm{Im}(z) \end{pmatrix} = F^{-1} \begin{pmatrix} x-q \\ \xi - p \end{pmatrix}.$$

1. Deduce a raising operator $B^\dagger$ for p_k.

2. Compute

$$\tau_y \circ (B^\dagger)^k = \sum_{\nu \le k} \binom{k}{\nu} \left(\frac{2}{\sqrt{\varepsilon}} Q^{-1} y \right)^{k-\nu} (B^\dagger)^\nu \circ \tau_y$$

for $(\tau_y f)(x) = f(x+y)$ for $y \in \mathbb{C}^d$.

3. Deduce

$$p_k(x+y) = \sum_{\nu \le k} \binom{k}{\nu} \left(\frac{2}{\sqrt{\varepsilon}} Q^{-1} y \right)^{k-\nu} p_\nu(x).$$

4. Use the sum rule for

$$\int \overline{p_k}(x+y_1)p_k(x+y_2)|\varphi_0(x)|^2 dx =$$

$$2^k k! \prod_{j=1}^{d} L_{k_j}\Big(-\tfrac{2}{\varepsilon}(\overline{Q^{-1}}y_1)_j(Q^{-1}y_2)_j\Big)\,.$$

5. Compute

$$\mathcal{W}^\varepsilon(\varphi_k)(x,\xi) = (2\pi\varepsilon)^{-d} \int \overline{\varphi_k(x+\tfrac{1}{2}y)}\varphi_k(x-\tfrac{1}{2}y)\mathrm{e}^{iy\cdot\xi/\varepsilon}dy.$$

For $z = -i(P^T(x-q) - Q^T(\xi - p))$,

$$\begin{aligned} \mathcal{W}^\varepsilon(\varphi_k[q,p,Q,P]) &= \frac{(-1)^{|k|}}{(\pi\varepsilon)^d} \prod_{j=1}^d L_{k_j}(\tfrac{2}{\varepsilon}|z_j|^2) \exp(-|z|^2/\varepsilon) \\ &= \mathcal{W}^\varepsilon(T_{q,p}\, R_F\, \varphi[0,0,\mathrm{Id}, i\mathrm{Id}]) \end{aligned}$$

with

$$T_{q,p} = \exp\Big(\tfrac{i}{\varepsilon}(p\cdot x - q\cdot(-i\varepsilon\nabla_x)\Big)$$

and R_F the metaplectic representation of F.

5. A Corollary

There exists $c \in \mathbb{C}$, $|c| = 1$ with

$$\varphi_k[q, p, Q, P] = c\, T_{q,p}\, R_F\, \varphi_k[0, 0, \mathrm{Id}, i\mathrm{Id}].$$

Let $W = W^T \in \mathbb{C}^{d\times d}$, $|W| \leq \mathrm{Id}$.

For $W = U|W|$, $B = U\mathrm{artanh}|W|$, set

$$D_B = \exp\Big(\tfrac{1}{2}(a^\dagger \cdot Ba^\dagger - a \cdot B^* a)\Big)\,.$$

We have $D_B = R_F$, if

$$\begin{aligned} Q &= (\mathrm{Id} + W)(\mathrm{Id} - W^*W)^{-1/2}, \\ P &= i(\mathrm{Id} - W)(\mathrm{Id} - W^*W)^{-1/2}. \end{aligned}$$

- ▷ Semiclassical wavepackets ('85, '98)
- ▷ Hagedorn wavepackets ('08)
- ▷ Generalized squeezed states ('92)
- ▷ Generalized coherent states ('97)

Thank you.