Boundaries and interfaces in asymptotics of hyperbolic systems of balance laws

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Boundaries and interfaces in asymptotics of hyperbolic systems of balance laws

Hyperbolic systems of balance laws with stiff effects

- hyperbolic limit
- parabolic limit

Pointwise effects and small layers

- Boundary conditions
- Coupling interfaces

Boundaries and interfaces in asymptotics of hyperbolic systems of balance laws

Two distinct problems:

I. Jin–Xin model with implicit equilibrium manifold on a bounded domain

- Asymptotic behavior of boundary conditions
- Approximate but explicit computation of the equilibrium manifold

with B. Perthame and M. Tournus

II. Interface coupling of a systems of balance laws with its parabolic limit

- The Goldstein–Taylor model and the heat equation
- The p-system and the nonlinear heat equation
- Interface coupling/domain decomposition/two-scale discontinuous rate

with A.-C. Boulanger, C. Cancès, H. Mathis and K. Saleh

Jin–Xin model with implicit equilibrium manifold

U-tube with porous walls (from kidney modeling):

$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \frac{1}{\varepsilon}\big(h(v_\varepsilon) - u_\varepsilon\big) \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \frac{1}{\varepsilon}\big(u_\varepsilon - h(v_\varepsilon)\big) \end{cases} \quad \text{on } [0,1]$$

BC's: $u_\varepsilon(t,0) = u_l$ and $v_\varepsilon(t,1) = \alpha u_\varepsilon(t,1) \quad (0 < \alpha \leqslant 1)$

and $\{h(v) = u\} \cap \{v = \alpha u\} = \emptyset$

Two main difficulties

1. Asymptotic boundary conditions: numerical boundary layers
2. Implicit equilibrium manifold $\mathscr{E} = \{u = h(v)\}$
   Assuming $h' > 1$, the limit $\varepsilon \to 0$ satisfies $u = h(v)$ and

$$\partial_t A(v) + \partial_x B(v) = 0$$

   where

$$A(v) = h(v) + v \quad \text{and} \quad B(v) = h(v) - v$$

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Numerical boundary layers for the Jin–Xin model

The classical Jin–Xin model:

(JX)
$$\begin{cases} \partial_t a_\varepsilon + \partial_x b_\varepsilon = 0 \\ \partial_t b_\varepsilon + \partial_x a_\varepsilon = \dfrac{1}{\varepsilon}\big(f(a_\varepsilon) - b_\varepsilon\big) \end{cases}$$

Assuming $0 < f' < 1$, the limit $\varepsilon \to 0$ satisfies

(CL)
$$\partial_t a + \partial_x f(a) = 0, \qquad b = f(a)$$

Numerical approximation by a splitting technique

- Upwind scheme for the PDE part $(a_\varepsilon, b_\varepsilon)_i^n \to (a_\varepsilon, b_\varepsilon)_i^{n+1/2}$
- Implicit scheme with explicit formula $(a_\varepsilon, b_\varepsilon)_i^{n+1/2} \to (a_\varepsilon, b_\varepsilon)_i^{n+1}$

$$\begin{cases} (a_\varepsilon)_i^{n+1} = (a_\varepsilon)_i^{n+1/2}, \\ (b_\varepsilon)_i^{n+1} = (b_\varepsilon)_i^{n+1/2} + \dfrac{\Delta t}{\varepsilon}\big(f((a_\varepsilon)_i^{n+1}) - (b_\varepsilon)_i^{n+1}\big). \end{cases}$$

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When $\varepsilon = 0$, it becomes the Rusanov scheme for (CL)

$$a_i^{n+1} = a_i^n - \frac{\Delta t}{2\Delta x}\big(f(a_{i+1}^n) - f(a_{i-1}^n) - (a_{i+1}^n - 2a_i^n + a_{i-1}^n)\big)$$

Numerical boundary layers for the Jin–Xin model

Define $u_\varepsilon = a_\varepsilon + b_\varepsilon$ and $v_\varepsilon = a_\varepsilon - b_\varepsilon$ to decouple the PDE part:

(JXd)
$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \dfrac{1}{\varepsilon}\big(f((u_\varepsilon + v_\varepsilon)/2) - (u_\varepsilon - v_\varepsilon)/2\big) \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = -\dfrac{1}{\varepsilon}\big((u_\varepsilon - v_\varepsilon)/2 - f((u_\varepsilon + v_\varepsilon)/2)\big) \end{cases}$$

Bounded domain $[0,1]$:

- Imposed entrance at $x = 0$:
  $u_\varepsilon(t,0) = u_l$
- Re-entrance at $x = 1$:
  $v_\varepsilon(t,1) = \alpha u_\varepsilon(t,1)$ (with $0 < \alpha \leqslant 1$)

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Assuming $0 < f' < 1$, the limit $\varepsilon \to 0$ satisfies

$$\begin{cases} \partial_t a + \partial_x f(a) = 0 \qquad \text{on } [0,1] \\ \big[a + f(a)\big](t,0) = u_l \end{cases}$$

At $x = 1$, presence of a relaxation boundary layer which vanishes as $\varepsilon \to 0$

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Numerical boundary layers for the Jin–Xin model

From the numerical point of view:

- Boundary conditions approximated by ghost cells
- But the Rusanov scheme

$$a_i^{n+1} = a_i^n - \frac{\Delta t}{2\Delta x}\big(f(a_{i+1}^n) - f(a_{i-1}^n) - (a_{i+1}^n - 2a_i^n + a_{i-1}^n)\big)$$

also introduces a numerical boundary layer
$\longrightarrow$ Bad numerical approximation of the relaxation boundary layer

Idea. Modify the numerical scheme in order to obtain the upwind scheme

$$a_i^{n+1} = a_i^n - \frac{\Delta t}{\Delta x}\big(f(a_i^n) - f(a_{i-1}^n)\big)$$

Solution. See [Chalons, Berthon, Turpault 2013]: convex combination w.r.t. ε

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Jin–Xin model with implicit equilibrium manifold

Jin–Xin model with implicit equilibrium manifold:

(IJX) $$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \dfrac{1}{\varepsilon}\big(h(v_\varepsilon) - u_\varepsilon\big) \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \dfrac{1}{\varepsilon}\big(u_\varepsilon - h(v_\varepsilon)\big) \end{cases}$$

Assuming $h' > 1$, the limit $\varepsilon \to 0$ satisfies $u = h(v)$ and

(ICL) $$\partial_t A(v) + \partial_x B(v) = 0$$

where $A(v) = h(v) + v$ and $B(v) = h(v) - v$

Alternative formulation:

$$\begin{cases} a_\varepsilon = u_\varepsilon + v_\varepsilon \\ b_\varepsilon = u_\varepsilon - v_\varepsilon \end{cases} \Longrightarrow \begin{cases} \partial_t a_\varepsilon + \partial_x b_\varepsilon = 0 \\ \partial_t b_\varepsilon + \partial_x a_\varepsilon = \dfrac{2}{\varepsilon}\big(h(a_\varepsilon - b_\varepsilon) - (a_\varepsilon + b_\varepsilon)\big) \end{cases}$$

Assuming $h' > 1$, the limit $\varepsilon \to 0$ satisfies $a + b = h(a - b)$ and

$$\partial_t a + \partial_x C(a) = 0$$

where $C(a) = B \circ A^{-1}(a)$

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Jin–Xin model with implicit equilibrium manifold

Whatever the formulation we choose:

- The limit equation requires to invert $A = h + \mathbf{I}$
- The usual splitting method for (IJX) also needs to invert a nonlinear function:

$$(b_\varepsilon)^{n+1} = (b_\varepsilon)^{n+1/2} + \frac{2\Delta t}{\varepsilon}\big(h((a_\varepsilon)^{n+1} - (b_\varepsilon)^{n+1}) - ((a_\varepsilon)^{n+1} + (b_\varepsilon)^{n+1})\big)$$

Construct a new numerical scheme for (IJX):

- which does not require any inversion of nonlinear functions
- which is a 3-point scheme
- which corresponds to the upwind scheme for the PDE part of (IJX)
- which becomes an upwind scheme for the limit equation (ICL)

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[S., Tournus 2015]

Construction of a class of such schemes, for instance:

(IAP)
$$\begin{cases} (u_\varepsilon)_i^{n+1} = (u_\varepsilon)_i^n - \dfrac{\Delta t}{\Delta x}\big((u_\varepsilon)_i^n - (u_\varepsilon)_{i-1}^n\big) + \dfrac{\Delta t}{\varepsilon + \Delta x}(S_{u_\varepsilon})_i^n \\ (v_\varepsilon)_i^{n+1} = (v_\varepsilon)_i^n + \dfrac{\Delta t}{\Delta x}\big((v_\varepsilon)_{i+1}^n - (v_\varepsilon)_i^n\big) - \dfrac{\Delta t}{\varepsilon + \Delta x}(S_{v_\varepsilon})_i^n \end{cases}$$

where

$$\begin{cases} (S_{u_\varepsilon})_i^n = h((v_\varepsilon)_{i-1}^n) - (u_\varepsilon)_{i-1}^n + (v_\varepsilon)_i^n - (v_\varepsilon)_{i-1}^n \\ (S_{v_\varepsilon})_i^n = h((v_\varepsilon)_i^n) - (u_\varepsilon)_i^n + (v_\varepsilon)_{i+1}^n - (v_\varepsilon)_i^n \end{cases}$$

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The limit of the scheme

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is

(ILS)
$$\begin{cases} a_i^{n+1} = a_i^n - \dfrac{\Delta t}{\Delta x}\big(B(v_i^n) - B(v_{i-1}^n)\big) \\ v_i^{n+1} = v_i^n - \dfrac{\Delta t}{\Delta x}\big(h(v_i^n) - u_i^n\big) \\ u_i^{n+1} = a_i^{n+1} - v_i^{n+1} \end{cases}$$

Jin–Xin model with implicit equilibrium manifold

The limit scheme

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must solve

$$\partial_t a + \partial_x (B \circ A^{-1})(a) = 0$$

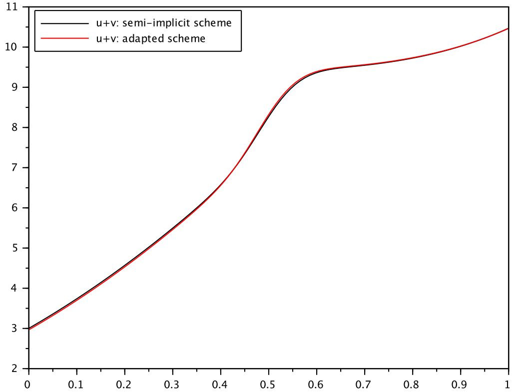
The second equation of the scheme corresponds to the numerical deviation from the equilibrium manifold

$$\mathscr{E} = \{u = h(v)\}$$

$\rightarrow$ Approximate Newton method
$\rightarrow$ Consistency in the sense of finite differences

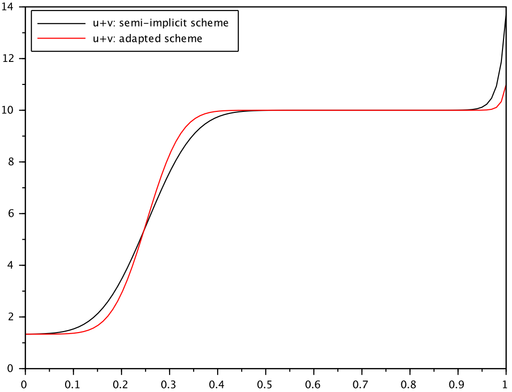
Semi-implicit scheme versus adapted scheme

Riemann problem with non-equilibrium right-hand boundary conditions: $\varepsilon = 1$



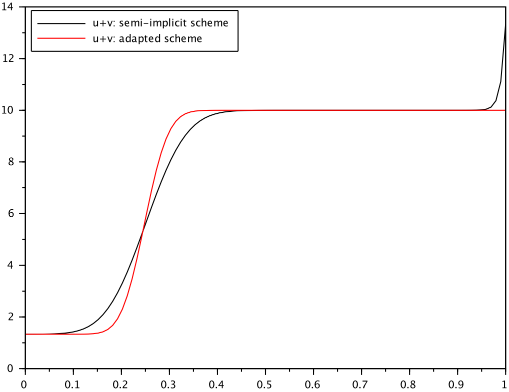
Semi-implicit scheme versus adapted scheme

Riemann problem with non-equilibrium right-hand boundary conditions: $\varepsilon = 10^{-2}$



Semi-implicit scheme versus adapted scheme

Riemann problem with non-equilibrium right-hand boundary conditions: $\varepsilon = 10^{-5}$



Jin–Xin model with implicit equilibrium manifold

- [Perthame, S., Tournus 2015]
  Convergence of the model when $\varepsilon \to 0$ (even with $h(v,x)$)
  - Adapted heterogeneous entropies
  - BV_t ($\Rightarrow \mathrm{BV}_x$) estimates $\to$ strong convergence
- [S., Tournus 2015]
  Construction of an explicit AP scheme
  - which tends to an upwind scheme
  - which approximately solve the implicit equilibrium
  - Complete analysis for $\varepsilon > 0$

- Analysis of the limit scheme: convergence towards the entropy solution?
- Extension to more complex models

Boundaries and interfaces in asymptotics of hyperbolic systems of balance laws

Two distinct problems:

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with B. Perthame and M. Tournus

II. Interface coupling of a systems of balance laws with its parabolic limit

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Interface coupling of a systems of balance laws with its parabolic limit

The coupling problem: $\varepsilon > 0$ at the left, $\varepsilon = 0$ at the right

$x < 0$	$x = 0$	$x > 0$

Hyperbolic system with relaxation	Associated parabolic limit
The Goldstein–Taylor model $\begin{cases} \varepsilon \partial_t v + \partial_x u = 0 \\ \varepsilon \partial_t u + a^2 \partial_x v = -\dfrac{\sigma}{\varepsilon} u \end{cases}$	The heat equation $\begin{cases} \partial_t v - \dfrac{a^2}{\sigma} \partial_{xx} v = 0 \\ u = 0 \end{cases}$
The p-system with damping $\begin{cases} \varepsilon \partial_t \tau - \partial_x u = 0 \\ \varepsilon \partial_t u + \partial_x P(\tau) = -\dfrac{\sigma}{\varepsilon} u \end{cases}$	The nonlinear heat equation $\begin{cases} \partial_t \tau + \dfrac{1}{\sigma} \partial_{xx} P(\tau) = 0 \\ u = 0 \end{cases}$

The Goldstein–Taylor model

Linear 2×2 system with linear dissipative term

(GT) $$\begin{cases} \varepsilon\partial_t v + \partial_x u = 0 \\ \varepsilon\partial_t u + a^2\partial_x v = -\dfrac{\sigma}{\varepsilon} u \end{cases}$$

- From the second equation: $u = -\varepsilon \dfrac{a^2}{\sigma}\partial_x v + \mathcal{O}(\varepsilon^2)$
- Inject in the first equation and divide by ε

When $\varepsilon \to 0$, one recover the linear heat equation

(HE) $$\begin{cases} \partial_t v - \dfrac{a^2}{\sigma}\partial_{xx} v = 0 \\ u = 0 \end{cases}$$

Passage from a hyperbolic regime to a parabolic regime. . .

Design of asymptotic-preserving schemes

Construct a numerical scheme for system (GT) which becomes a numerical scheme for system (HE) when $\varepsilon \to 0$

- Control of the numerical diffusion compared to the parabolic limit
- From a hyperbolic CFL condition to a parabolic CFL condition

Here, we follow [Gosse, Toscani 2003]:

1. Space localization of the source term (well-balanced schemes, LeRoux et al.)
2. Implicit discretization of the source to guarantee the asymptotic stability

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AP scheme: space localization of the source term

1. Given a mesh with Δt and Δx, approximate solution $(W_i^n)_{i,n} = (v_i^n, u_i^n)_{i,n}$:

$$\begin{cases} v_i^{n+1} = v_i^n - \dfrac{\Delta t}{\Delta x}\big(F_v(W_i^n, W_{i+1}^n) - F_v(W_{i-1}^n, W_i^n)\big) \\ u_i^{n+1} = v_i^n - \dfrac{\Delta t}{\Delta x}\big(F_u(W_i^n, W_{i+1}^n) - F_u(W_{i-1}^n, W_i^n)\big) + \Delta t\ S_i^n \end{cases}$$

or $u_i^{n+1} = v_i^n - \dfrac{\Delta t}{\Delta x}\big(F_u^-(W_i^n, W_{i+1}^n) - F_u^+(W_{i-1}^n, W_i^n)\big)$

2. Solve at each interface the extended Riemann problem

$$\begin{cases} \partial_t v + \partial_x \dfrac{u}{\varepsilon} = 0 \\ \partial_t u + \partial_x \dfrac{a^2 v}{\varepsilon} + \dfrac{\sigma}{\varepsilon^2} u\ \partial_x \chi = 0 \\ \partial_t \chi = 0 \end{cases} \quad \text{with} \quad \begin{cases} W(0,x) = \begin{cases} W_i^n & \text{for } x < 0 \\ W_{i+1}^n & \text{for } x > 0 \end{cases} \\ \chi(0,x) = \begin{cases} \chi_l & \text{for } x < 0 \\ \chi_r & \text{for } x > 0 \end{cases} \end{cases}$$

with $\chi_r - \chi_l = \Delta x$

AP scheme: space localization of the source term

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with $\chi_r - \chi_l = \Delta x$

AP scheme: space localization of the source term

3. . . . and obtain (remark the jump of v at $x = 0$)

$$W(t,x) = \begin{cases} W_l & \text{for } x/t < -a/\varepsilon \\ (\bar{v}^-, \bar{u}/K_\varepsilon) & \text{for } -a/\varepsilon < x/t < 0 \\ (\bar{v}^+, \bar{u}/K_\varepsilon) & \text{for } 0 < x/t < a/\varepsilon \\ W_r & \text{for } x/t > a/\varepsilon \end{cases}$$

with $K_\varepsilon = 1 + \dfrac{\sigma \Delta x}{2a\varepsilon}$ and

$$\bar{u}(W_l, W_r) = \frac{u_l + u_r}{2} - \frac{a}{2}(v_r - v_l)$$

$$\bar{v}^-(W_l, W_r) = v_l - \frac{1}{a}(\bar{u}(W_l, W_r)/K_\varepsilon - u_l)$$

$$\bar{v}^+(W_l, W_r) = v_r + \frac{1}{a}(\bar{u}(W_l, W_r)/K_\varepsilon - u_r)$$

AP scheme: space localization of the source term

4. The numerical scheme writes

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right]$$

$$u_i^{n+1} = u_i^n - \frac{a^2 \Delta t}{\varepsilon \Delta x}\left[\bar{v}^-(W_i^n, W_{i+1}^n) - \bar{v}^+(W_{i-1}^n, W_i^n)\right]$$

or equivalently, with $\bar{v}(W_l, W_r) = \dfrac{v_l + v_r}{2} - \dfrac{1}{2a}(u_r - u_l)$,

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BUT the resulting source term is **explicitly approximated**!

$\Longrightarrow$ CFL condition: $\Delta t \leqslant \dfrac{2\varepsilon}{\sigma}\left(\varepsilon + \dfrac{\sigma \Delta x}{2a}\right)$, which $\to 0$ when $\varepsilon \to 0$!

AP scheme: implicit modification of the source term

4. The classical well-balanced scheme:

$$
\begin{aligned}
v_i^{n+1} &= v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right] \\
u_i^{n+1} &= u_i^n - \frac{a^2 \Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{v}(W_i^n, W_{i+1}^n) - \bar{v}(W_{i-1}^n, W_i^n)\right] - \frac{\sigma \Delta t}{\varepsilon^2 K_\varepsilon} u_i^n
\end{aligned}
$$

5. Implicit discretization of the source term:

$$
\begin{aligned}
v_i^{n+1} &= v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right] \\
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5. Implicit discretization of the source term, but explicit formula:

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right]$$
$$\left(1 + \frac{\sigma \Delta t}{\varepsilon^2 K_\varepsilon}\right) \times u_i^{n+1} = u_i^n - \frac{a^2 \Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{v}(W_i^n, W_{i+1}^n) - \bar{v}(W_{i-1}^n, W_i^n)\right]$$

AP scheme

The final scheme (recall that $K_\varepsilon = 1 + \frac{\sigma \Delta x}{2a\varepsilon}$):

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right]$$

$$u_i^{n+1} = \left(1 + \frac{\sigma \Delta t}{\varepsilon^2 K_\varepsilon}\right)^{-1}\left(u_i^n - \frac{a^2 \Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{v}(W_i^n, W_{i+1}^n) - \bar{v}(W_{i-1}^n, W_i^n)\right]\right)$$

When $\varepsilon \to 0$, one obtains (except for $n = 0$):

$$v_i^{n+1} = v_i^n + \frac{a^2}{\sigma}\frac{\Delta t}{\Delta x^2}\left[v_{i+1}^n - 2v_i^n + v_{i-1}^n\right], \qquad u_i^{n+1} = 0$$

Proposition ([Gosse, Toscani 2003])

This numerical scheme is asymptotic preserving since

- *it is consistent with* (GT) *when* $\varepsilon > 0$ *and with* (HE) *when* $\varepsilon = 0$
- *it is* $\mathbf{L}^2$*-stable under the CFL condition* $\quad \Delta t \leqslant \varepsilon \frac{\Delta x}{a} + \frac{\sigma}{2a^2}\Delta x^2$

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The p-system with damping

The p-system with linear dissipative term, with $P' > 0$,

(PS)
$$\begin{cases} \varepsilon \partial_t \tau - \partial_x u = 0 \\ \varepsilon \partial_t u + \partial_x P(\tau) = -\dfrac{\sigma}{\varepsilon} u \end{cases}$$

- From the second equation: $u = -\varepsilon \dfrac{1}{\sigma} \partial_x P(\tau) + \mathcal{O}(\varepsilon^2)$
- Inject in the first equation and divide by ε

When $\varepsilon \to 0$, one recover the nonlinear heat equation

(NHE)
$$\begin{cases} \partial_t \tau + \dfrac{1}{\sigma} \partial_{xx} P(\tau) = 0 \\ u = 0 \end{cases}$$

Passage from a hyperbolic regime to a parabolic regime...

Design of asymptotic-preserving schemes

Construct a numerical scheme for system (PS) which becomes a numerical scheme for system (NHE) when $\varepsilon \to 0$

- Control of the numerical diffusion compared to the parabolic limit
- From a hyperbolic CFL condition to a parabolic CFL condition

Here, we extend [Gosse, Toscani 2003] to the nonlinear case:

1. Use approximate Riemann solver (HLL, relaxation. . .)
2. Space localization of the source term (well-balanced schemes, LeRoux et al.)
3. Implicit discretization of the source to guarantee the asymptotic stability

Note that schemes of [Chalons, Coquel, Godlewski, Raviart, S. 2010] and [Berthon, Turpault 2010] correspond to steps 1+2 (see also [Chalons, Girardin, Kokh 2013] for large time-step methods)

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2. Solve at each interface the extended approximate Riemann problem

$$\begin{cases} \partial_t \tau - \partial_x \dfrac{u}{\varepsilon} = 0 \\ \partial_t u + \partial_x \dfrac{\pi}{\varepsilon} + \dfrac{\sigma}{\varepsilon^2} u\ \partial_x \chi = 0 \\ \partial_t \pi + \partial_x \dfrac{a^2}{\varepsilon} u = 0 \\ \partial_t \chi = 0 \end{cases} \quad \text{with} \quad \begin{cases} W(0,x) = \begin{cases} W_i^n & \text{for } x < 0 \\ W_{i+1}^n & \text{for } x > 0 \end{cases} \\ \chi(0,x) = \begin{cases} \chi_l & \text{for } x < 0 \\ \chi_r & \text{for } x > 0 \end{cases} \end{cases}$$

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AP scheme

Same calculations + Implicit discretization of the source term...

$$\tau_i^{n+1} = \tau_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{u}(W_i^n, W_{i+1}^n) - \bar{u}(W_{i-1}^n, W_i^n)\right]$$

$$u_i^{n+1} = \left(1 + \frac{\sigma \Delta t}{\varepsilon^2 K_\varepsilon}\right)^{-1}\left(u_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x}\left[\bar{\pi}(W_i^n, W_{i+1}^n) - \bar{\pi}(W_{i-1}^n, W_i^n)\right]\right)$$

When $\varepsilon \to 0$, one obtains (except for $n = 0$):

$$\tau_i^{n+1} = \tau_i^n - \frac{\Delta t}{\sigma \Delta x^2}\left[P(\tau_{i+1}^n) - 2P(\tau_i^n) + P(\tau_{i-1}^n)\right], \qquad u_i^{n+1} = 0$$

Proposition ([Boulanger, Cancès, Mathis, Saleh, S. 2012])

This numerical scheme is asymptotic preserving since

- *it is consistent with* (PS) *when* $\varepsilon > 0$ *and with* (NHE) *when* $\varepsilon = 0$
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The coupling problem

The coupling problem: $\varepsilon > 0$ at the left, $\varepsilon = 0$ at the right

$x < 0$ $\qquad$ $x = 0$ $\qquad$ $x > 0$

The Goldstein–Taylor model	The heat equation
$\begin{cases} \partial_t v + \frac{1}{\varepsilon}\partial_x u = 0 \\ \partial_t u + \frac{a^2}{\varepsilon}\partial_x v = -\frac{\sigma}{\varepsilon^2} u \end{cases}$	$\begin{cases} \partial_t v - \frac{a^2}{\sigma}\partial_{xx} v = 0 \\ u = 0 \end{cases}$

Basic requirements:

- When $\varepsilon \gg 1$
  Hyperbolic solution at the left and the parabolic solution at the right
- When $\varepsilon \ll 1$
  Hyperbolic and parabolic are similar, the coupling should also be

$\Longrightarrow$ Mixing of BC's
$\Longrightarrow$ Recover the parabolic scheme when $\varepsilon \to 0$

The coupling problem: a first simple idea

Coupling conditions, from parabolic/parabolic coupling:

- Continuity of the unknown:
  $v(t, 0^-) = v(t, 0^+)$
- Continuity of the flux, i.e. global conservation of v:
  $\frac{u}{\varepsilon}(t, 0^-) = -\frac{a^2}{\sigma}\partial_x v(t, 0^+)$

At the numerical interface of coupling $x_{1/2} = 0$ between cells 0 and 1:

- Ghost-cell method:

$$W_0^n = (v_0^n, u_0^n) \quad \Big| \quad W_1^n = (v_0^n, 0)$$

- Compute the common flux for v using the hyperbolic model:

$$F_{1/2}^n = F_{\text{Hyp}}\big(W_0^n, (v_1^n, 0)\big) = \frac{1}{\varepsilon K_\varepsilon}\bar{u}\big(W_0^n, (v_1^n, 0)\big)$$

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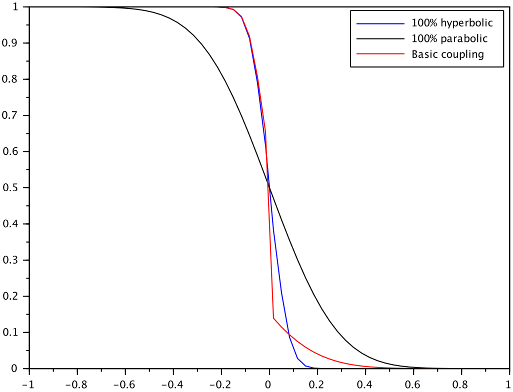
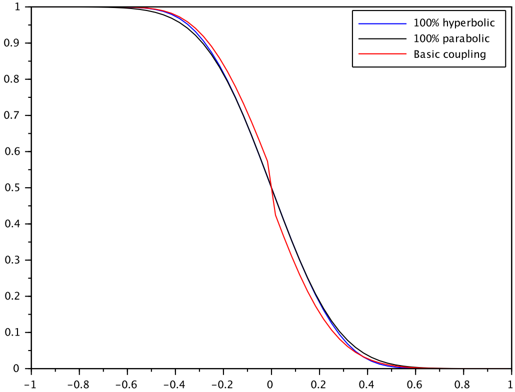
The coupling problem: a first simple idea

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 1$



The coupling problem: a first simple idea

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.1$



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At the numerical interface of coupling $x_{1/2} = 0$:

- Solve a partial Riemann problem at the left:

$$u^*_{1/2} - u^n_0 = a(v^n_0 - v^*_{1/2})$$

- Continuity of the flux:

$$\frac{u^*_{1/2}}{\varepsilon} = -\frac{a^2}{\sigma}\frac{v^n_1 - v^*_{1/2}}{\Delta x/2}$$

The coupling problem: partial Riemann problem

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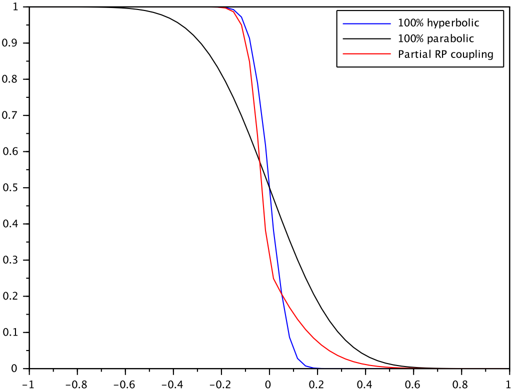
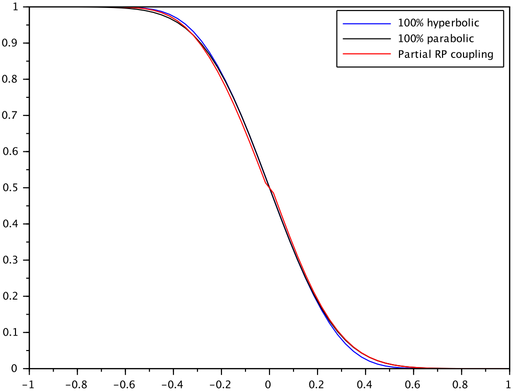


The coupling problem: partial Riemann problem

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.1$



The coupling problem for the p-system with damping

The coupling problem: $\varepsilon > 0$ at the left, $\varepsilon = 0$ at the right

$x < 0$	$x = 0$	$x > 0$
The p-system with damping $\begin{cases} \partial_t \tau - \frac{1}{\varepsilon}\partial_x u = 0 \\ \partial_t u + \frac{1}{\varepsilon}\partial_x P(\tau) = -\frac{\sigma}{\varepsilon^2} u \end{cases}$		The nonlinear heat equation $\begin{cases} \partial_t \tau + \frac{a^2}{\sigma}\partial_{xx} P(\tau) = 0 \\ u = 0 \end{cases}$

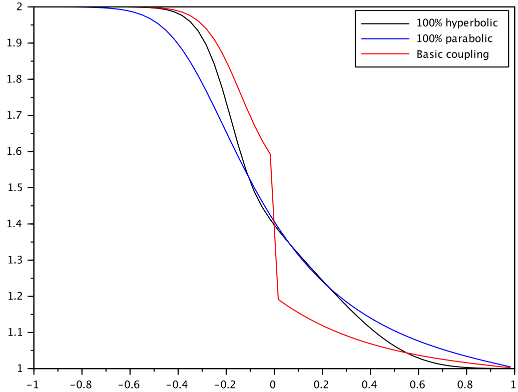
Same requirements and ideas...

- First idea: use the AP numerical flux
- Second idea: use a partial Riemann problem

Same consequences!

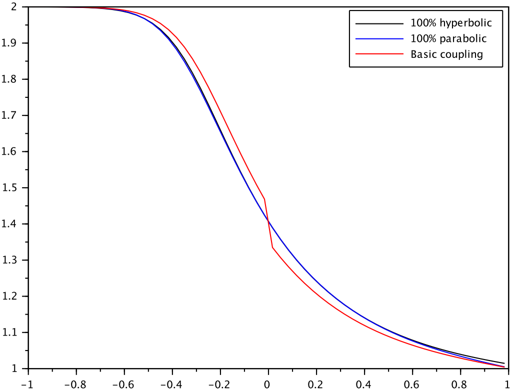
The coupling problem: using the AP flux

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.5$



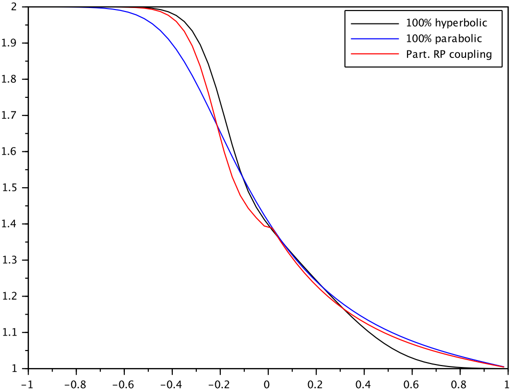
The coupling problem: using the AP flux

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.1$



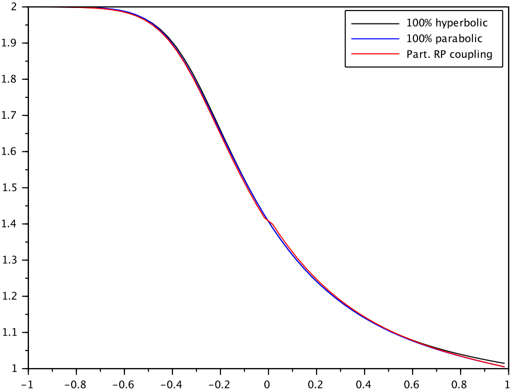
The coupling problem: partial Riemann problem

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.5$



The coupling problem: partial Riemann problem

Riemann problem with Dirichlet boundary conditions: $\varepsilon = 0.1$



Interfacial coupling and parabolic limit

- Goldstein–Taylor model and the p-system with damping
- Construction of asymptotic preserving schemes

- Interfacial coupling between two numerical schemes which perfectly match
  1. Direct use of the numerical flux of the AP scheme
  2. Partial Riemann problem in the hyperbolic part

But. . .

- What are the rigorous coupling conditions?
- How to extend this coupling to more complex systems?
- What happens when the two numerical schemes do not perfectly match?

[Salvarani 1999], [Salvarani, Golse 2007], [Golse, Jin, Levermore 2003], [Lemou, Méhats 2012], [Vasseur 2012], [Coquel, Jin, Liu, Wang 2015]. . .

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Than you for your attention