Suppression of chemotactic explosion by mixing

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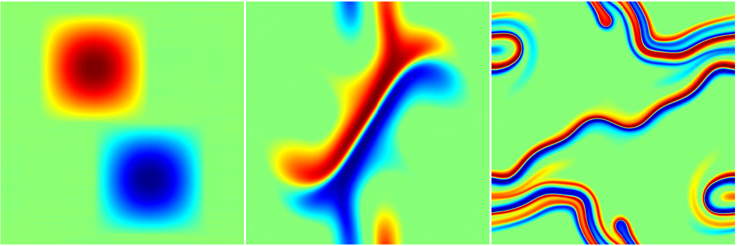
Background: Mixing

Question: How to mix two immiscible dyes?

- Can you mix it "perfectly"?
- At what cost?



Model: Transport Equation

- Can you mix it “perfectly”?
- At what cost?

Concentration of dye satisfies:

$$\partial_t \theta + u \nabla \theta = 0. \tag{1}$$

θ: smooth, periodic, spatial mean-zero, bounded.
u: divergence free.

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θ: smooth, periodic, spatial mean-zero, bounded.
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Goal: choose u that “best mixes” θ under following physical constraints.

- Fixed energy, $\|u\|_{L^2} = 1$.
- Fixed enstrophy, $\|\nabla u\|_{L^2} = 1$.

Definition of Mixing

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Quantify how "mixed" θ is:
Naive approach: Variance.

$$Var(\theta) = \|\theta - <\theta>\|_{L^2}^2 = \|\theta\|_{L^2}^2 \to 0.$$

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Does not work!

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Proposition. Suppose the spatial mean-zero function $\theta(x,t)$ is bounded uniformly in $L^2(\mathbb{T}^d)$ for all $t > 0$. Then

$$\lim_{t\to\infty} \int_{\mathbb{T}^d} g(x)\theta(x,t) = 0 \quad \forall g \in L^2(\mathbb{T}^d)$$

$$\iff \lim_{t\to\infty} \|\theta\|_{\dot{H}^{-a}} = 0, \forall a > 0.$$

Here recall that $\|\theta\|_{\dot{H}^{-a}} = \||k|^{-a}\hat{\theta}(k)\|_{l^2}$ for mean-zero θ.

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"Definition": (Lin,Thiffeault,Doering 2011) The smaller $\|\theta\|_{H^{-1}}$ is, the better mixing θ gets.

Previous results

Theorem.(Lin,Thiffeault,Doering 2011) Let θ be the solution to (1). Under the constraint $\|u\|_{L^2} = U$, for some constant $C(U)$ we have

$$\|\theta\|_{H^{-1}} \geq \|\theta_0\|_{H^{-1}} - C(U)\|\theta_0\|_{L^\infty} t.$$

Theorem.(Lunasin,Lin,Novikov,Mazzucato,Doering 2012) The linear lower bound for fixed energy constraint is sharp.

Remark. In these papers, numerical simulation suggests that for enstrophy-constraint flow we may have exponential decay of H^{-1} norm.

Enstrophy-constrained case

Theorem (G. Iyer, A. Kiselev, X. Xu, 2014)

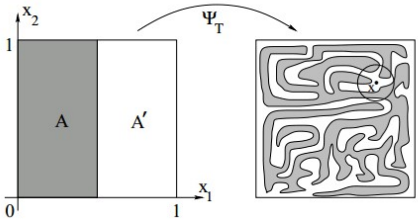
Let θ solve (1) and $p > 1$. There exists constant c_0, c_1 depend only on p, d and the initial data θ_0, such that

$$\|\theta\|_{H^{-1}} \geq c_0 \exp\left(-c_1 \int_0^t \|\nabla u(s)\|_{L^p} ds\right).$$

Remarks.

1. This means in finite time, it is impossible for $\|\theta\|_{H^{-1}}$ to get zero for fixed enstrophy constraint.
2. The proof based on the Crippa-DeLellis' work related to Bressan's mixing conjecture and the connection between H^{-1} mixing and Bressan's δ-mixing.
3. Christian Seis got the similar result independently by using totally different method.

Idea of the proof



Definition

Let $\kappa \in (0, \frac{1}{2})$ be fixed. For $\delta > 0$, we say a set $H \subseteq \mathbb{T}^d$ is δ-mixed if

$$\kappa \leq \frac{m(H \cap B(x,\delta))}{m(B(x,\delta))} \leq 1 - \kappa \quad \text{for every } x \in \mathbb{T}^d.$$

Idea of the proof

Conjecture (Bressan, 2003)

Let A to be the left half of the torus, and Ψ be the flow generated by an incompressible vector field u. If after time T the image of A under the flow Ψ is δ-mixed, then there exists a constant C such that

$$\int_0^T ||\nabla u(\cdot, t)||_{L^1}\, dt \geq \frac{|\log \delta|}{C}.$$

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Theorem (G. Crippa, C. De Lellis, 2008)

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$$\int_0^T ||\nabla u(\cdot, t)||_{L^p} \, dt \geq \frac{|\log \delta|}{C},$$

for all $p > 1$.

Idea of the proof

Slightly extension of δ-mixed setting:

Definition

Let $\kappa \in (0, \frac{1}{2})$ be fixed. For $\delta > 0$, we say a set $H \subseteq \mathbb{T}^d$ is δ-semi-mixed if

$$\frac{m(H \cap B(x,\delta))}{m(B(x,\delta))} \leq 1 - \kappa \quad \text{for every } x \in \mathbb{T}^d.$$

Fact: Start with arbitrary sets, not just A.

Idea of the proof

Lemma

$$\|\theta\|_{H^{-1}} \leq c_0(\lambda)\|\theta\|_{L^\infty}\delta^2 \Longrightarrow A_\lambda = \{\theta > \lambda\|\theta\|_{L^\infty}\} \text{ is } \delta\text{-semi-mixed.}$$

1. Set $\delta \approx \|\theta(t)\|_{H^{-1}}^{1/2}$.
2. Use Crippa-De Lellis' rearrangement cost lemma: $\int_0^t ||\nabla u(\cdot, t)||_{L^p} dt \geq \frac{m(A_\lambda)^{1/p}}{C_p} \log(\frac{2r_0}{\delta})$.

Optimal mixing flow

Question: Is that sharp?

Optimal mixing flow

Question: Is that sharp?
Answer: Yes. Answered by Y. Yao and A. Zlatos in 2014, by constructing a family of specific flows.
Meanwhile, G. Alberti, G. Crippa, and A. Mazzucato give a different construction leading to similar result.

Question: Can this insight into mixing mechanisms be useful in other applications?

Keller-Segel equation

Keller-Segel equation describes a population of bacteria or mold that secrete a chemical and are attracted by it. In one version of the simplified parabolic-elliptic form, this equation can be written in $\mathbb{R}^2$ as

$$\begin{cases} \partial_t \rho - \triangle \rho + \nabla \cdot (\rho \nabla c) = 0, \\ -\triangle c = \rho, \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{2}$$

Finite time blow-up

Assume ρ is smooth. Let $m_2(t) = \int_{\mathbb{R}^2} |x|^2 \rho(t,x) dx$ be the second moment, $m_0 = \int_{\mathbb{R}^2} \rho_0(x) dx$ be the total mass. Then, there exists some constant m_{crit} such that

$$\frac{d}{dt} m_2(t) = 4 m_0 (1 - \frac{m_0}{m_{crit}}).$$

Big mass $\implies$ Finite time blow up!

Main theorem

$$\begin{cases} \partial_t \rho - \triangle \rho + \nabla \cdot (\rho \nabla c) = 0, \\ -\triangle c = \rho, \\ \rho(x,0) = \rho_0(x). \end{cases}$$

Theorem (A. Kiselev, X. Xu, 2015)

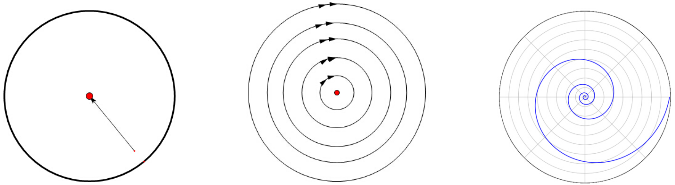
Consider the equation

$$\begin{aligned} &\partial_t \rho + (u \cdot \nabla)\rho - \triangle \rho + \nabla \cdot (\rho \nabla (-\triangle)^{-1}(\rho - \bar{\rho})) = 0, \\ &\rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}^d. \end{aligned} \tag{3}$$

Then, given any initial data $\rho_0 \geq 0$, $\rho_0 \in C^\infty(\mathbb{T}^d)$, $d = 2$ or 3, there exist smooth incompressible flows u such that the unique solution $\rho(x,t)$ of (3) is globally regular in time.

Remarks

Remark.
1. This is not trivial!



Fact: Suppose we add advection term $(u \cdot \nabla \rho)$ in (2) and take $u \cdot x = 0$, then the second moment will not change.
2. The proof of this theorem based on the idea in the paper about relaxation enhancing flow, by P. Constantin, A. Kiselev, L. Ryzhik and A. Zlatos in 2008.

Step 1: L^2 criterion

Lemma

Suppose that $\rho_0 \in C^\infty(\mathbb{T}^d)$, $\rho_0 \geq 0$, and suppose that $u \in C^\infty$ is divergence free, $d = 2$ or $d = 3$. Assume $[0, T]$ is the maximal interval of existence of unique smooth solution $\rho(x, t)$ of equation (3). Then we must have

$$\int_0^t \|\rho(\cdot, \tau) - \bar{\rho}\|_{L^2(\mathbb{T}^d)}^{\frac{4}{4-d}} d\tau \xrightarrow{t \to T} \infty. \tag{4}$$

Remark. Only need to control L^2 norm of ρ.

Step 2: H^1 condition

Lemma

Let $\rho(x,t) \geq 0$ be smooth local solution to (3) set on $\mathbb{T}^d$, $d = 2$ or 3. Suppose that $\|\rho(\cdot,t) - \bar{\rho}\|_{L^2} \equiv B > 0$ for some $t \geq 0$. Then there exists a universal constant C_1 such that

$$\|\rho(\cdot, t+\tau) - \bar{\rho}\|_{L^2} \leq 2B \text{ for every } 0 \leq \tau \leq C_1 \min(1, \bar{\rho}^{-1}, B^{-\frac{4}{4-d}}). \quad (5)$$

Moreover, there exists a universal constant C_0 such that if in addition

$$\|\rho(\cdot,t)\|_{H^1}^2 \geq B_1^2 \equiv C_0 B^{\frac{12-2d}{4-d}} + 2\bar{\rho}B^2, \quad (6)$$

then $\partial_t \|\rho(\cdot,t)\|_{L^2} < 0$.

Step 3: approximation lemma

Lemma

Suppose u is divergence free and Lipschitz in x, η is the solution to the transport equation:

$$\partial_t \eta + (Au \cdot \nabla)\eta = 0, \quad \eta(x,0) = \rho_0(x). \tag{7}$$

Then, suppose that the unique local smooth solution $\rho(x,t)$ exists for $t \in [0,T]$. Then for every $t \in [0,T]$ we have

$$\frac{d}{dt}\|\rho - \eta\|_{L^2}^2 \leq -\|\rho\|_{H^1}^2 + 4\|\rho_0\|_{H^1}^2 F(t)^2 + C\|\rho - \bar{\rho}\|_{L^2}^2 \left(\|\rho - \bar{\rho}\|_{L^2}^{\frac{12}{6-d}} + \bar{\rho}^2 \right). \tag{8}$$

Here $F(t) \in L^\infty[0,\infty)$ and depend only on u.

Remark. After time rescaling ($t = A\tau$), the difference of ρ and η is small.

Step 4: H^{-1} condition

Lemma

Suppose u is divergence free and Lipschitz in x, η is the solution to the transport equation:

$$\partial_t \eta + (u \cdot \nabla)\eta = 0, \quad \eta(x,0) = \rho_0(x). \tag{9}$$

Then for any $\sigma > 0$, integer N, there exists time T so that

$$\frac{1}{T}\int_0^T \|(Id - P_N)\eta(\cdot,t)\|_{L^2}^2 dt \geq (1-\sigma)\|\rho_0(\cdot)\|_{L^2}^2,$$

provided that

$$\frac{1}{T}\int_0^T \|\eta(\cdot,t)\|_{H^{-1}}^2 dt \xrightarrow{T\to\infty} 0.$$

Here P_N is the projection to the first N frequencies.

Summary

Step 1. We only need to control L^2 norm of the solution.

Step 2. If the H^1 norm of the solution is "big", then the L^2 norm will decrease.

Step 3. The solution to the nonlinear differential equation is close to the transport equation in L^2 if A is big enough.

Step 4. On the Fourier side, the velocity u will make the solution to transport equation concentrate on high frequencies.

Examples

Remarks.

Here we can choose three kinds of velocity u: the optimal mixing flow u^{YZ}, the relaxation enhancing flow u^{RE} and the cellular flow u^{L}.

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Here we can choose three kinds of velocity u: the optimal mixing flow u^{YZ}, the relaxation enhancing flow u^{RE} and the cellular flow u^L.
A: magnitude of u.

u^{YZ}: We have an explicit form of A in terms of ρ_0. However, u^{YZ} depends on time.

u^{RE}: We don't have an explicit form of A. Nevertheless, u^{RE} does not depend on time.

u^L: We have an explicit form of A. u^L does not depend on time. L depends on the size of initial data ρ_0.

Cellualr flow

Lemma (G. Iyer, X, $\geq$ 2016)

Assume

$$\partial_t \xi + (u_L \cdot \nabla)\xi = 0 \,, \quad x \in \mathbb{T}^2, \xi(\cdot, 0) = \xi_0. \tag{10}$$

Here u_L is a cellular flow of cell size L. In addition assume $\int_{\mathbb{T}^2} \xi_0 = 0$. Then for all $C > 0$, there exists $L_0 > 0$ and $T = T(L_0)$ such that for all $L < L_0$ we have

$$\frac{1}{T}\int_0^T \|\xi\|_{H^1} \geq C\|\xi\|_{L^2}.$$

Remark. $L \approx \max\{\|\rho_0\|_{L^2}, \|\rho_0\|_{L^1}\}^2$.

Corollary

The following corollary is true for both u^{YZ} and u^{RE}.

Corollary

For every $\delta > 0$ and $\kappa > 0$, there exists $A_1 = A_1(\rho_0, u, \kappa, \delta)$ such that if $A \geq A_1$, then

$$\|\rho^A(\cdot, t) - \bar{\rho}\|_{L^2} \leq \|\rho_0 - \bar{\rho}\|_{L^2} e^{-\kappa t} \tag{11}$$

for all $t \geq \delta$.

Generalization

- Generalized model, e.g., 3D Navier-Stokes equation.
- Coupled system, e.g., Keller-Segel equation coupled with Stokes equation.

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