Equilibria of diffusing and self-attracting particles

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Non-linear Diffusion & Non-local Interaction

Particle density $\rho(t,\cdot) \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$, $m > 0$, satisfies

$$\frac{\partial \rho}{\partial t} = \frac{1}{N}\Delta \rho^m + 2\chi \nabla \cdot \left(\rho \nabla \left(W_k * \rho\right)\right) , \quad t > 0 , \quad x \in \mathbb{R}^N ,$$

$$\rho \geq 0, \quad \int \rho(x)\, dx = 1, \quad \int x\rho(x)\, dx = 0.$$

Interaction kernel:

$$W_k(x) = \begin{cases} \dfrac{|x|^k}{k} , & \text{if} \quad k \in (-N, N) \setminus \{0\} \\ \log|x| , & \text{if} \quad k = 0 \end{cases} .$$

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$$\frac{\partial \rho}{\partial t} = \frac{1}{N} \underbrace{\Delta \rho^m}_{\text{Repulsion}} + 2\chi \underbrace{\nabla \cdot \left(\rho \nabla \left(W_k * \rho\right)\right)}_{\text{Attraction}}, \quad t > 0\,, \quad x \in \mathbb{R}^N\,,$$

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Free Energy

The energy functional corresponding to this reaction-diffusion equation is given by

$$\mathcal{F}_{m,k}[\rho] = \mathcal{U}_m[\rho] + \chi \mathcal{W}_k[\rho] ,$$

where

$$\mathcal{U}_m[\rho] = \begin{cases} \dfrac{1}{N(m-1)} \int_{\mathbb{R}^N} \rho^m(x)\, dx , & \text{if} \quad m \neq 1 \\ \dfrac{1}{N} \int_{\mathbb{R}^N} \rho \log \rho \, dx , & \text{if} \quad m = 1 \end{cases}$$

$$\mathcal{W}_k[\rho] = \iint_{\mathbb{R}^N \times \mathbb{R}^N} W_k(x-y)\rho(x)\rho(y)\, dxdy .$$

Gradient flow structure

We can write our reaction-diffusion equation as the formal gradient flow of our free energy, when $\mathcal{P}$ is endowed with the Wasserstein-2 distance $\mathbf{W}$:

$$\partial_t \rho(t) = -\nabla_{\mathbf{W}} \mathcal{F}_{m,k}[\rho(t)] = -\nabla \cdot \left(\rho \, \nabla \frac{\delta \mathcal{F}_{m,k}}{\delta \rho}[\rho] \right)$$

Entropy dissipation:

$$\frac{d}{dt} \mathcal{F}_{m,k}[\rho(t)] = -\int_{\mathbb{R}^N} \rho \left| \nabla \left(\frac{m}{N(m-1)} \rho^{m-1} + 2\chi W_k * \rho \right) \right|^2 dx$$

Questions: Existence of equilibria? Are they unique? Do we converge to them? If yes, do we have an explicit rate? Otherwise, what is the asymptotic behaviour of solutions?

Dilations

Given ρ, we look at its dilations

$$\rho_\lambda(x) = \lambda^N \rho(\lambda x), \qquad x \in \mathbb{R}^N,\ \lambda > 0.$$

Each of the two contributions to the free energy is homogeneous:

$$\mathcal{F}_{m,k}[\rho_\lambda(t)] = \lambda^{N(m-1)} \mathcal{U}_m[\rho(t)] + \lambda^{-k} \chi \mathcal{W}_k[\rho(t)]$$

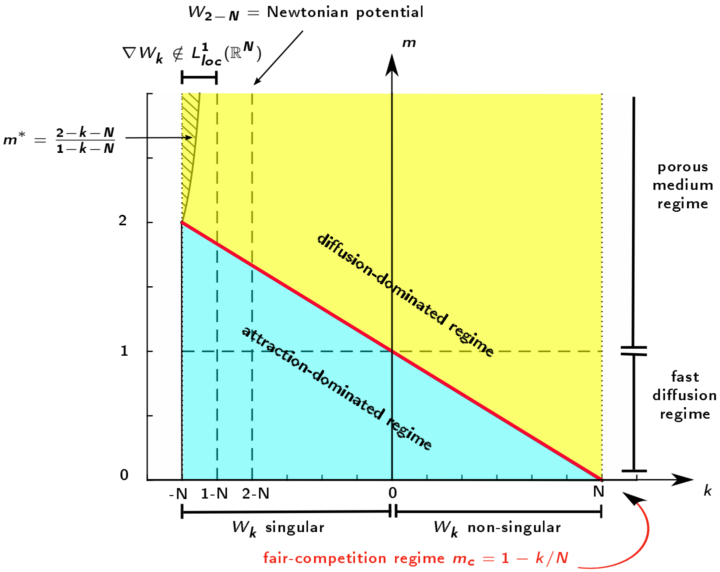
Three regimes:

- $\mathbf{N(m-1) = -k}$: Fair-competition regime
  $\longrightarrow$ critical mass and KS-like dichotomy for cases
  $k = 2 - N$, $m = 2 - 2/N$, $N \geq 3$ [Blanchet, Carrillo, Laurençot 2009],
  $k = 0$, $m = 1$, $N = 2$ [Blanchet, Campos, Calvez, Carlen, Carrillo, Dolbeault, Egaña, Figalli, Masmoudi, Mischler, Perthame, ...]

Attraction vs Repulsion

- $\mathbf{N(m-1) > -k}$: Diffusion-dominated regime
  $\longrightarrow$ Solutions exist globally in time and are bounded uniformly in time [Calvez, Carrillo 2006], [Sugiyama 2007]
  $\longrightarrow$ Stationary states are radially symmetric if $2 - N \leq k < 0$ [Carrillo, Hittmeir, Volzone, Yao 2016]
  $\longrightarrow$ Existence and uniqueness of minimisers $k = 0$, $m > 1$, $N \geq 2$ [Carrillo, Castorina, Volzone 2015]
  $\longrightarrow$ Asymptotic behaviour is given by stationary solutions $k = 0$, $m > 1$, $N = 2$ [Carrillo, Hittmeir, Volzone, Yao 2016]
  $\longrightarrow$ Little knowledge about asymptotic behaviour and minimisers in general
- $\mathbf{N(m-1) < -k}$: Attraction-dominated regime
  $\longrightarrow$ For any $\chi > 0$, blow-up can occur, but there also exist global in time regular solutions under some smallness assumptions
  $\longrightarrow$ classification blow-up/global existence and long-time behaviour of radial solutions depending on initial data and m for cases
  $k = 2 - N$, $m = 2N/(N+2)$, $N \geq 3$ [Chen, Liu, Wang 2012] and
  $k = 2 - N$, $0 < m < 2 - 2/N$, $N \geq 3$ [Bian, Liu 2013]

Regimes



W_{2-N} = Newtonian potential
$\nabla W_k \notin L^1_{loc}(\mathbb{R}^N)$
m
$m^* = \frac{2-k-N}{1-k-N}$
2
1
0
diffusion-dominated regime
attraction-dominated regime
porous medium regime
fast diffusion regime
-N
1-N
2-N
0
N
k
W_k singular
W_k non-singular
fair-competition regime $m_c = 1 - k/N$

Fair-Competition Regime

This means:

- $N(m-1)+k=0,\ k \in (-N,N),\ m \in (0,2)$.
- $\mathcal{F}_k[\rho_\lambda] = \lambda^{-k}\mathcal{F}_k[\rho]$.

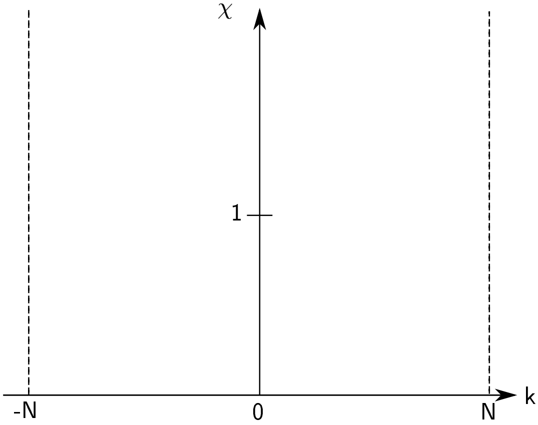
Goal: Understand relation

$$\text{stationary states} \leftrightarrow \text{global minimisers}$$

Logarithmic case ($k=0,\ m=1$):
$\rightarrow$ modified Keller-Segel model (1D), classical Keller-Segel model (2D)

- $0<\chi<1$: No stationary states. Solutions converge to a unique self-similar profile.
- $\chi=1$: Existence of infinitely many stationary states.
- $\chi>1$: Finite-time blow-up.

Fair-Competition Regime



Goal: Understand the bigger picture

Porous medium case: $k < 0,\ m \in (1, 2)$

Fast diffusion case: $k > 0,\ m \in (0, 1)$

Porous Medium Case $k < 0$

- $\mathcal{F}_k[\rho_\infty] = 0$ for any stationary state ρ_∞ with $|x|^2 \rho_\infty \in L^1(\mathbb{R}^N)$
- By variant of HLS with best constant C_*: for any $\chi > 0$,

$$\mathcal{F}_k[\rho] \geq \frac{1 - \chi C_*}{N(m-1)} ||\rho||_m^m ,$$

- Define critical interaction strength $\chi_c := 1/C_*$.
- In the critical case $\chi = \chi_c$: $\mathcal{F}_k[\rho] \geq 0$
  $\longrightarrow$ (stationary states with bounded 2nd moment $\Rightarrow$ global minimisers)
- In the sub-critical case $0 < \chi < \chi_c$: no stationary states exist.

Questions: Are global minimisers regular enough to be stationary states? If there are no stationary states, do we have self-similar profiles? Can we characterise the asymptotic behaviour of solutions?

Critical Case $\chi = \chi_c$ $(k < 0)$

Results:

- There exist a global minimiser of $\mathcal{F}_k$.
- Global minimisers of $\mathcal{F}_k$ are stationary states.
- Global minimisers of $\mathcal{F}_k$ are radially symmetric non-increasing, compactly supported and uniformly bounded.

Theorem (Variant of HLS in 1D)

Let $N = 1$, $k \in (-1, 0)$, $m = 1 - k$.

$$\exists \text{ stationary state } \rho_\infty \Rightarrow \mathcal{F}_k[\rho] \geq 0 \text{ with equality iff } \rho \text{ is a dilation of } \rho_\infty.$$

- Consequence: Uniqueness of stationary states up to dilations.
- Proof for existence of global minimisers uses HLS inequality as key ingredient.

OK, so we have existence of infinitely many, compactly supported stationary states.

What next?

Long-time Behaviour $N = 1$ $(\chi = \chi_c,\ k < 0)$

- We have: $\frac{d}{dt} W(\rho(t), \rho_\infty)^2 \leq (m-1)\mathcal{F}[\rho(t)] \to$ no information!
- We can write $\rho dx = \psi' \# \rho_\infty dx$ for ψ the Brenier map, $\psi'' \geq 0$. For $a, b \in \mathbb{R}$, define

$$\langle \psi''[a,b] \rangle := \int_0^1 \psi''\left((1-s)a + sb\right)\, ds\,.$$

  Then we can show that

$$\langle \psi''[a,b] \rangle \in \left(0, 1 + \frac{1}{m}\right)\ \forall a, b \in \mathbb{R}\,,\ \forall t > 0 \quad \Longrightarrow \quad \frac{d}{dt} W(\rho(t), \rho_\infty)^2 \leq 0$$

  with equality iff $\rho = \rho_\infty$.

  $\to$ no rate of convergence!

Subcritical Case $0 < \chi < \chi_c$ ($k < 0$)

- No stationary states exist. $\mathcal{F}_k$ has no global minimisers.

Self-Similar Profiles: Rescale $u(t,x) := \alpha(t)\rho\left(\beta(t), \alpha(t)x\right)$, where

$$\alpha(t) = e^{t}, \quad \beta(t) = \begin{cases} \frac{1}{2-k}\left(e^{(2-k)t} - 1\right), & \text{if } k \neq 2, \\ t, & \text{if } k = 2. \end{cases}$$

We obtain the rescaled reaction-diffusion equation

$$\partial_t u = \frac{1}{N}\Delta u^m + 2\chi\nabla \cdot \left(u\nabla\left(W_k * u\right)\right) + \nabla \cdot (xu)$$

and the rescaled energy

$$\mathcal{F}_{\text{resc}}[u] = \frac{1}{N(m-1)}\int u^m(x)\,dx + \chi\iint W_k(x-y)u(x)u(y)\,dxdy + \frac{1}{2}\int |x|^2 u(x)\,dx$$

$\longrightarrow$ Stationary states of rescaled eqn = self-similar profiles of original eqn.

Subcritical Case $0 < \chi < \chi_c$ $(k < 0)$

Results in rescaled variables:

- All stationary states of the rescaled eqn are continuous and compactly supported.
- There exist a global minimiser of $\mathcal{F}_{\mathrm{resc}}$.
- Global minimisers of $\mathcal{F}_{\mathrm{resc}}$ are stationary states of the rescaled eqn.
- Global minimisers of $\mathcal{F}_{\mathrm{resc}}$ are radially symmetric non-increasing and uniformly bounded.

Theorem (Functional Inequality in 1D)

Let $N = 1$, $k \in (-1, 0)$, $m = 1 - k$.
$\exists$ *stationary state* ρ_∞ *of rescaled eqn* $\Rightarrow$ $\mathcal{F}_{\mathrm{resc}}[\rho] \geq \mathcal{F}_{\mathrm{resc}}[\rho_\infty]$ *with equality iff* $\rho = \rho_\infty$.

$\longrightarrow$ known inequality?
$\longrightarrow$ stability estimates?

- Consequence: Uniqueness of stationary states in rescaled variables.

Long-time Behaviour $N = 1$ $(\chi < \chi_c,\ k < 0)$

We can show for $N = 1$:

Proposition

Let $N = 1$, $k \in (-1, 0)$, $m = 1 - k$. If ρ_∞ is a stationary state of the rescaled eqn, and ψ the Brenier map, $\rho dx = \psi' \# \rho_\infty dx$, $\psi'' \geq 0$, then under the assumption that

$$\langle \psi''[a, b] \rangle \in \left(0, 1 + \frac{1}{m}\right) \ \forall a, b \in \mathbb{R},\ \forall t > 0$$

we have

$$\frac{d}{dt} W(\rho(t), \rho_\infty)^2 \leq -2W(\rho(t), \rho_\infty)^2$$

with equality iff $\rho = \rho_\infty$.

$\longrightarrow$ rate of convergence does not depend on χ!!

Fast diffusion case $k > 0$

- HLS inequality is not valid
- No radially symmetric non-increasing stationary states with kth moment bounded.
  No radially symmetric non-increasing global minimisers of $\mathcal{F}_k$.
  $\longrightarrow$ seek self-similar solutions
- No critical χ !!!
- In rescaled variables: ρ_∞ radially symmetric non-increasing stationary state
  - $\rho_\infty \in L^1\left(\mathbb{R}^N\right) \Leftrightarrow 0 < k < 2$, that is $(N-2)/N < m < 1$.
  - $|x|^2 \rho_\infty \in L^1\left(\mathbb{R}^N\right) \Leftrightarrow 0 < k < 2N/(2+N)$, that is $N/(2+N) < m < 1$.
  - $|x|^k \rho_\infty \in L^1\left(\mathbb{R}^N\right) \Leftrightarrow 0 < k < k^*(N) \in (1,2)$.

Fast diffusion case $k > 0$

Theorem (Existence of stationary states)

Let $\chi > 0$, $k \in (0, 1]$ and let F be the set of continuous radially symmetric non-increasing functions in $L^1_+(\mathbb{R}^N)$ with unit mass, bounded kth moment, and decaying at infinity. Then there exists a stationary state $\rho_\infty \in F$ for the rescaled equation.

$\longrightarrow$ Pf: rewrite Euler-Lagrange condition as fixed point of a compact operator, and then use Schauder's fixed point theorem.

Theorem (Functional Inequality in 1D)

Let $N = 1$, $k \in (0, 1)$, $m = 1 - k$.
$\exists$ stationary state ρ_∞ of rescaled eqn $\Rightarrow \mathcal{F}_{\mathrm{resc}}[\rho] \geq \mathcal{F}_{\mathrm{resc}}[\rho_\infty]$ with equality iff $\rho = \rho_\infty$.

$\longrightarrow$ same inequality as for porous medium case

- Consequence: Uniqueness of stationary states in rescaled variables.

Long-time Behaviour $N = 1$ $(k > 0)$

$\longrightarrow$ We have exponential convergence to equilibrium, this time without stability condition!!!

Proposition

Let $N = 1$, $k \in (0,1)$, $m = 1 - k$. If ρ_∞ is a stationary state of the rescaled eqn, then

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- Rate of convergence independent of χ
- This shows why there are no stationary states in original variables: the rescaled density would converge to a dirac delta $\longrightarrow$ contradiction!

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The bigger picture: thoughts about uniqueness...

- Property of strictly convex functions: $\exists \implies !$
- **Question**: What about the convexity properties of $\mathcal{F}_k$ and $\mathcal{F}_{\text{resc}}$?

Uniqueness Story

Theorem (McCann, 1997)

If $\mathcal{F}_k$ is strictly displacement convex, then it has at most one minimiser.

- **Problem:** Our $\mathcal{F}_k$ is not necessarily displacement convex!

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Previous Results: In 1D, we recover convexity property $\exists \implies !$

$\longrightarrow$ continuation of seminal paper [McCann, 1997]

$\longrightarrow$ true for general k, N, m?

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- $k < 0$, $\chi = \chi_c$: infinitely many global minimisers of $\mathcal{F}_k$ by homogeneity
  $\rightarrow$ uniqueness up to dilations? True for $k = 2 - N$, $N \geq 2$ [Yao, 2014].
- $k < 0$, $\chi < \chi_c$: uniqueness of radially symmetric stationary states in rescaled variables?

What's next...?

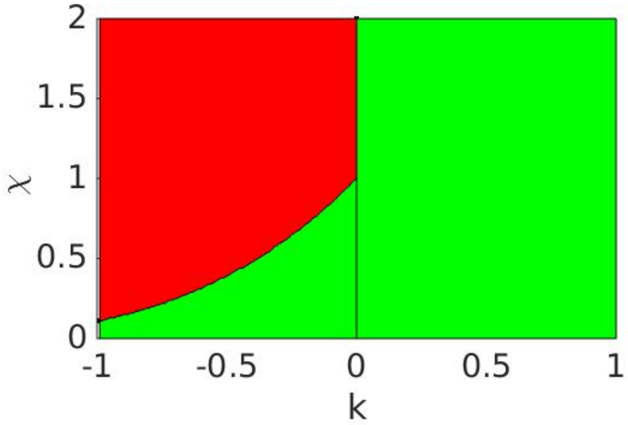
Work in progress:

- **Uniqueness of stationary states and self-similar profiles in the porous medium fair-competition regime $\mathbf{k \in (-N, 0)}$, $\mathbf{m = 1 - k/N}$** (with V. Calvez): Uniqueness of global minimisers of $\mathcal{F}_k$ (if $\chi = \chi_c$) modulo dilations and of $\mathcal{F}_{\mathrm{resc}}$ (if $\chi < \chi_c$) in radial variables? Asymptotic behaviour?
- **The porous medium diffusion-dominated regime $\mathbf{k \in (-N, 2 - N)}$, $\mathbf{m > 1 - k/N}$, $\mathbf{N \geq 3}$** (with J.A. Carrillo, E. Mainini and B. Volzone):
  $\exists$ global minimisers of $\mathcal{F}_k$ for any $\chi > 0$. All stationary states are radially symmetric. All global minimisers are stationary states. Uniqueness in 1D.
- **The smooth kernel diffusion-dominated regime $\mathbf{k \in (0, N)}$, $\mathbf{m > 1 - k/N}$** (with J. Dolbeault and R. Frank): For large enough $k \in (0, N)$ it is possible that stationary states exist. Reversed HLS type inequality? Existence of global minimisers of $\mathcal{F}_k$?
- **Duality and stability estimates for related functional inequalities $\mathbf{k < 0}$** (with E. Carlen): more on $\mathcal{F}_{\mathrm{resc}}[\rho] \geq \mathcal{F}_{\mathrm{resc}}[\rho_\infty]$.
- **Aggregation-dominated regime?**

...maaaaaaaaaaaaaany open questions remain!

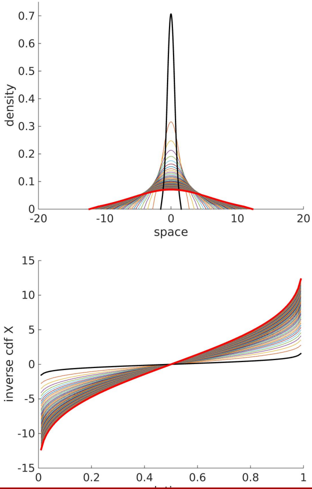
Numerics: Parameterspace $N = 1$

- Numerical Method: [Blanchet, Calvez, Carrillo 2008]

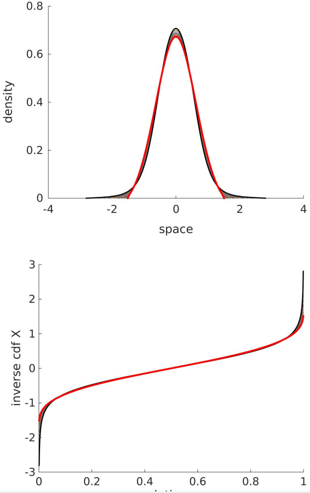


Numerics: Stationary States

$\chi = 0.2,\ k = -0.5,\ resc = 0$

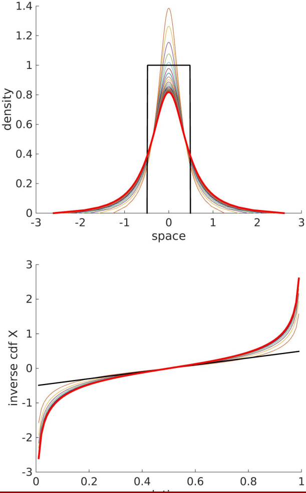


$\chi = 0.2,\ k = -0.5,\ resc = 1$

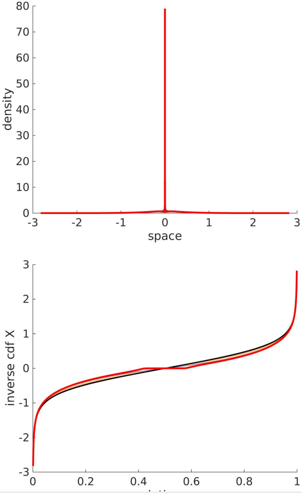


Numerics: Stationary States

$\chi = 0.8,\ k = 0.2,\ resc = 1$

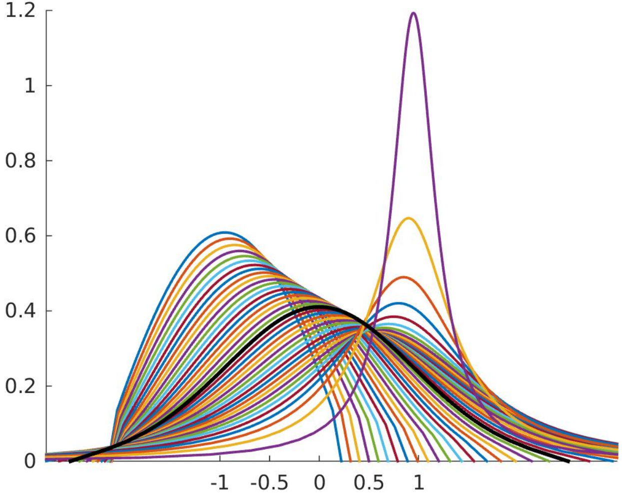


$\chi = 1.0,\ k = -0.5,\ resc = 0$



Numerics: $\chi = 0.05$, $resc = 1$

- k varies from $+0.95$ to -0.95 in 0.05 steps.



Numerics: $\chi = 0.8$, $resc = 1$

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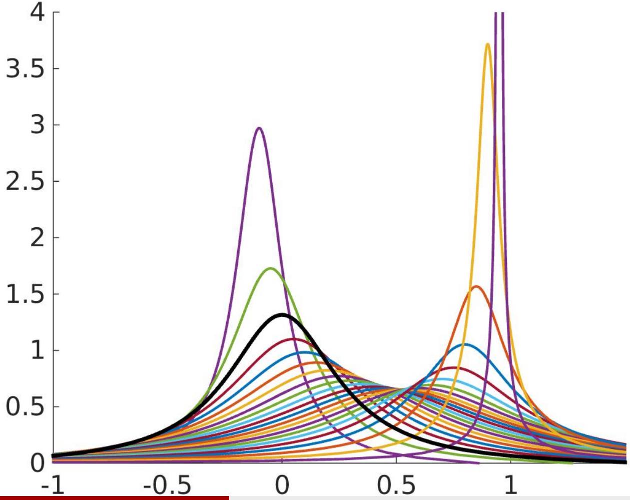
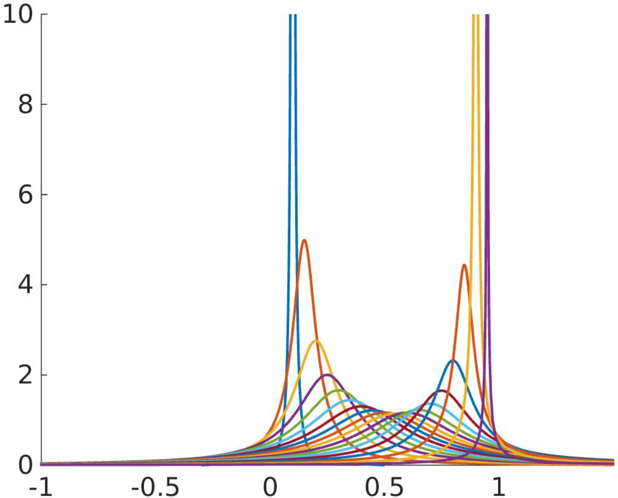


Numerics: $\chi = 1.2$, $resc = 1$

- k varies from $+0.95$ to -0.95 in 0.05 steps.



Thank you for your attention!