Control of Stochastic Behaviors in Robotic Swarms using PDE Models

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Problem Statement
We study a task allocation problem for a swarm of robots where a target distribution of spatial coverage by the swarm is desired. We model the population dynamics of the robots using a set of advection reaction diffusion partial differential equations (PDEs). The task allocation problem is then framed and solved as an optimal control problem.

Microscopic Model
- Agent primitives based on stochastic differential equation formalism
- Robots’ changes in state are modeled as a Chemical Reaction Network (CRN) in which the species are \( F \), a flying robot; \( H_j \), a robot that is hovering over a flower of type \( j \); and \( V_j \), an instance of a robot visit to a flower of type \( j \)
  \[
  H_j \overset{k_j}{\rightarrow} F + V_j
  \]
- Robot \( i \) has position \( \mathbf{x}_i(t) = [x_i(t) \ y_i(t)]^T \) at time \( t \).
- Time-dependent velocity field \( \mathbf{v}(t) = [v_x(t) \ v_y(t)]^T \)
- Robots’ motion over time step \( \Delta t \) modeled using the standard-form Langevin equation:
  \[
  \mathbf{x}_i(t + \Delta t) - \mathbf{x}_i(t) = \mathbf{v}(t) \Delta t + (2D \Delta t)^{1/2} \mathbf{Z}(t)
  \]

Macrosopic Model
- \( \Omega \in \mathbb{R}^2 \) is an open bounded subset with Lipschitz continuous boundary \( \partial \Omega \). \( Q = \Omega \times (0, T) \) and \( \Sigma = \partial \Omega \times (0, T) \).
- \( \mathbf{f} \in \mathbb{R}^2 \) is the outward normal to \( \partial \Omega \).
- There are \( n_f \) types of flowers.
- \( H_j : \Omega \rightarrow \{0, 1\} \) are indicator functions that model the presence or absence of flower type \( j \) over the domain \( \Omega \).
- \( y_1(x, t), y_2(x, t) \) and \( y_3(x, t) \) are the states representing the density fields of flying robots, hovering robots and flower visits at \( (x, t) \in Q \).

Optimal Control Problem
\[
\min_{(y_1, y_2, y_3, u) \in \mathcal{U}} \int_0^T \left[ \frac{1}{2} ||W_y(y, \cdot, T) - y_1 ||^2_{L^2(\Omega)} + \frac{\lambda}{2} ||u||^2_{L^2(0,T)} \right] dt
\]
\[
\mathcal{U}_{ad} = \{ u \in L^2(0, T)^{m+2} ; u_i^{min} \leq u_i \leq u_i^{max} \text{ a.e. in } (0, T) \}
\]
\[
\mathbf{v}(t) = (u_1, u_2) = (\mathbf{b}_j, u_i)_{i \leq 3} \text{ for } 3 \leq i \leq m+2 \text{ and } m = n_f
\]
- Unique weak solution exists satisfying the PDE as an equality in \( Y^* = L^2(0,T) \).
- Existence of optimal control can be proven using standard arguments based on weak compactness of closed bounded sets in the corresponding infinite dimensional spaces and embedding arguments.
- Directional Derivatives of the control to state map exist along directions \( \mathbf{h} \in L^m(0,T)^{m+2} \).
- Adjoint equation characterizes the first order necessary conditions for the optimal control:
  \[
  \frac{\partial p_1}{\partial t} = \nabla \cdot (D_V p_1 + v(t)p_1) + \sum_{j \in \mathcal{I}} k_j H_j (-p_1 + p_2 + p_3) \text{ in } Q,
  \]
  \[
  \frac{\partial p_2}{\partial t} = k_f p_1 - k_f p_2 \text{ in } Q,
  \]
  \[
  \frac{\partial p_3}{\partial t} = 0 \text{ in } Q,
  \]
  \[
  \mathbf{n} \cdot \nabla p_1 = 0 \text{ on } \Sigma,
  \]
  \[
  p(T) = W^* (W_y(y(\cdot, T) - y_1))
  \]

Simulation Results
The top-left figure is a snapshot of the robot states for a sample test case. The bottom-left figure shows the optimized control parameters for a test case with two flower types and a spatially non-uniform target distribution of robot activity. The bottom-right figure shows the optimized control parameters for a test case with one flower type and uniform target distribution of robot activity over the two rightmost rows.

This framework can also be used to map features of interest when the task spatial distribution (indicator functions \( H_i \)) is not known a priori [2]