Dynamics of a heavy quantum tracer particle in a boson gas

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Goal: Derivation of mean field equations for heavy tracer particle in a gas of bosons displaying Bose-Einstein condensation.

[Fröhlich-Zhou, F-Soffer-Zhou] Classical tracer particle interacting with nonlinear Hartree eq and Hamiltonian friction.

Without tracer particle:

[Hepp] Derivation of Hartree eq from quantum dyn. in Fock space

[Rodnianski-Schlein] Convergence rates for mean field limit

[Grillakis-Machedon, G-M-Margetis], [Schlein et al] Much improved convergence rates

Other approaches: [Spohn], [Erdös-Schlein-Yau], [Kirkpatrick-Schlein-Staffilani], [C-Pavlovic], [X. Chen-Holmer], [Fröhlich et al], [Pickl], ...

QFT: Description of field of indistinguishable quantum particles

Wave function for one particle: $f \in L^2(\mathbb{R}^3)$

Wave function for two indistinguishable particles (bosons):

$$\frac{1}{2}\Big(f_1 \otimes f_2 + f_2 \otimes f_1 \Big)(x_1, x_2) \in \mathrm{Sym}_2(L^2(\mathbb{R}^3))^{\otimes 2}$$

n indistinguishable particles:

$$\underbrace{\frac{1}{n!}\sum_{\pi \in S_n}}_{\mathrm{Sym}_n} \big(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)} \big)(x_1, \ldots, x_n) \in \mathcal{F}_n = \mathrm{Sym}_n(L^2(\mathbb{R}^3))^{\otimes n}$$

Describe states with fluctuating particle # by vectors in boson Fock space

$$\mathcal{F} \;=\; \bigoplus_{n \geq 0} \mathcal{F}_n$$

Zero particle space: $\mathcal{F}_0 \;\;=\;\; \mathbb{C}$. Vacuum vector $\Omega = (1, 0, 0, \ldots)$.

Introduce creation and annihilation operators

$$a^+(f) \quad = \quad \mathrm{Sym}_{n+1}\, f \,\otimes\, \bullet \qquad : \; \mathcal{F}_n \to \mathcal{F}_{n+1}$$

$$a(f) \quad = \quad \left\langle\, f \,,\, \bullet \,\right\rangle_{L^2_{x_n}(\mathbb{R}^3)} \qquad : \; \mathcal{F}_n \to \mathcal{F}_{n-1}$$

under the condition that

$$a(f)\,\Omega \;=\; 0 \qquad \forall\, f \in L^2(\mathbb{R}^3)$$

Then, n bosons with wave functions $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)$:

$$\Phi^{(n)} \;=\; a^+(f_1) \cdots a^+(f_n)\,\Omega \;=\; \mathrm{Sym}_n\, f_1 \otimes \cdots \otimes f_n \;\in\; \mathcal{F}_n$$

Linear span of such states, for $n \geq 0$, is dense in $\mathcal{F}$.

$$\Phi \;=\; \big(\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n)}, \ldots\big) \;\in\; \mathcal{F}$$

Inner product: $\big\langle \Phi, \Psi \big\rangle_{\mathcal{F}} \;=\; \sum_n \big\langle \Phi^{(n)}, \Psi^{(n)} \big\rangle_{\mathcal{F}_n}$

Adjoints $a^+(f) = (a(f))^*$. Canonical commutation relations

$$[a(f), a^+(g)] \;=\; \big\langle f, g \big\rangle_{L^2} \quad , \quad [a^\sharp(f), a^\sharp(g)] \;=\; 0$$

$a^+(f)$, $a(f)$ are linear/antilinear in $f \in L^2(\mathbb{R}^3)$. Can write

$$a(f) \quad = \quad \int dx\, f^*(x)\, a_x$$

$$a^+(f) \quad = \quad \int dx\, f(x)\, a_x^+ \quad = \quad (a(f))^*$$

Operator-valued distributions a_z^+, a_x, CCR

$$\left[a_x, a_y^+\right] \quad = \quad \delta(x-y)$$

$$\left[a_x, a_y\right] \quad = \quad 0 \; = \; \left[a_x^+, a_y^+\right].$$

Fock vacuum $\Omega \in \mathcal{F}$, with $a_x \Omega = 0 \; \forall x \in \mathbb{R}^3$.

Definition of the model

Consider a heavy quantum mechanical tracer particle coupled to a field of identical scalar bosons with two-particle interactions.

Hilbert space for the quantum tracer particle $L^2(\mathbb{R}^3)$.

Boson Fock space

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{F}_n$$

with n-particle Hilbert space

$$\mathcal{F}_n := (L^2(\mathbb{R}^3))^{\otimes_{sym} n}$$

Creation- and annihilation operators, canonical commutation relations

$$[a_x, a_y^+] = \delta(x - y) \quad , \quad [a_x, a_y] = 0 \quad , \quad [a_x^*, a_y^*] = 0 \, ,$$

Fock vacuum $\Omega \in \mathcal{F}$, with $a_x \Omega = 0$ for all $x \in \mathbb{R}^3$.

Boson number operator and kinetic energy operator

$$N_b := \int dx\, a_x^+ a_x \quad , \quad T := \frac{1}{2} \int dx\, a_x^+ (-\Delta_x a_x)$$

Hilbert space of the coupled system

$$\mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}\,.$$

Initial data $\Phi_0 \in \mathcal{H}$ with expected particle number, $\Big\langle \Phi_0, \mathbf{1} \otimes N_b \Phi_0 \Big\rangle = N$.

Moreover, we assume that the mass of the heavy tracer particle is N.

Assume bosons interact via a mean field interaction potential $\frac{1}{2N} v$.

Accordingly, the Hamiltonian of the system is given by

$$\begin{aligned} \mathcal{H}_N \quad := \quad & -\frac{1}{2N} \Delta_X \otimes \mathbf{1} + \mathbf{1} \otimes T + \int dx\, w(X - x) \otimes a_x^+ a_x \\ & + \mathbf{1} \otimes \frac{\lambda}{2N} \int dx dy\, a_x^+ a_x v(x - y) a_y^+ a_y \end{aligned}$$

where w is the potential energy coupling tracer particle and bosons.

Formal similarity to translation-invariant model in non-relativistic Quantum Electrodynamics (QED) describing a quantum mechanical electron coupled to the quantized electromagnetic radiation field.

[Fröhlich, AHP 73] Infrared problem for massless bosons

[Bach-F-Sigal, Adv Math 98] Renormalization Group analysis of spectral problems in non-relativistic QED.

[C, JFA 08], [B-C-F-S, JFA 07] Infrared mass renormalization

[C-F, PSPM 07] Infrared representations without IR cutoff

[C-Pizzo-F, CMP 10, JMP 09] Scattering states without IR cutoff

[B-C-F-F-S, AHP 13] Effective dynamics of electron in non-relat. QED

Momentum operator for the boson field

$$P_b := \int dx\, a_x^+ (i\nabla_x a_x)$$

Define the conserved total momentum operator

$$P_{tot} = i\nabla_X \otimes \mathbf{1} + \mathbf{1} \otimes P_b$$

Hamiltonian is translation invariant, $[\mathcal{H}_N, P_{tot}] = 0$.

Consider the decomposition of $\mathfrak{H}$ as a fiber integral w.r.t. P_{tot}.

$$\mathfrak{H} = \int_{\mathbb{R}^3}^{\oplus} dk\, \mathfrak{H}_k$$

Fiber Hilbert spaces $\mathfrak{H}_k$ isomorphic to $\mathcal{F}$, invariant under $e^{-it\mathcal{H}_N}$.

Given $k \in \mathbb{R}^3$, consider value Nk of conserved total momentum P_{tot}.

The restriction of $\mathcal{H}_N$ to $\mathfrak{H}_k$ is given by the fiber Hamiltonian

$$\begin{aligned} \mathcal{H}_N(k) \quad := \quad & \frac{1}{2N}(Nk - P_b)^2 + T + \int dx\, w(x) a_x^+ a_x \\ & + \frac{\lambda}{2N} \int dx dy\, a_x^+ a_x v(x-y) a_y^+ a_y \end{aligned}$$

The origin of the coordinate system sits at the expected location of the tracer particle, so that $X = 0$.

From now on, identify $\mathfrak{H}$ with $L^2(\mathbb{R}^3, \mathcal{F})$, and omit the tensor products.

The solution of the Schrödinger equation on $\mathfrak{H}$ has the following form.

Proposition *Given $u \in L^2(\mathbb{R}^3)$ and $\Psi^{\mathcal{F}}_{k,0} \in \mathcal{F}$, let*

$$\Phi_{u,0}(X) := \int dk\, \widehat{u}(k) e^{iX\cdot(Nk - P_b)} \Psi^{\mathcal{F}}_{k,0} \in \mathfrak{H}\,.$$

Then,

$$\Phi_u(t, X) := \int dk\, \widehat{u}(k) e^{iX\cdot(Nk - P_b)} \Psi^{\mathcal{F}}_{k}(t)$$

solves

$$i\partial_t \Phi_u = \mathcal{H}_N \Phi_u$$

on $\mathfrak{H}$ with initial data $\Phi_u(0, X) = \Phi_{u,0}(X) \in \mathfrak{H}$, iff $\Psi^{\mathcal{F}}_k(t) \in \mathcal{F}$ solves

$$i\partial_t \Psi^{\mathcal{F}}_k(t) = \mathcal{H}_N(k) \Psi^{\mathcal{F}}_k(t)$$

on $\mathcal{F}$ with initial data $\Psi^{\mathcal{F}}_k(0) = \Psi^{\mathcal{F}}_{k,0} \in \mathcal{F}$.

Mean field limit

Weyl operator associated to $\phi \in L^2(\mathbb{R}^3)$

$$\mathcal{W}[\sqrt{N}\phi] \quad := \quad \exp\Big(\sqrt{N} \int dx \,\big(\phi(x) a_x^+ - \overline{\phi(x)} a_x \big) \Big)$$

Given $\phi_0 \in H^1(\mathbb{R}^3)$, consider solution of Schrödinger equation on $\mathcal{F}$

$$e^{-it\mathcal{H}_N(k)} \, \mathcal{W}[\sqrt{N}\phi_0] \, \Omega$$

with initial data given by coherent state

$$\mathcal{W}[\sqrt{N}\phi_0] \, \Omega = \Big(\frac{(\sqrt{N}\phi_0)^{\otimes n}}{n!} \Big)_{n \in \mathbb{N}_0}$$

Expected particle number is N.

Comparison dynamics:

Let $v \in C^2(\mathbb{R}^3)$.

Assume that for some $T > 0$, $\phi_t \in L^\infty_t H^3_x([0,T) \times \mathbb{R}^3)$ is the solution of

$$i\partial_t \phi_t \;=\; -\Big(k - (\phi_t, i\nabla\phi_t)\Big)\, i\nabla\phi_t - \frac{1}{2}\Delta\phi_t + w\phi_t + \lambda(v * |\phi_t|^2)\phi_t\,, \qquad (1)$$

with initial data $\phi_0 \in H^3_x(\mathbb{R}^3)$.

Time-dependent mean-field Hamiltonian; self-adjoint, bilinear in a^+, a,

$$\mathcal{H}^{\phi_t}_{mf}(k) \ := \ \mathcal{H}^{\phi_t}_{Har}(k) + \mathcal{H}^{\phi_t}_{cor}$$

with "diagonal" Hartree Hamiltonian commuting with N_b

$$\begin{aligned} \mathcal{H}^{\phi_t}_{Har}(k) \quad &:= \quad -\Big(k - (\phi_t, i\nabla\phi_t)\Big) \cdot P_b + T + \int dx\, w(x) a_x^+ a_x \\ &\quad\quad +\lambda \int |\phi_t(x)|^2 v(x-y) a_y^+ a_y dx dy \end{aligned}$$

and "off-diagonal" Hamiltonian not preserving particle number,

$$\begin{aligned} \mathcal{H}^{\phi_t}_{cor} \quad &:= \quad \frac{1}{2}\Big(a^+(i\nabla\phi_t) + a(i\nabla\phi_t)\Big)^2 \\ &\quad\quad +\lambda \int v(x-y)\phi_t(x)\overline{\phi_t(y)}\, a_x^+ a_y\, dx dy \\ &\quad\quad +\frac{\lambda}{2} \int v(x-y)\Big(\phi_t(x)\phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x)\phi_t(y)} a_y a_x\Big) dx dy \,. \end{aligned}$$

Obtain the unitary flow $\mathcal{V}(t,s)$ generated by $\mathcal{H}^{\phi_t}_{mf}(k)$,

$$i\partial_t \mathcal{V}(t,s) \ = \ \mathcal{H}^{\phi_t}_{mf}(k)\, \mathcal{V}(t,s) \quad , \quad \mathcal{V}(s,s) = \mathbf{1} \,.$$

Theorem *Let* $k \in \mathbb{R}^3$*. Assume* $v \in C^2(\mathbb{R}^3)$*, and that for some* $T > 0$*,* $\phi_t \in L_t^\infty H_x^3([0,T) \times \mathbb{R}^3)$ *is the solution of*

$$i\partial_t \phi_t \;=\; -\Big(k - (\phi_t, i\nabla\phi_t)\Big)\, i\nabla\phi_t - \frac{1}{2}\Delta\phi_t + w\phi_t + \lambda(v * |\phi_t|^2)\phi_t\,,$$

with initial data $\phi_0 \in H_x^3(\mathbb{R}^3)$*. Let*

$$\begin{aligned} S(t,t') \quad &:= \quad N\int_{t'}^{t} ds\Big(-\frac{1}{2}k^2 + \frac{1}{2}(\phi_s, i\nabla\phi_s)^2 \\ &\qquad\qquad + \frac{\lambda}{2}\int |\phi_s(x)|^2 v(x-y) |\phi_s(y)|^2 dxdy \Big)\,. \end{aligned}$$

Then, the following (mean field) limit holds, strongly in $\mathcal{F}$

$$\lim_{N\to\infty} \Big\| e^{-it\mathcal{H}_N(k)} \mathcal{W}[\sqrt{N}\phi_0]\,\Omega - e^{-iS(t,0)}\mathcal{W}[\sqrt{N}\phi_t]\mathcal{V}(t,0)\,\Omega \Big\|_{\mathcal{F}} = 0$$

Remarks:

In $\mathcal{V}(t,s)$, Bogoliubov translation not split from Bogoliubov rotation.

The time-dependent mean-field Hamiltonian is similar to the quasifree nonlinear approximation of the Hamiltonian in I.M. Sigal's talk.

[Lewin-Nam-Schlein '15], [Grillakis-Machedon '17]: Similar construction in different context (to optimize convergence rate of mean field limit for pure Hartree dynamics).

Analysis of the generalized Hartree equation

Expected boson momentum $j_\phi(t) := (\phi(t), i\nabla\phi(t))$.

Expected trajectory of tracer particle

$$X_\phi(t) \ = \ \int_0^t ds(k - j_\phi(s))$$

Theorem *Assume* $\|w\|_{W_x^{2,\frac{3}{2}}} < \infty$, *and*

$$\|w\|_{W_x^{1,\frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1,\frac{3}{2}}} < 1\,. \tag{2}$$

Then, there exists a unique global mild solution to

$$i\partial_t\phi = -\Big(k - j_\phi(t)\Big)\, i\nabla\phi - \frac{1}{2}\Delta\phi + w\phi + \lambda(v * |\phi|^2)\phi\,,$$

with initial data $\phi(t=0) = \phi_0 \in H_x^1$, *satisfying*

$$\|\phi\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3)} + \|\tau_{X_\phi}\phi\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)} < \infty$$

where $(\tau_{X_\phi} f)(t,x) := f(x + X_\phi(t))$.

In particular, $|\partial_t X_\phi(t)| < C\|\phi_0\|_{H_x^1}$, *uniformly in* $t \in \mathbb{R}$.

Proof idea; bosons coupled to classical tracer particle

For any $\phi \in L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)$, $|j_\phi(t)| \leq \|\phi\|_{L_t^\infty H_x^1} < C$.

Thus, $|X_\phi(t)|$ is bounded for every finite $t \in \mathbb{R}$.

Hence, $e^{iX_\phi(t)\cdot i\nabla} : H_x^1 \to H_x^1$ is unitary for every $t \in \mathbb{R}$.

Define

$$\psi(t,x) \ := \ e^{iX_\phi(t)\cdot i\nabla}\phi(t,x) \ = \ \phi\Big(t, x - X_\phi(t)\Big)$$

Clearly, by unitarity,

$$j_\phi(t) = j_\psi(t)\,.$$

Therefore,

$$X_\phi(t) = X_\psi(t)\,,$$

and

$$i\partial_t\psi = e^{iX_\phi(t)\cdot i\nabla}\Big(-\frac{1}{2}\Delta + w + \lambda(v * |\phi|^2)\Big)e^{-iX_\phi(t)\cdot i\nabla}\psi\,,$$

Remarks

The first term on the r.h.s. has been canceled by $(i\partial_t X_\phi(t))\phi$ from the time derivative. Noting that the operator $-\Delta$ is translation invariant, and

$$
\begin{aligned}
&\Big(e^{iX_\phi(t)\cdot i\nabla}(v * |\phi|^2)e^{-iX_\phi(t)\cdot i\nabla}\Big)(t,x) \\
&= \int v\Big(x - X_\phi(t) - y\Big)\,|\phi(t,y)|^2\,dy \\
&= \int v(x-y)\,|\phi(t, y - X_\phi(t))|^2\,dy \\
&= \big(v * |\psi|^2\big)(t,x)\,,
\end{aligned}
$$

We find that ψ satisfies the nonlinear Hartree equation

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + w_\psi \psi + \lambda(v * |\psi|^2)\psi \quad , \quad \psi(t=0) = \psi_0 \equiv \phi_0$$

where

$$w_\psi(t,x) := w\Big(x - X_\psi(t)\Big) \tag{3}$$

is the potential w, translated by $X_\psi(t)$.

The proof of the theorem therefore follows from GWP for ψ. □

Note that $X_\psi(t)$ can be written as

$$X_\psi(t) = kt - (\psi, x\psi)$$

and that it satisfies the Ehrenfest dynamics

$$\begin{aligned} \partial_t^2 X_\psi(t) &= \Big(\psi\,,\, \nabla(w_\psi + \lambda v * |\psi|^2)\psi\Big) \\ &= \int dx (\nabla w)(x - X_\psi(t))|\psi(x)|^2\,. \end{aligned} \tag{4}$$

The term involving v is zero because v is even.

Describes *classical* tracer particle along trajectory $X_\psi(t) \in \mathbb{R}^3$, coupled to boson field.

[Fröhlich-Zhou, F-Soffer-Z] Proof of emergence of Hamiltonian friction for models of similar type.

In particular, we find $\partial_t^2 X_\psi(t) = 0$ in the special case where $\phi_0 = Q_k$ is the minimizer of generalized Hartree functional

$$
\begin{aligned}
\mathcal{E}_k[\phi] \quad &:= \quad \frac{1}{N} \Big\langle \, \Phi_{N,\phi} \, , \, \mathcal{H}_N(k) \, \Phi_{N,\phi} \, \Big\rangle \\
&= \quad \frac{1}{2}\Big(k - \int dx \, \overline{\phi(x)} i \nabla_x \phi(x) \Big)^2 + \frac{1}{2} \int dx \, |\nabla \phi(x)|^2 \\
&\qquad + \int dx \, w(x) \, |\phi(x)|^2 + \frac{\lambda}{2} \int dx \, dy \, |\phi(x)|^2 v(x-y) |\phi(y)|^2 \\
&= \quad \frac{1}{2}\Big(k - (\phi \, , \, i\nabla\phi) \Big)^2 + \mathcal{E}_0[\phi]
\end{aligned}
$$

with $\|Q_k\|_{L^2} = 1$.

It follows that Q_k is the nonlinear ground state

$$
\mu_k Q_k = -\Big(k - (Q_k \, , \, i\nabla Q_k) \Big) i\nabla Q_k - \frac{1}{2} \Delta Q_k + w Q_k + \lambda (v * |Q_k|^2) Q_k
$$

with $\|Q_k\|_{L^2} = 1$.

(Value of μ_k obtained from taking inner product with Q_k).

We have $Q_k = e^{-i\frac{k}{2}x} Q_0$ where Q_0 is the rotationally symmetric minimizer of the standard Hartree functional, with $\|Q_0\|_{L^2_x} = 1$.

Due to rotational symmetry of Q_0, we find that $X_\psi(t) = \frac{k}{2} t$, and that with $\psi(t,x) = Q_k(x - \frac{k}{2} t)$, the r.h.s of (4) is zero, so that $\partial_t^2 X_\psi(t) = 0$.

$\square$

Sketch of proof of mean field limit

We have

$$\Big\| e^{-it\mathcal{H}_N(k)}\mathcal{W}[\sqrt{N}\phi_0]\,\Omega - e^{-iS(t,0)}\mathcal{W}[\sqrt{N}\phi_t]\mathcal{V}(t,0)\,\Omega \Big\|_{\mathcal{F}}^2 = 2(\,1 - M(t)\,)$$

where

$$\begin{aligned} M(t) \quad &:= \quad Re\Big\langle\, e^{-it\mathcal{H}_N(k)}\mathcal{W}[\sqrt{N}\phi_0]\,\Omega\,,\, e^{-iS(t,0)}\mathcal{W}[\sqrt{N}\phi_t]\mathcal{V}(t,0)\,\Omega \Big\rangle \\ &= \quad Re\Big\langle\, \Omega\,,\, \mathcal{W}^*[\sqrt{N}\phi_0]e^{it\mathcal{H}_N(k)}e^{-iS(t,0)}\mathcal{W}[\sqrt{N}\phi_t]\mathcal{V}(t,0)\,\Omega \Big\rangle\,. \end{aligned}$$

One can easily verify that given (2), we have

$$i\partial_t\mathcal{W}[\sqrt{N}\phi_t] \;=\; [\,\mathcal{H}_{Har}^{\phi_t}(k)\,,\,\mathcal{W}[\sqrt{N}\phi_t]\,]\,.$$

We consider the unitary flow

$$\mathcal{U}(t,s) := \mathcal{W}^*[\sqrt{N}\phi_s]e^{i(t-s)\mathcal{H}_N(k)-iS(t,s)}\mathcal{W}[\sqrt{N}\phi_t]$$

and introduce the selfadjoint operator

$$\begin{aligned}
\mathcal{L}_N^{\phi_t}(k) &:= \mathcal{W}^*[\sqrt{N}\phi_t]\Big(\mathcal{H}_N(k) - \partial_t S(t,0)\Big)\mathcal{W}[\sqrt{N}\phi_t] \\
&\quad -\mathcal{W}^*[\sqrt{N}\phi_t][\mathcal{H}_{Har}^{\phi_t}(k), \mathcal{W}[\sqrt{N}\phi_t]] - \mathcal{H}_{mf}^{\phi_t}(k) \\
&= \mathcal{W}^*[\sqrt{N}\phi_t]\mathcal{H}_N(k)\mathcal{W}[\sqrt{N}\phi_t] - \partial_t S(t,0) \\
&\quad -\mathcal{W}^*[\sqrt{N}\phi_t]\mathcal{H}_{Har}^{\phi_t}(k)\mathcal{W}[\sqrt{N}\phi_t] - \mathcal{H}_{cor}^{\phi_t}(k)\,. \qquad (5)
\end{aligned}$$

Then, it is clear that

$$i\partial_t\Big(\mathcal{U}_N(t,0)\mathcal{V}(t,0)\,\Omega\Big) = -\mathcal{U}_N(t,0)\mathcal{L}_N^{\phi_t}(k)\,\mathcal{V}(t,0)\,\Omega\,.$$

A straightforward but somewhat lengthy calculation shows that

$$\begin{aligned}
\mathcal{L}_N^{\phi_t}(k) &= \frac{1}{2\sqrt{N}}\Big(P_b\cdot\big(a^+(i\nabla\phi_t)+a(i\nabla\phi_t)\big)+\big(a^+(i\nabla\phi_t)+a(i\nabla\phi_t)\big)\cdot P_b\Big)\\
&\quad+\frac{1}{2N}P_b^2\\
&\quad+\frac{\lambda}{\sqrt{N}}\int v(x-y)a_x^+\Big(\overline{\phi_t(y)}a_y+\phi_t(y)a_y^+\Big)a_x\,dxdy\\
&\quad+\frac{\lambda}{2N}\int v(x-y)\,a_x^+a_y^+a_ya_x\,dxdy\,.
\end{aligned}$$

Evidently,

$$\begin{aligned}
M(t) &= Re\Big\langle\,\Omega\,,\,\mathcal{U}_N(t,0)\,\mathcal{V}(t,0)\,\Omega\Big\rangle\\
&= M(0)+Re\int_0^t ds\,\partial_s M(s)\\
&= 1-Re\Big\{\,i\int_0^t ds\,\Big\langle\,\Omega\,,\,\mathcal{U}_N(w,0)\mathcal{L}_N^{\phi_s}(k)\,\mathcal{V}(s,0)\,\Omega\Big\rangle\,\Big\}\,.
\end{aligned}$$

It follows from the unitarity of $\mathcal{U}_N(t,0)$ that

$$\Big| \Big\langle \Omega\,,\, \mathcal{U}_N(t,0)\mathcal{L}_N^{\phi_t}(k)\, \mathcal{V}(t,0)\,\Omega \Big\rangle \Big| \leq \Big\| \mathcal{L}_N^{\phi_t}(k)\, \mathcal{V}(t,0)\,\Omega \Big\|_{\mathcal{F}} .$$

We prove that

$$\Big\| \mathcal{L}_N^{\phi_t}(k)\, \mathcal{V}(t,0)\,\Omega \Big\|_{\mathcal{F}} \leq C_0 \frac{e^{C_1 t}}{\sqrt{N}} ,$$

for some constants C_0, C_1 depending on $\|v\|_{C^2(\mathbb{R}^3)}$ and $\|\phi_t\|_{L^\infty_t H^3_x([0,T)\times\mathbb{R}^3)}$. Hence, we find that

$$|M(t)-1| \,\leq\, C_0 \int_0^t \frac{e^{C_1 s}}{\sqrt{N}} ds \,<\, \frac{C_0}{C_1} \frac{e^{C_1 t}}{\sqrt{N}} .$$

We therefore conclude that for any $t>0$, the lhs of (5) converges to zero in the limit $N\to\infty$.

Thank you for your attention !