An asymptotic preserving and well-balanced scheme for a chemotaxis model

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jointed work with

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Main motivations

☐ Hyperbolic systems of conservation law with source term

$$\partial_t w + \partial_x f(w) = \frac{1}{\nu} S(w) \qquad\qquad w \in \Omega$$

- Steady states

$$\partial_x f(w) = \frac{1}{\nu} S(w) \qquad \Longleftrightarrow \qquad \text{Manifold given by} \quad \mathcal{M} = \{w \in \Omega;\ g(w) = 0\}$$

- Asymptotic diffusive regime: $\nu \to 0$

☐ Finite volume schemes

- Numerical approximations of the weak solutions
- Robustness
- Exact capture of (a part of) $\mathcal{M}$
- Asymptotic preserving

Outline

□ A chemotaxis model (with A. Crestetto and F. Foucher)

Godunov type well-balanced scheme

- Model and main properties
- Godunov type scheme
- Characterization of the approximate Riemann solver
- Robustness and well-balanced properties
- Asymptotic preserving property

□ Asymptotic convergence rate (with M. Bessemoulin and H. Mathis)

- relative entropies to estimate the asymptotic convergence rate
- A continuous estimation by C. Lattanzio and A. Tzavaras
- Extension to a semi-discrete scheme

A model of Chemotaxis (Ribot et al 2012 - 2014)

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p(\rho)) = -\chi \rho \partial_x \Phi - \alpha \rho u \\ \partial_t \Phi - D \partial_{xx} \Phi = a\rho - b\Phi \end{cases}$$

χ, α, D, a, b given parameters

$\Omega = \{w \in \mathbb{R}^3;\ \rho \geq 0, u \in \mathbb{R}, \Phi \geq 0\}$

$p(\rho) = \delta \rho^\gamma$

☐ Steady states at rest

$$\begin{cases} u = 0 \\ e(\rho) - \chi \Phi = \text{cste} \end{cases} \qquad e(\rho) = \delta \frac{\gamma}{\gamma - 1} \rho^{\gamma - 1} + \text{cste}$$

solutions of $D\partial_{xx}\Phi - b\Phi = a\rho$

if $\rho = 0$: $\quad \phi(x) = A \cosh\left(x\sqrt{b}\right) + B \sinh\left(x\sqrt{b}\right)$,

if $\rho > 0,\ C < 0$: $\quad \phi(x) = A \cos\left(x\sqrt{|C|}\right) + B \sin\left(x\sqrt{|C|}\right) - \phi_p, \quad \rho(x) = \frac{\chi}{2\delta}\left(\phi(x) - K\right)$,

if $\rho > 0,\ C > 0$: $\quad \phi(x) = A \cosh\left(x\sqrt{C}\right) + B \sinh\left(x\sqrt{C}\right) - \phi_p, \quad \rho(x) = \frac{\chi}{2\delta}\left(\phi(x) - K\right)$,

where A and B are some constants, $C = \frac{1}{D}\left(b - \frac{a\chi}{2\delta}\right)$ and $\phi_p = \frac{Ka\chi}{2\delta b - a\chi}$.

□ Asymptotic behavior

Rescaling: $t \to t/\nu$ (long time) and $\alpha \to \alpha/\nu$ (dominant friction)

Limit $\nu \to 0$ to get a diffusive regime

$$\begin{cases} \partial_t \rho = \partial_x(\partial_x p - \chi \rho \partial_x \Phi) \\ D\partial_{xx}\Phi = b\Phi - a\rho \end{cases}$$

□ Objectives

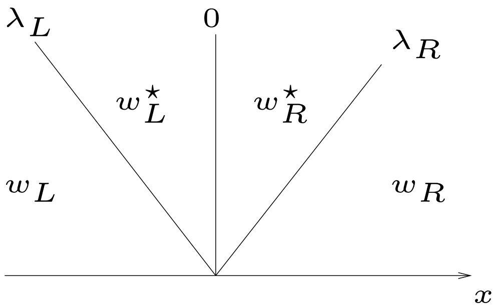
- Robustness ($\rho \geq 0$ and $\Phi \geq 0$)
- Steady state preserving (well-balanced)
- Asymptotic preserving

Godunov type strategy (CB and Chalons 2015)

Godunov type scheme

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p(\rho)) = -\chi \rho \partial_x \Phi - \alpha \rho u \end{cases} \qquad \Phi \text{ given}$$

☐ Approximate Riemann solver: $\tilde{w}$



$\lambda_L < 0 < \lambda_R$ HLL type solver

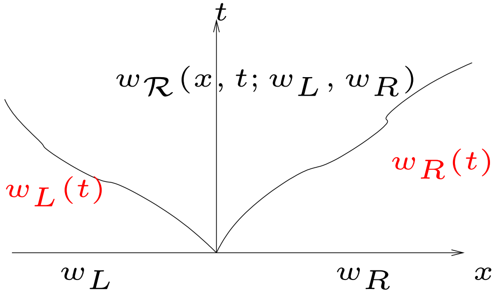
Source term → stationary contact wave

- Harten-Lax-van Leer consistency condition

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \tilde{w}\,(x, \Delta t; w_L, w_R)\,dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} w_{\mathcal{R}}\,(x, \Delta t; w_L, w_R)\,dx$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \tilde{w}\,(x, \Delta t; w_L, w_R)\,dx = \frac{1}{2}(w_L + w_R) + \frac{\Delta t}{\Delta x}(\lambda_L(w_L - w_L^\star) + \lambda_R(w_R^\star - w_R))$$

Because of the source term, $w_{\mathcal{R}}$ stays unknown



constant is not a natural solution

$$\int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} \Big(\partial_t w + \partial_x f(w) = S(w) \Big) dx dt$$

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} w_{\mathcal{R}} \left(x, \Delta t; w_L, w_R\right) dx = \frac{1}{2}(w_L + w_R) + \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} S(w) dx dt$$
$$- \frac{1}{\Delta x} \int_0^{\Delta t} f(w_{\mathcal{R}}(\Delta x/2, t)) dt + \frac{1}{\Delta x} \int_0^{\Delta t} f(w_{\mathcal{R}}(-\Delta x/2, t)) dt$$

<u>Approximation</u>

$$w_{\mathcal{R}}(-\Delta x/2, t) \simeq w_L$$
$$w_{\mathcal{R}}(\Delta x/2, t) \simeq w_R$$

As a consequence

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \rho_{\mathcal{R}}\left(x,\Delta t; w_L, w_R\right) dx \simeq \frac{1}{2}(\rho_L+\rho_R) - \frac{\Delta t}{\Delta x}(\rho_R u_R - \rho_L u_L)$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} (\rho u)_{\mathcal{R}}\left(x,\Delta t; w_L, w_R\right) dx \simeq \frac{1}{2}(\rho_L u_L+\rho_R u_R) - \frac{\Delta t}{\Delta x}(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L)$$

$$+ \Delta t S_{\mathcal{R}} - \alpha \int_0^{\Delta t} \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} (\rho u)_{\mathcal{R}}\left(x,t; w_L, w_R\right) dx dt$$

$$S_{\mathcal{R}} = \frac{1}{\Delta t \Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} \chi \rho_{\mathcal{R}} \partial_x \Phi dx dt$$

- Approximation $S_{\mathcal{R}} \simeq S^\star$ to be defined (independently from Δt)
- Approximation

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} (\rho u)_{\mathcal{R}}\left(x,\Delta t; w_L, w_R\right) dx \simeq \mathcal{F}(\Delta t)$$

solution of the following integral equation

$$\mathcal{F}(\Delta t) = \frac{1}{2}(\rho_L u_L+\rho_R u_R) - \frac{\Delta t}{\Delta x}(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L) + \Delta t S^\star - \alpha \int_0^{\Delta t} \mathcal{F}(t) dt$$

□ Consistency conditions

$$\lambda_L(\rho_L - \rho_L^\star) + \lambda_R(\rho_R^\star - \rho_R) = \rho_L u_L - \rho_R u_R$$

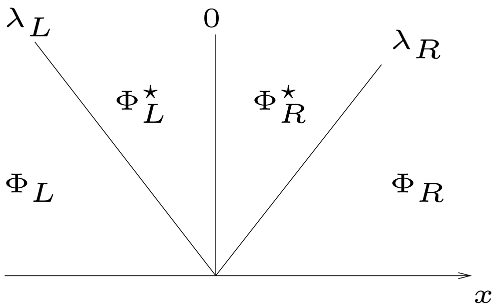
$$\lambda_L(\rho_L u_L - \rho_L^\star u_L^\star) + \lambda_R(\rho_R^\star u_R^\star - \rho_R u_R) =$$
$$\frac{1}{\alpha\Delta t}\left(\mathrm{e}^{-\alpha\Delta t} - 1\right)\left(\frac{\alpha}{2}(\rho_L u_L + \rho_R u_R)\Delta x - (\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L) + \Delta x S^\star\right)$$

□ Flux continuity: $\rho_L^\star u_L^\star = \rho_R^\star u_R^\star$

□ Well-balanced conditions

$$S^\star = \frac{\chi}{\Delta x}\frac{p_R - p_L}{e_R - e_L}(\Phi_R - \Phi_L)$$
$$e_L\frac{\rho_L^\star}{\rho_L} - \chi\phi_L = e_R\frac{\rho_R^\star}{\rho_R} - \chi\phi_R$$

□ Approximate Riemann solver: $\tilde{w}$



$$\begin{cases} \partial_t \Phi + \partial_x \Psi = a\rho - b\Phi \\ \Psi = \partial_x \Phi \qquad\qquad \rho \text{ given} \end{cases}$$

□ Harten-Lax-van Leer consistency condition

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \tilde{\Phi}\left(x, \Delta t; w_L, w_R\right) dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \Phi_{\mathcal{R}}\left(x, \Delta t; w_L, w_R\right) dx$$

Main difficulty: Evaluate $\Phi_{\mathcal{R}}$

$$\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} \left(\partial_t \Phi - D\partial_x \Psi\right) dxdt = a\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} \rho_{\mathcal{R}}(x,t)dxdt - b\int\int \Phi_{\mathcal{R}}(x,t)dxdt$$

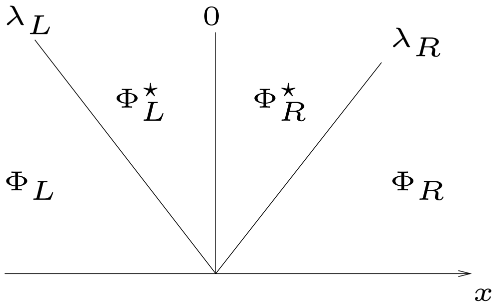
Approximations

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \rho_{\mathcal{R}}(x,t)dx \simeq \frac{1}{2}(\rho_L + \rho_R) - \frac{t}{\Delta x}(\rho_R u_R - \rho_L u_L)$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} \partial_x \Psi(x,t)dx \simeq \frac{\Delta t}{\Delta x}(\Psi_R - \Psi_L)$$

Godunov type scheme

□ Approximate Riemann solver: $\tilde{w}$



$$\begin{cases} \partial_t \Phi + \partial_x \Psi = a\rho - b\Phi \\ \Psi = \partial_x \Phi \qquad\qquad \rho \text{ given} \end{cases}$$

□ Harten-Lax-van Leer consistency condition

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \Phi_{\mathcal{R}}(x,t)dx \simeq \mathcal{G}(\Delta t)$$

$\mathcal{G}(\Delta t)$ solution of the following integral equation

$$\mathcal{G}(\Delta t) = \frac{1}{2}(\Phi_L + \Phi_R) + D\frac{\Delta t}{\Delta x}(\Psi_R - \Psi_L) + a\left(\frac{\Delta t}{2}(\rho_L + \rho_R) - \frac{\Delta t^2}{2\Delta x}(\rho_R u_R - \rho_L u_L)\right) - b\int_0^{\Delta t} \mathcal{G}(t)dt$$

□ Godunov type scheme

$$\begin{cases} w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(w_i^n - w^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(w_i^n - w^\star_{i+\frac{1}{2},L}\right)\right) \\ \Phi_i^{n+1} = \Phi_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(\Phi_i^n - \Phi^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(\Phi_i^n - \Phi^\star_{i+\frac{1}{2},L}\right)\right) \end{cases}$$

□ Definition of Ψ_i^n

$$\Psi_i^n = \frac{1}{2\Delta x}(\Phi_{i+1}^n - \Phi_{i-1}^n) \times \mathcal{E}(\Delta x)$$

$$\mathcal{E}(\Delta x) \quad \text{is consitent with 1} \qquad \lim_{\Delta x \to 0} \mathcal{E}(\Delta x) = 1$$

We impose

$$\mathcal{E}(\Delta x) = \begin{cases} \frac{\Delta x^2}{2} \frac{b}{\cos\left(\sqrt{b}\Delta x\right) - 1} & \text{if } \rho = 0 \\ \frac{\Delta x^2}{2} \frac{C}{\cos\left(\sqrt{|C|}\Delta x\right) - 1} & \text{if } \rho > 0,\ C < 0 \\ \frac{\Delta x^2}{2} \frac{C}{\cosh\left(\sqrt{C}\Delta x\right) - 1} & \text{if } \rho > 0,\ C > 0 \end{cases}$$

to recover the steady states

Theorem

- Robustness (adopting a local cut-off, Chalons et al 2014)

$$\left.\begin{aligned} \rho^{\star}_{L,R} &= \min(\max(0, \rho^{\star}_{L,R}), 2\rho^{HLL}) \\ \Phi^{\star}_{L,R} &= \min(\max(0, \Phi^{\star}_{L,R}), 2\Phi^{HLL}) \end{aligned}\right. \quad \Rightarrow \quad \rho_i^{n+1} \geq 0 \quad \text{and} \quad \Phi_i^{n+1} \geq 0$$

- Well-balance property

Asymptotic diffusive regime

☐ Rescalling

$$u = \nu u^\nu, \qquad t = t^\nu/\nu \qquad \text{and} \qquad \alpha = \alpha^\nu/\nu,$$

↪ Rescalled system given by

$$\begin{cases} \nu\partial_{t^\nu}\rho^\nu + \nu\partial_x\left(\rho^\nu u^\nu\right) = 0, \\ \nu^2\partial_{t^\nu}\left(\rho^\nu u^\nu\right) + \partial_x\left(\nu^2\rho^\nu(u^\nu)^2 + p(\rho^\nu)\right) = \chi\rho^\nu\partial_x\phi^\nu - \alpha^\nu\rho^\nu u^\nu, \\ \nu\partial_{t^\nu}\phi^\nu - D\partial_{xx}\phi^\nu = a\rho^\nu - b\phi^\nu, \end{cases}$$

↪ Chapman-Enskog expansion

$$\rho^\nu = \rho^0 + O(\nu) \qquad \rho^\nu u^\nu = \rho^0 u^0 + O(\nu) \qquad \phi^\nu = \phi^0 + O(\nu).$$

↪ Zero-order governed by

$$\begin{cases} \partial_{t^\nu}\rho^0 + \partial_x\left(\dfrac{\chi}{\alpha^\nu}\rho^0\partial_x\phi^0 - \dfrac{1}{\alpha^\nu}\partial_x p(\rho^0)\right) = 0, \\ D\partial_{xx}\phi^0 = b\phi^0 - a\rho^0, \\ \rho^0 u^0 = -\dfrac{1}{\alpha^\nu}\partial_x p(\rho^0) + \dfrac{\chi}{\alpha^\nu}\rho^0\partial_x\phi^0. \end{cases}$$

Asymptotic preserving scheme

$$\begin{cases} w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(w_i^n - w^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(w_i^n - w^\star_{i+\frac{1}{2},L}\right)\right) \\ \Phi_i^{n+1} = \Phi_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(\Phi_i^n - \Phi^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(\Phi_i^n - \Phi^\star_{i+\frac{1}{2},L}\right)\right) \end{cases}$$

☐ Rescalling

$$(u_n) = \nu (u_i^n)^\nu, \qquad \Delta t = \Delta t^\nu/\nu \qquad \text{and} \qquad \alpha = \alpha^\nu/\nu,$$

↪ Simplification

$$\lambda_R = -\lambda_L = \lambda \qquad\qquad \lambda \simeq u + \sqrt{p'(\rho)} = O(1)$$

↪ CFL restriction

$$\frac{\Delta t}{\nu \Delta x} \max_{i\in\mathbb{Z}} \left(\lambda_{i+1/2}\right) \leq \frac{1}{2},$$

☐ In the limite $\nu \to 0$

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \frac{e\left(\rho_i^n\right)}{\rho_i^n} \Big(\ell_{i-1/2} \left((\rho u)_i^n - (\rho u)_{i-1}^n \right) + \ell_{i+1/2} \Big((\rho u)_{i+1}^n - (\rho u)_i^n \Big) \Big)$$

$$+ \lambda C_{CFL} \Big(\ell_{i+1/2} \left(e\left(\rho_{i+1}^n\right) - e\left(\rho_i^n\right) \right) - \ell_{i+1/2} \left(e\left(\rho_i^n\right) - e\left(\rho_{i-1}^n\right) \right) \Big)$$

$$+ \lambda C_{CFL} \chi \Big(\ell_{i+1/2} (\phi_{i+1}^n - \phi_i^n) - \ell_{i-1/2} (\phi_i^n - \phi_{i-1}^n) \Big)$$

$$(\rho u)_i^n = \frac{1}{\alpha} \left(-\frac{p(\rho_{i+1}^n) - p(\rho_{i-1}^n)}{2\Delta x} + \frac{\chi}{2} (\{\rho \partial_x \phi\}_{i-1/2}^n + \{\rho \partial_x \phi\}_{i-1/2}^n) \right)$$

$$+ \frac{1}{2} \left(\lambda C_{CFL} - \frac{1}{2} \right) \left((\rho u)_{i+1}^n - 2(\rho u)_i^n + (\rho u)_{i-1}^n \right) + \left((\rho u)_i^n - (\rho u)_i^{n+1} \right)$$

$$D \frac{(\partial_x \phi)_{i+1}^n - (\partial_x \phi)_{i-1}^n}{2\Delta x} = b \frac{\phi_{i-1}^n + 2\phi_i^n + \phi_{i+1}^n}{4} - a \frac{\rho_{i-1}^n + 2\rho_i^n + \rho_{i+1}^n}{4} + (\phi_i^{n+1} - \phi_i^n)$$

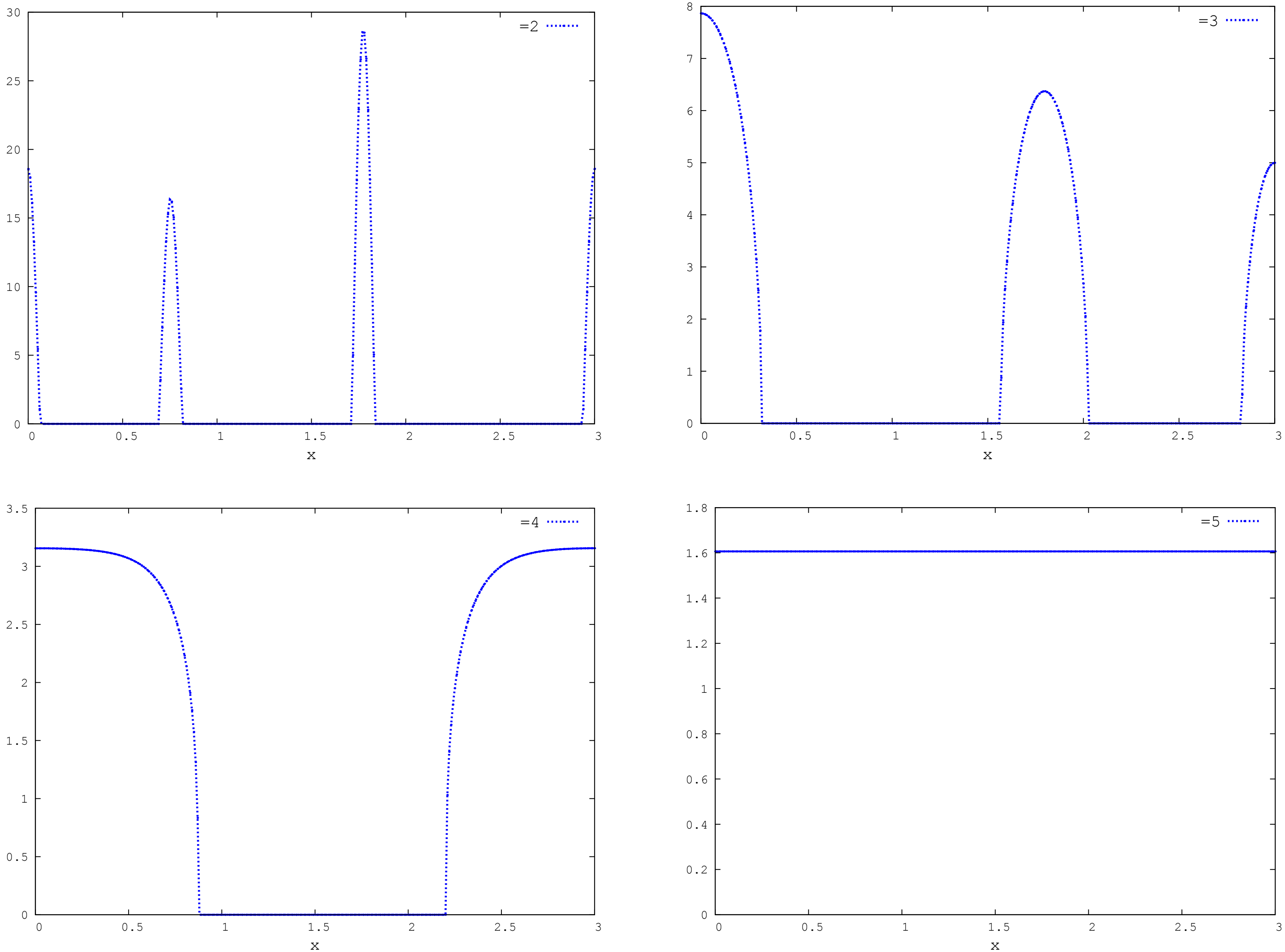
where we have set

$$\ell_{i+1/2} = \frac{1}{e\left(\rho_i^n\right) / \rho_i^n + e\left(\rho_{i+1}^n\right) / \rho_{i+1}^n}$$

The scheme is consitent with the asymptotic regime

Numerical results

Influence of γ at $\chi = 10$, $L = 3$, $\Delta x = 0.01$.



Convergence rate as $\nu \to 0$

☐ Simpler model: p-system

Rescalling

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x p(\tau) = -\sigma u \end{cases} \qquad \begin{aligned} t &\to t/\epsilon \\ u &\to \epsilon u^\epsilon \\ \alpha &\to \alpha/\epsilon \end{aligned} \qquad \begin{cases} \partial_t \tau^\epsilon - \partial_x u^\epsilon = 0 \\ \epsilon^2 \partial_t u^\epsilon + \partial_x p(\tau^\epsilon) = -\sigma u^\epsilon \end{cases}$$

☐ Asymptotic regime

$$\begin{cases} \partial_t \bar{\tau} - \partial_x \bar{u} = 0 \\ \partial_t p(\bar{\tau}) = -\sigma\, \bar{u} \end{cases}$$

☐ Relative entropy

$$\eta^\varepsilon(\tau, u | \overline{\tau}, \overline{u}) = \frac{\varepsilon^2}{2}(u - \overline{u})^2 - P(\tau|\overline{\tau}) \quad \text{with} \quad P(\tau|\overline{\tau}) = P(\tau) - P(\overline{\tau}) - p(\overline{\tau})(\tau - \overline{\tau})$$

To satisfy

$$\partial_t \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u}) + \frac{1}{\varepsilon^2}\partial_x \psi(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u}) = -\sigma(u^\varepsilon - \overline{u})^2 + \frac{1}{\sigma}\partial_{xx} p(\overline{\tau})\, p(\tau^\varepsilon | \overline{\tau}) + \frac{\varepsilon^2}{\sigma}\partial_{xt} p(\overline{\tau})(u^\varepsilon - \overline{u})$$

□ Lattanzio and Tzavaras estimation

Introduce

$$\phi(t) = \int_{\mathbb{R}} \eta(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u}) dx$$

Assume

- $\overline{\tau} \geq c > 0$
- $\|\partial_{xx} p(\overline{\tau})\|_{L^\infty(Q_T)} \leq K < +\infty$
- $\|\partial_{xt} p(\overline{\tau})\|_{L^2(Q_T)} \leq K < +\infty$

Then

$$\phi(t) \leq C(\phi(0) + \varepsilon^4) \qquad t \in [0, T)$$

□ Objective: Recover this estimation with numerical approximations

Semi-discrete scheme

□ p-system: semi-discrete scheme $(\tau_i(t), u_i(t))$

$$\frac{d\tau_i}{dt} = \frac{1}{2\Delta x}(u_{i+1} - u_{i-1}) + \frac{\lambda}{2\Delta x}(\tau_{i+1} - 2\tau_i + \tau_{i-1})$$
$$\frac{du_i}{dt} = -\frac{1}{2\varepsilon^2\Delta x}(p(\tau_{i+1}) - p(\tau_{i-1})) - \frac{\sigma}{\varepsilon^2}u_i + \frac{\lambda}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1})$$

□ asymptotic regime: semi-discrete scheme $(\overline{\tau}_i(t), \overline{u}_i(t))$

$$\frac{d\overline{\tau}_i}{dt} = \frac{1}{2\Delta x}(\overline{u}_{i+1} - \overline{u}_{i-1}) + \frac{\lambda}{2\Delta x}(\overline{\tau}_{i+1} - 2\overline{\tau}_i + \overline{\tau}_{i-1})$$
$$\overline{u}_i = -\frac{1}{\sigma}\frac{1}{2\Delta x}(p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1}))$$

□ Relative entropy: $\eta_i = \eta(\tau_i, u_i | \overline{\tau}_i, \overline{u}_i)$

$$\frac{d\eta_i}{dt} + \frac{1}{\varepsilon^2}\frac{\psi_{i+1/2} - \psi_{i-1/2}}{\Delta x} = -\sigma(u_i - \overline{u}_i)^2 + \frac{1}{\sigma}\frac{p(\overline{\tau}_{i+2}) - 2p(\overline{\tau}_i) + p(\overline{\tau}_{i-2})}{(2\Delta x)^2} p(\tau_i|\overline{\tau}_i) + \frac{\varepsilon^2}{\sigma}\frac{d}{dt}\left(\frac{p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})}{2\Delta x}\right)(u_i - \overline{u}_i) + \mathcal{R}_i^\tau + \mathcal{R}_i^u$$

$$\mathcal{R}_i^\tau = \varepsilon^2 \frac{\lambda}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1})(u_i - \overline{u}_i)$$

$$\mathcal{R}_i^u = \frac{\lambda}{2\Delta x}\left((p(\overline{\tau}_i) - p(\tau_i))(\tau_{i+1} - 2\tau_i + \tau_{i-1}) + p'(\overline{\tau}_i)(\overline{\tau}_{i+1} - 2\overline{\tau}_i + \overline{\tau}_{i-1})(\tau_i - \overline{\tau}_i)\right)$$

□ By summation: $\phi(t) = \sum_{i\in\mathbb{Z}} \eta_i(t)\Delta x$

$$\phi(t) - \phi(0) = -\sigma \int_0^t \sum_{i\in\mathbb{Z}} (u_i - \overline{u}_i)^2 \Delta x ds + \frac{1}{\sigma}\int_0^t \sum_{i\in\mathbb{Z}} \frac{p(\overline{\tau}_{i+2}) - 2p(\overline{\tau}_i) + p(\overline{\tau}_{i-2})}{(2\Delta x)^2} p(\tau_i|\overline{\tau}_i)\Delta x ds$$
$$+ \frac{\varepsilon^2}{\sigma}\int_0^t \sum_{i\in\mathbb{Z}} \frac{d}{dt}\left(\frac{p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})}{2\Delta x}\right)(u_i - \overline{u}_i)\Delta x ds + \int_0^t \sum_{i\in\mathbb{Z}} (\mathcal{R}_i^\tau + \mathcal{R}_i^u)\Delta x ds$$

Sketch of the proof of the convergence rate

☐ Estimations of Lattanzio and Tzavaras

- $$\frac{1}{\sigma}\int_0^t \sum_{i\in\mathbb{Z}} \frac{p(\overline{\tau}_{i+2}) - 2p(\overline{\tau}_i) + p(\overline{\tau}_{i-2})}{(2\Delta x)^2}\, p(\tau_i|\overline{\tau}_i)\Delta x ds \leq \frac{C}{\sigma}\|D_{xx}p(\overline{\tau})\|_\infty \int_0^t \phi(s)ds$$

- $$\frac{\varepsilon^2}{\sigma}\int_0^t \sum_{i\in\mathbb{Z}} \frac{d}{dt}\left(\frac{p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})}{2\Delta x}\right)(u_i - \overline{u}_i)\Delta x ds$$

$$\leq \frac{\sigma}{2}\int_0^t \sum_{i\in\mathbb{Z}} (u_i - \overline{u}_i)^2 \Delta x ds + C\|D_{xt}p(\overline{\tau})\|_2 \varepsilon^4$$

☐ Estimation of the viscous terms

- $$\int_0^t \sum_{i\in\mathbb{Z}} \mathcal{R}_i^u \Delta x ds \leq \frac{\lambda\theta}{2}\int_0^t \sum_{i\in\mathbb{Z}} (u_i - \overline{u}_i)^2 \Delta x ds + C(\theta, \|D_{xx}\overline{u}\|_2)\varepsilon^4$$

- $$\int_0^t \sum_{i\in\mathbb{Z}} \mathcal{R}_i^\tau \Delta x ds \leq C(\|D_x\overline{\tau}\|_\infty, \|D_{xx}\overline{\tau}\|_\infty)\int_0^t \phi(s)ds$$

Theorem: Assume $\overline{\tau} \geq c > 0$ and

- $$\|D_{tx} p(\overline{\tau})\|_{L^2} := \left(\int_0^T \sum_{i\in\mathbb{Z}} \Delta x \left| \frac{d}{dt} \left(\frac{p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})}{2\Delta x} \right) \right|^2 ds \right)^{1/2} \leq K$$
- $$\|D_{xx} p(\overline{\tau})\|_{L^\infty} := \sup_{t\in[0,T]} \sup_{i\in\mathbb{Z}} \left| \frac{p(\overline{\tau}_{i+2}) - 2p(\overline{\tau}_i) + p(\overline{\tau}_{i-2})}{(2\Delta x)^2} \right| \leq K$$
- $$\|D_{xx} \overline{\tau}\|_{L^\infty} := \left(\int_0^T \sum_{i\in\mathbb{Z}} \Delta x \left(\frac{\overline{\tau}_{i+1} - 2\overline{\tau}_i + \overline{\tau}_{i-1}}{\Delta x^2} \right)^2 ds \right)^{1/2} \leq K$$
- $$\|D_x \overline{\tau}\|_{L^\infty} := \sup_{t\in[0,T]} \sup_{i\in\mathbb{Z}} \frac{|\overline{\tau}_{i+1} - \overline{\tau}_i|}{\Delta x} \leq K$$
- $$\|D_{xx} \overline{u}\|_{L^2} := \left(\int_0^T \sum_{i\in\mathbb{Z}} \Delta x \left(\frac{\overline{u}_{i+1} - 2\overline{u}_i + \overline{u}_{i-1}}{\Delta x^2} \right)^2 ds \right)^{1/2} \leq K$$

Then

$$\phi(t) \leq \phi(0) + C\varepsilon^4 + C \int_0^t \phi(s) ds$$

to get the convergence rate

$$\phi(t) \leq \left(\phi(0) + C\varepsilon^4\right) \mathrm{e}^{CT} \qquad \forall t \leq T$$

☐ Jin-Pareschi-Toscani scheme (98)

↪ Refomulation

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x p(\tau) = -\dfrac{1}{\varepsilon^2}\left(\sigma\, u + (1-\varepsilon^2)\partial_x p(\tau)\right) \end{cases}$$

↪ Splitting scheme

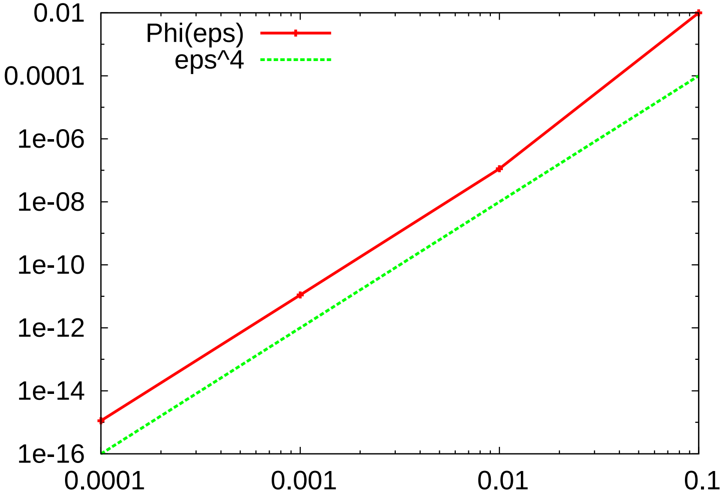
$$\begin{cases} \tau_i^{n+\frac12} = \tau_i^n - \dfrac{\Delta t}{\Delta x}\left(\mathcal{G}^{\tau}_{i+\frac12} - \mathcal{G}^{\tau}_{i-\frac12}\right) \\ u_i^{n+\frac12} = u_i^n - \dfrac{\Delta t}{\Delta x}\left(\mathcal{G}^{u}_{i+\frac12} - \mathcal{G}^{u}_{i-\frac12}\right) \end{cases}$$

$$\begin{cases} \tau_i^{n+1} = \tau_i^{n+\frac12} \\ \dfrac{u_i^{n+1} - u_i^{n+\frac12}}{\Delta t} = -\dfrac{1}{\varepsilon^2}\left(\sigma\, u_i^{n+1} + (1-\varepsilon^2)\dfrac{p(\tau^{n+1}_{i+\frac12}) - p(\tau^{n+1}_{i-\frac12})}{\Delta x}\right) \end{cases} \qquad \tau^{n+1}_{i+\frac12} = \frac{\left(\tau_i^{n+1} + \tau_{i+1}^{n+1}\right)}{2}$$

↪ Expected result

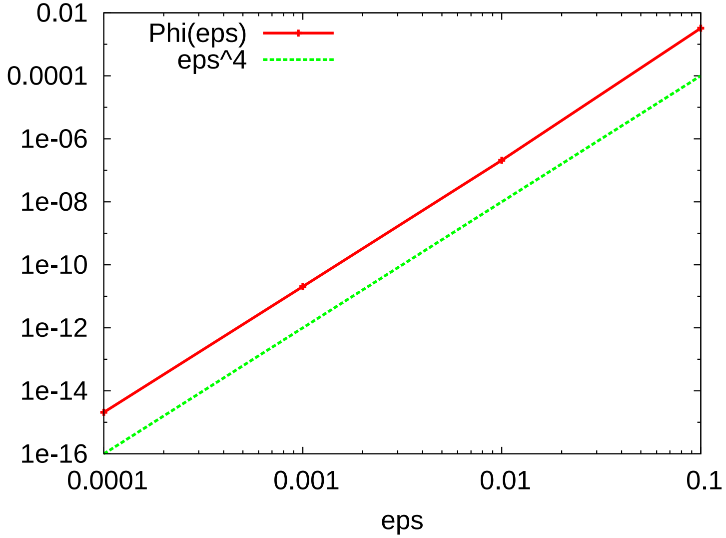
$$\phi^n = \sum_{i\in\mathbb{Z}} \eta(\tau_i^n, u_i^n | \overline{\tau}_i^n, \overline{u}_i^n)$$
$$\leq \left(\phi^0 + C\varepsilon^4\right) \mathrm{e}^{CN} \qquad \forall n \leq N$$

Numerical experiments



$$\tau_0(x) = \begin{cases} 2 & \text{si} \quad x < 0 \\ 1 & \text{si} \quad x > 0 \end{cases}$$

$$u_0(x) = \delta_0$$



$$\tau_0(x) = \exp(-100x^2) + 1$$

$$u_0(x) = \partial_x \tau_0(x)$$

Number of cells: 800

Final time of simulations: $T = 10^{-2}$

Thanks for your attention