Reverse Hardy-Littlewood-Sobolev inequalities

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Functional Inequalities

The HLS inequality

Theorem ((Lieb 1983))

For any $-N < \lambda < 0$, there exists a constant $\mathcal{C}_{HLS} = \mathcal{C}_{HLS}(N, \lambda, q) > 0$ such that any $f \in \mathrm{L}^p(\mathbb{R}^N)$ and $g \in \mathrm{L}^q(\mathbb{R}^N)$ satisfy

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda f(x)\, g(y)\, dx\, dy \le \mathcal{C}_{HLS} \|f\|_p \|g\|_q$$

$$\frac{1}{p} + \frac{1}{q} = 2 + \frac{\lambda}{N}, \qquad p, q > 1$$

Sharp inequality: Let $f = g = \rho \ge 0$ and $p = q = \frac{2N}{2N+\lambda}$, then

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \rho(x)\, \rho(y)\, dx\, dy \le \mathcal{C}_{HLS} \left(\int_{\mathbb{R}^N} \rho^q\, dx \right)^{2/q}.$$

The reverse HLS inequality

Theorem ((Dou, Zhu 2015)(Ngô, Nguyen 2017))

For any $\lambda > 0$, there exists a constant $\mathcal{C}_{RHLS} = \mathcal{C}_{RHLS}(N, \lambda, q) > 0$ such that any non-negative $f \in \mathrm{L}^p(\mathbb{R}^N)$ and $g \in \mathrm{L}^q(\mathbb{R}^N)$ satisfy

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, f(x)\, g(y)\, dx\, dy \;\geq \mathcal{C}_{RHLS} \|f\|_p \|g\|_q$$

$$\frac{1}{p} + \frac{1}{q} = 2 + \frac{\lambda}{N}\,, \qquad p, q \in (0,1)$$

Convention: $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N} |\rho(x)|^p\, dx < \infty$ for any $p > 0$.

Sharp inequality: Let $f = g = \rho \geq 0$ and $p = q = \frac{2N}{2N+\lambda}$, then

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x)\, \rho(y)\, dx\, dy \;\geq \mathcal{C}_{RHLS} \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{2/q} .$$

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2\,N - q\,(2\,N + \lambda)}{N\,(1-q)}$$

Define

$$\mathcal{C}_{N,\lambda,q} := \inf \left\{ \frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}} \right\} ,$$

where the inf is taken over ρ such that $0 \leq \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ \rho \not\equiv 0$.

$\longrightarrow$ Recover sharp reversed HLS inequality for $\alpha = 0$.

Questions:

- Is $\mathcal{C}_{N,\lambda,q} = 0$ or positive?
- Do ρ exist that achieve the inf?

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2\,N - q\,(2\,N + \lambda)}{N\,(1-q)}$$

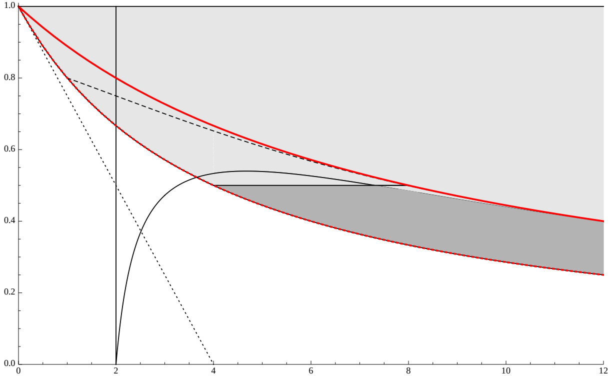
Theorem

Let $\lambda > 0$. The inequality

$$I_\lambda[\rho] \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \tag{1}$$

holds for any $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ with $\mathcal{C}_{N,\lambda,q} > 0$ if and only if $q > N/(N+\lambda)$.

If either $N = 1$, 2 or if $N \geq 3$ and $q \geq \min\left\{1 - 2/N \,,\, 2N/(2N+\lambda)\right\}$, then there is a radial nonnegative optimizer $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$.



$N = 4$, *region of the parameters* (λ, q) *for which* $\mathcal{C}_{N,\lambda,q} > 0$
Optimal functions exist in the light grey area

Free energy point of view

A toy model

Assume that u solves the *fast diffusion with external drift* V given by

$$\frac{\partial u}{\partial t} = \Delta u^q + \nabla \cdot \big(u \nabla V\big)$$

To fix ideas: $V(x) = 1 + \frac{1}{2}|x|^2 + \frac{1}{\lambda}|x|^\lambda$. *Free energy* functional

$$\mathcal{F}[u] := \int_{\mathbb{R}^N} V\,u\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} u^q\,dx$$

- Under the mass constraint $M = \int_{\mathbb{R}^N} u\,dx$, smooth minimizers are

$$u_\mu(x) = \big(\mu + V(x)\big)^{-\frac{1}{1-q}}$$

- The equation can be seen as a gradient flow

$$\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\int_{\mathbb{R}^N} u\left|\tfrac{q}{1-q}\nabla u^{q-1} - \nabla V\right|^2 dx$$

A toy model (continued)

If $\lambda = 2$, the so-called *Barenblatt profile* u_μ has finite mass if and only if

$$q > q_c := \frac{N-2}{N}$$

- For $\lambda > 2$, the integrability condition is $1 - 2/N > q > 1 - \lambda/N$ but $q = q_c$ is a threshold for the regularity: the mass of $u_\mu = (\mu + V)^{1/(1-q)}$ is

$$M(\mu) := \int_{\mathbb{R}^N} u_\mu \, dx \le M_\star = \int_{\mathbb{R}^N} \left(\tfrac{1}{2} |x|^2 + \tfrac{1}{\lambda} |x|^\lambda\right)^{-\frac{1}{1-q}} dx$$

- If one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u \, dx = M$ for an arbitrary $M > M_\star$, the limit of a minimizing sequence is the measure

$$\left(M - M_\star\right) \delta + u_{-1}$$

The nonlinear model: heuristics

$$V = \rho * W_\lambda\,, \quad W_\lambda(x) := \tfrac{1}{\lambda}\,|x|^\lambda$$

is motivated by the study of the nonnegative solutions of the evolution equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^q + \nabla \cdot (\rho\, \nabla W_\lambda * \rho)$$

Optimal functions for (RHLS) are energy minimizers for the *free energy* functional

$$\begin{aligned}\mathcal{F}[\rho] : &= \frac{1}{2} \int_{\mathbb{R}^N} \rho\,(W_\lambda * \rho)\, dx - \frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q\, dx \\ &= \frac{1}{2\lambda}\, I_\lambda[\rho] - \frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q\, dx\end{aligned}$$

under a *mass* constraint $M = \int_{\mathbb{R}^N} \rho\, dx$ while smooth solutions obey to

$$\frac{d}{dt} \mathcal{F}[\rho(t,\cdot)] = - \int_{\mathbb{R}^N} \rho \left| \tfrac{q}{1-q} \nabla \rho^{q-1} - \nabla W_\lambda * \rho \right|^2 dx$$

Minimization: free energy vs quotient

$$\mathcal{F}[\rho] = -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx + \frac{1}{2\,\lambda} I_\lambda[\rho]$$

$$\mathsf{Q}_{q,\lambda}[\rho] := \frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}}$$

$$\mathcal{C}_{N,\lambda,q} := \inf\left\{\mathsf{Q}_{q,\lambda}[\rho] :\ 0 \le \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ \rho \not\equiv 0\right\}\,,$$

If $N/(N+\lambda) < q < 1$, $\rho_\ell(x) := \ell^{-N}\,\rho(x/\ell)/\|\rho\|_1$

$$\mathcal{F}[\rho_\ell] = -\,\ell^{(1-q)\,N}\,\mathsf{A} + \ell^\lambda\,\mathsf{B}$$

has a minimum at $\ell = \ell_\star$ and

$$\mathcal{F}[\rho] \ge \mathcal{F}[\rho_{\ell_\star}] = -\,\kappa_\star\left(\mathsf{Q}_{q,\lambda}[\rho]\right)^{-\frac{N\,(1-q)}{\lambda-N\,(1-q)}}$$

Proposition

$\mathcal{F}$ is bounded from below if and only if $\mathcal{C}_{N,\lambda,q} > 0$

Reverse Hardy-Littlewood-Sobolev inequality

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2\,N - q\,(2\,N + \lambda)}{N\,(1-q)}$$

Theorem

Let $\lambda > 0$. The inequality

$$I_\lambda[\rho] \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \qquad (2)$$

holds for any $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ with $\mathcal{C}_{N,\lambda,q} > 0$ if and only if $q > N/(N+\lambda)$.

If either $N = 1$, 2 or if $N \geq 3$ and $q \geq \min\left\{1 - 2/N \,,\, 2N/(2N+\lambda)\right\}$, then there is a radial nonnegative optimizer $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$.

The conformally invariant case $q = 2N/(2N+\lambda)$

$$I_\lambda[\rho] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \;\geq \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{2/q}$$

$$q = 2N/(2N+\lambda) \quad\Longleftrightarrow\quad \alpha = 0$$

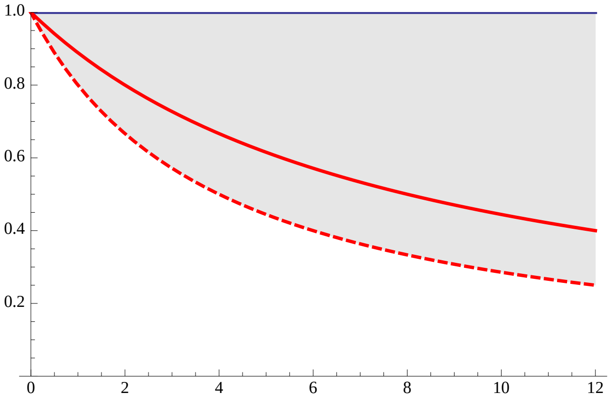
(Dou, Zhu 2015) (Ngô, Nguyen 2017)

The optimizers are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1+|x|^2\right)^{-N/q} \quad \forall\, x\in\mathbb{R}^N$$

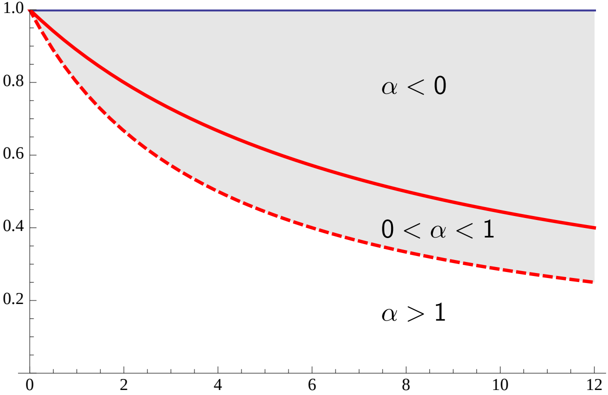
and the value of the optimal constant is

$$\mathcal{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}}\,\frac{\Gamma\left(\frac{N}{2}+\frac{\lambda}{2}\right)}{\Gamma\left(N+\frac{\lambda}{2}\right)}\,\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1+\frac{\lambda}{N}}$$



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$
The plain, red curve is the conformally invariant case $\alpha = 0$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \geq \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$



A Carlson type inequality

Lemma

Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$

$$\left(\int_{\mathbb{R}^N} \rho\, dx\right)^{1-\frac{N(1-q)}{\lambda\, q}} \left(\int_{\mathbb{R}^N} |x|^\lambda \rho\, dx\right)^{\frac{N(1-q)}{\lambda\, q}} \geq c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\, dx\right)^{\frac{1}{q}}$$

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}} \left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}$$

Equality is achieved if and only if

$$\rho(x) = \left(1+|x|^\lambda\right)^{-\frac{1}{1-q}}$$

up to translations, dilations and constant multiples

(Carlson 1934) (Levine 1948)

Proposition

Let $\lambda > 0$. If $N/(N+\lambda) < q < 1$, then $\mathcal{C}_{N,\lambda,q} > 0$

By rearrangement inequalities: prove the reverse HLS inequality for symmetric non-increasing ρ's so that

$$\int_{\mathbb{R}^N} |x-y|^\lambda \, \rho(y)\, dx \ge \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \quad \text{for all} \quad x \in \mathbb{R}^N$$

implies

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \int_{\mathbb{R}^N} \rho \, dx$$

In the range $\frac{N}{N+\lambda} < q < 1$

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\, dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N} \rho \, dx \, dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \ \ge \ c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{\frac{2-\alpha}{q}}$$

and conclude with Carlson's inequality.

The case $\lambda = 2$

Corollary

Let $\lambda = 2$ and $N/(N+2) < q < 1$. Then the optimizers for (RHLS) are given by translations, dilations and constant multiples of

$$\rho(x) = \left(1 + |x|^2\right)^{-\frac{1}{1-q}}$$

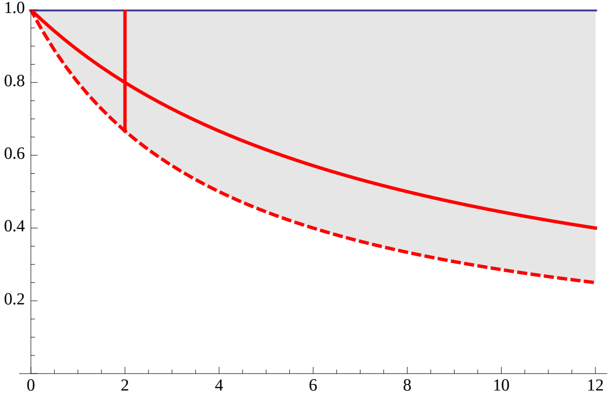
and the optimal constant is

$$\mathcal{C}_{N,2,q} = \frac{1}{2}\, c_{N,2,q}^{\frac{2\,q}{N\,(1-q)}}$$

By rearrangement inequalities it is enough to prove (RHLS) for symmetric non-increasing ρ's, and so $\int_{\mathbb{R}^N} x\,\rho\,dx = 0$. Therefore

$$I_2[\rho] = 2 \int_{\mathbb{R}^N} \rho\,dx \int_{\mathbb{R}^N} |x|^2 \rho\,dx$$

and the optimal function is optimal for Carlson's inequality.



$N = 4$, *region of the parameters* (λ, q) *for which* $\mathcal{C}_{N,\lambda,q} > 0$. *The dashed, red curve is the threshold case* $q = N/(N + \lambda)$

The threshold case $q = N/(N+\lambda)$ and below

Proposition

If $0 < q \leq N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0$.

- Case $0 < q < N/(N+\lambda)$ shown in (Carrillo, Delgadino, Patacchini 2018).
- Alternative proof that can be extended to the threshold case $q = N/(N+\lambda)$ ($i.e.$ $\alpha = 1$)

The threshold case $q = N/(N+\lambda)$ and below

Proposition

If $0 < q \le N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0$.

Let ρ, $\sigma \ge 0$ such that $\int_{\mathbb{R}^N} \sigma\,dx = 1$, smooth (+ compact support)

$$\rho_\varepsilon(x) := \rho(x) + M\,\varepsilon^{-N}\,\sigma(x/\varepsilon)$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon\,dx = \int_{\mathbb{R}^N} \rho\,dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx \quad \text{as} \quad \varepsilon \to 0_+$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx \quad \text{as} \quad \varepsilon \to 0_+$$

If $0 < q < N/(N+\lambda)$, *i.e.*, $\alpha > 1$, take ρ_ε as a trial function,

$$\mathcal{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx}{\left(\int_{\mathbb{R}^N} \rho\,dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{(2-\alpha)/q}} =: \mathcal{Q}[\rho, M]$$

and let $M \to +\infty$.

A relaxed inequality

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \ \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx + M \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \qquad (3)$$

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality (3) holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (RHLS) and admits an optimizer (ρ, M).

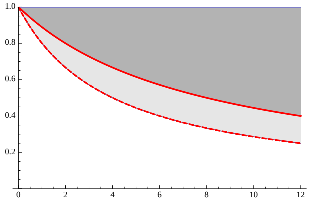
- Heuristically, this is the extension of (RHLS)

$$I_\lambda[\rho] \ \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q}$$

  to measures of the form $\rho + M\,\delta$.
- Recover original problem for $M = 0$.

Existence of minimizers and relaxation

Existence of a minimizer: first case



The $\alpha < 0$ case: dark grey region

Proposition

If $\lambda > 0$ and $\frac{2N}{2N+\lambda} < q < 1$, there is a minimizer ρ for $\mathcal{C}_{N,\lambda,q}$.

The limit case $\alpha = 0$, $q = \frac{2N}{2N+\lambda}$ is the *conformally invariant* case: see (Dou, Zhu 2015) and (Ngô, Nguyen 2017)

A minimizing sequence ρ_j can be taken radially symmetric non-increasing by rearrangement, and such that

$$\int_{\mathbb{R}^N} \rho_j(x)\,dx = \int_{\mathbb{R}^N} \rho_j(x)^q\,dx = 1 \quad \text{for all } j \in \mathbb{N}$$

Since $\rho_j(x) \le C \min\left\{|x|^{-N},\, |x|^{-N/q}\right\}$ by Helly's selection theorem we may assume that $\rho_j \to \rho$ a.e., so that

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] \quad \text{and} \quad 1 \ge \int_{\mathbb{R}^N} \rho(x)\,dx$$

by Fatou's lemma. Pick $p \in (N/(N+\lambda), q)$ and apply (RHLS) with the same λ and $\alpha = \alpha(p)$:

$$I_\lambda[\rho_j] \ge \mathcal{C}_{N,\lambda,p} \left(\int_{\mathbb{R}^N} \rho_j^p\,dx\right)^{(2-\alpha(p))/p}$$

Hence the ρ_j are uniformly bounded in $\mathrm{L}^p(\mathbb{R}^N)$: $\rho_j(x) \le C'\,|x|^{-N/p}$,

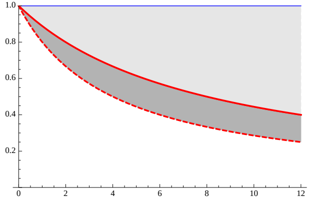
$$\int_{\mathbb{R}^N} \rho_j^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx = 1$$

by dominated convergence.

Existence of a minimizer: second case

If $N/(N+\lambda) < q < 2N/(2N+\lambda)$ we consider the *relaxed inequality*

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \ \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx + M \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q}$$



The $0 < \alpha < 1$ case: dark grey region

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (RHLS) and admits an optimizer (ρ, M).

Sketch Proof

Let (ρ_j, M_j) be a minimizing sequence with ρ_j radially symmetric non-increasing by rearrangement, such that

$$\int_{\mathbb{R}^N} \rho_j \, dx + M_j = \int_{\mathbb{R}^N} \rho_j^q = 1$$

- Local estimates + Helly's selection theorem: $\rho_j \to \rho$ almost everywhere and $M_j \to M := L + \lim_{j\to\infty} M_j$, so that $\int_{\mathbb{R}^N} \rho \, dx + M = 1$, and $\int_{\mathbb{R}^N} \rho(x)^q \, dx = 1$.
- μ_j are tight: up to a subsequence, $\mu_j \to \mu$ weak * and $d\mu = \rho \, dx + L \, \delta$

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \geq I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \,,$$

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} |x|^\lambda \, \rho_j \, dx \geq \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx$$

- Conclusion: $\liminf_{j\to\infty} \mathcal{Q}[\rho_j, M_j] \geq \mathcal{Q}[\rho, M]$.

Optimizers are positive

$$\mathcal{Q}[\rho, M] := \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx}{\left(\int_{\mathbb{R}^N} \rho \, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{(2-\alpha)/q}}$$

Lemma

Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \geq 0$ is an optimal function for some $M > 0$, then ρ is radial (up to a translation), monotone non-increasing and positive a.e. on $\mathbb{R}^N$

If ρ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure, then

$$\mathcal{Q}\big[\rho, M + \varepsilon \, \mathbb{1}_E\big] = \mathcal{Q}[\rho, M] \left(1 - \frac{2-\alpha}{q} \, \frac{|E|}{\int_{\mathbb{R}^N} \rho(x)^q \, dx} \, \varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction if (ρ, M) is a minimizer of $\mathcal{Q}$.

Euler–Lagrange equation

Euler–Lagrange equation for a minimizer (ρ_*, M_*)

$$\frac{2\int_{\mathbb{R}^N}|x-y|^\lambda\,\rho_*(y)\,dy+M_*|x|^\lambda}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\,\rho_*\,dy}-\frac{\alpha}{\int_{\mathbb{R}^N}\rho_*\,dy+M_*}-\frac{(2-\alpha)\,\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}=0$$

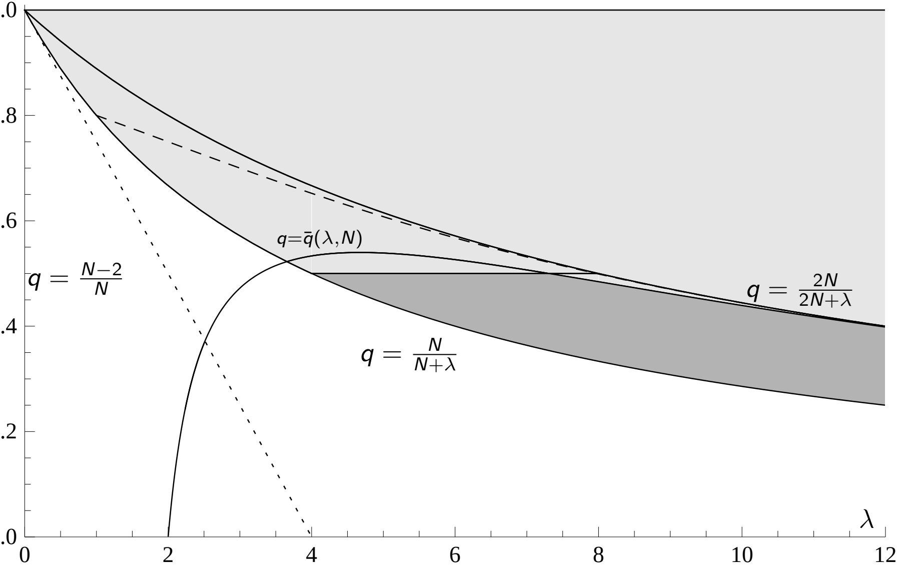
We can reformulate the question of the optimizers of (RHLS) as:

When is it true that $M_*=0$?

We already know that $M_*=0$ if

$$\frac{2N}{2N+\lambda}<q<1$$

Regions of no concentration and regularity of measure valued minimizers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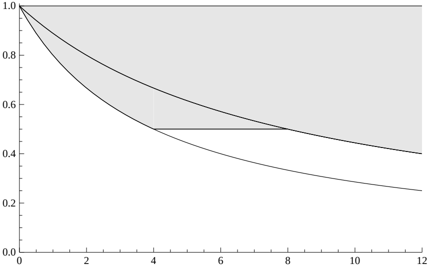
.8
.6
.4
.2
.0
0
2
4
6
8
10
12
λ
$q = \frac{N-2}{N}$
$q=\bar{q}(\lambda,N)$
$q = \frac{2N}{2N+\lambda}$
$q = \frac{N}{N+\lambda}$

No concentration 1



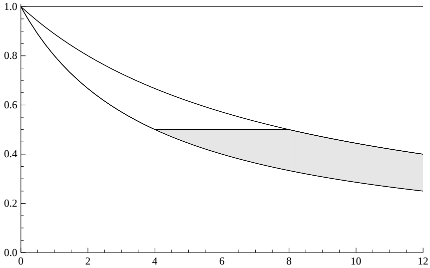
Proposition

Let $N \geq 1$, $\lambda > 0$ *and* $\dfrac{N}{N+\lambda} < q < \dfrac{2N}{2N+\lambda}$

If $N \geq 3$ *and* $\lambda > 2N/(N-2)$, *assume further that* $q \geq \dfrac{N-2}{N}$

If (ρ_*, M_*) *is a minimizer, then* $M_* = 0$.

Regularity and concentration



Proposition

If $N \geq 3$, $\lambda > 2N/(N-2)$ *and*

$$\frac{N}{N+\lambda} < q < \min\left\{\frac{N-2}{N},\ \frac{2N}{2N+\lambda}\right\},$$

and $(\rho_*, M_*) \in \mathrm{L}^{N(1-q)/2}(\mathbb{R}^N) \times [0, +\infty)$ *is a minimizer, then* $M_* = 0$

Regularity

Proposition

Let $N \geq 1$, $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$
Let (ρ_*, M_*) be a minimizer

1. *If $\int_{\mathbb{R}^N} \rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, then $M_* = 0$ and ρ_*, bounded and*

$$\rho_*(0) = \left(\frac{(2-\alpha) I_\lambda[\rho_*] \int_{\mathbb{R}^N} \rho_*\,dx}{\left(\int_{\mathbb{R}^N} \rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \int_{\mathbb{R}^N} \rho_*\,dx - \alpha I_\lambda[\rho_*]\right)}\right)^{\frac{1}{1-q}}$$

2. *If $\int_{\mathbb{R}^N} \rho_*\,dx = \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, then $M_* = 0$ and ρ_* is unbounded*
3. *If $\int_{\mathbb{R}^N} \rho_*\,dx < \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, then ρ_* is unbounded and*

$$M_* = \frac{\alpha I_\lambda[\rho_*] - 2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \;\int_{\mathbb{R}^N} \rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx} > 0$$

Ingredients of the proof

- Vary $\mathcal{Q}[\rho_*, M]$ with respect to M and make use of:

Lemma

For constants A, $B > 0$ and $0 < \alpha < 1$, define

$$f(M) = \frac{A+M}{(B+M)^\alpha} \quad \textit{for} \quad M \geq 0$$

Then f attains its minimum on $[0, \infty)$ at $M = 0$ if $\alpha A \leq B$ and at $M = (\alpha A - B)/(1-\alpha) > 0$ if $\alpha A > B$

- Vary $\mathcal{Q}[\rho, M_*]$ with respect to ρ and make use of the Euler-Lagrange condition to derive a condition for the boundedness of ρ_*.

No concentration 2

For any $\lambda \geq 1$ we deduce from

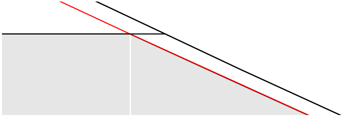
$$|x-y|^\lambda \leq \big(|x|+|y|\big)^\lambda \leq 2^{\lambda-1}\left(|x|^\lambda+|y|^\lambda\right)$$

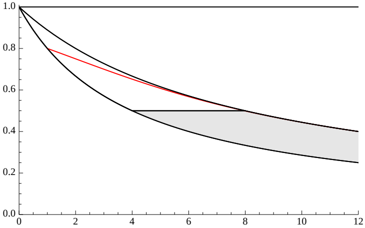
that

$$I_\lambda[\rho] < 2^\lambda \int_{\mathbb{R}^N} |x|^\lambda \,\rho\, dx \int_{\mathbb{R}^N} \rho(x)\, dx$$

For all $\alpha \leq 2^{-\lambda+1}$, we infer that $M_* = 0$ if

$$q \geq \frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda}$$



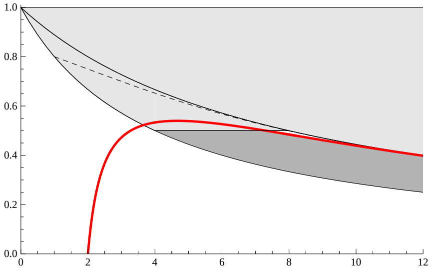


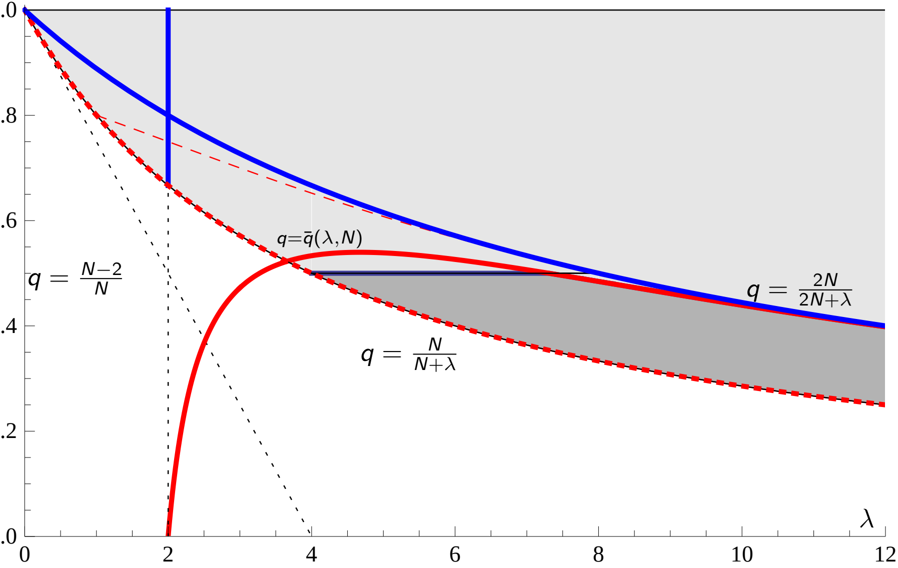
No concentration 3

Layer cake representation (superlevel sets are balls)

$$I_\lambda[\rho] \le 2\, A_{N,\lambda} \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx \int_{\mathbb{R}^N} \rho(x)\, dx$$

$$A_{N,\lambda} := \sup_{0\le R,S<\infty} \frac{\iint_{B_R\times B_S} |x-y|^\lambda\, dx\, dy}{|B_R| \int_{B_S} |x|^\lambda\, dx + |B_S| \int_{B_R} |y|^\lambda\, dy}$$





.0
.8
.6
.4
.2
.0
0
2
4
6
8
10
12
λ
$q=\bar{q}(\lambda,N)$
$q = \frac{N-2}{N}$
$q = \frac{2N}{2N+\lambda}$
$q = \frac{N}{N+\lambda}$

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Thank you for your attention !