Error estimate of a random particle blob method for the Keller-Segel equation

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Background of the Keller-Segel system

Keller-Segel system was proposed by Evelyn. F. Keller and Lee A. Segel, in 1970's, as

$$\begin{cases} \rho_t = \nu\Delta\rho - \mathrm{div}(\rho\nabla c), & x \in \mathbb{R}^n, t \geq 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^n, t \geq 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

where $\nu > 0$ and $0 \leq \rho_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

$\rho(t,x)$ represents the bacteria density and $c(t,x)$ represents the chemical substance concentration. The model is used to describe the collective motion of cells or the evolution of the density of bacteria.

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Here, we can solve $c = \Phi * \rho(t,x)$ with Newton potential $\Phi(x)$ and set the attractive force

$$F(x) = \nabla \Phi(x)$$

. Moreover we define the drift term

$$G(t,x) := \nabla c(t,x) = \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy$$

In addition, one has $-\Delta G(t,x) = \nabla \rho(t,x)$.

Setup of the problem

Assumption 1

1. $\rho_0(x)$ has a compact support D with $D \subseteq B(R_0)$;
2. $0 \leq \rho_0 \in H^k(\mathbb{R}^d)$ with $k \geq \frac{3d}{2} + 1$.

In fact, the above assumption is sufficient for the existence of the unique local solution to (1) with the following regularity

$$||\rho||_{L^\infty(0,T;H^k(\mathbb{R}^d))}, ||\partial_t \rho||_{L^\infty(0,T;H^{k-2}(\mathbb{R}^d))} \leq C(||\rho_0||_{H^k(\mathbb{R}^d)}) \tag{2}$$

$$||G||_{L^\infty\left(0,T;W^{k-\frac{d}{2},\infty}(\mathbb{R}^d)\right)}, ||\partial_t G||_{L^\infty\left(0,T;W^{k-\frac{d}{2}-2,\infty}(\mathbb{R}^d)\right)} \leq C(||\rho_0||_{H^k(\mathbb{R}^d)}) \tag{3}$$

Self-consistent SDE

The above regularity make sure that the following stochastic differential equation (SDE):

$$X(t) = X(0) + \int_0^t \int_{\mathbb{R}^d} F(X(s) - y)\rho(s, y)dyds + \sqrt{2\nu}B(t) \qquad (4)$$

has a unique strong solution $X(t)$, where $X(0) = \alpha \in D$ and $B(t)$ is a standard Brownian motion.

Relation between the SDE and the KS equation

Let $g(t,x;0,\alpha)$ is the fundamental solution (Green's function) of the following PDE:

$$\begin{cases} u_t = \nu \triangle u - \nabla \cdot (uG), \\ u(0,x) = \delta_\alpha(x), \ \alpha \in D. \end{cases} \quad (5)$$

Then $g(t,x;0,\alpha)$ is the transition probability density of the self-consistent stochastic process $X(t)$, i.e., $g(t,x;0,\alpha)$ is the density that a particle reached the position x at time t from position α at time 0. Moreover,

$$\rho(t,x) = \int_{\mathbb{R}^d} g(t,x;0,\alpha)\rho_0(\alpha)d\alpha. \quad (6)$$

is the solution to the KS equation with initial data $\rho_0(x)$

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We take h as a grid size and decompose the domain D into the union of non-overlapping cells $C_i = X_i(0) + [-\frac{h}{2}, \frac{h}{2}]^d$ with center $X_i(0) = hi := \alpha_i \in D$, i.e. $D \subset \bigcup_{i \in I} C_i$, where $I = \{i\} \subset \mathbb{Z}^d$ is the index set for cells. The total number of cells is given by $N = \sum_{i \in I} \approx \frac{|D|}{h^d}$.

Suppose $X_i(t)$ is the strong solution to the following SDE

$$X_i(t) = X_i(0) + \int_0^t G(s, X_i(s))ds + \sqrt{2\nu}B_i(t) \quad i \in I \qquad (7)$$

with the initial data $X_i(0) = \alpha_i = hi$ where $B_i(t)$ are independent standard Brownian motions.

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If N is large, then the empirical measure

$$\mu_N(t,x) := \sum_{j\in I} \delta(x - X_j(t))\rho_0(\alpha_j)h^d$$

should be an approximation to the density $\rho(t,x)$ in the following sense,

$$\int_{\mathbb{R}^d} \varphi(x)\rho(t,x)dx \approx \int_{\mathbb{R}^d} \varphi(x)\mu_N(t,x)dx = \sum_{j\in I} \varphi(X_j(t))\rho_0(\alpha_j)h^d$$

Actually, we prove that

$$||\mathbb{E}[\mu_N(t)] - \rho||_{H^{-(d+1)}(\mathbb{R}^d)} \leq Ch^{d+1}.$$

(J.-G. Liu and Y. Zhang. Convergence of diffusion-drift many particle systems in probability under sobolev norm, 2015.)

Stochastic system of the interacting particle system

In particular, if F were sufficiently regular, we could approximate $G(s, X_i(s))$ by

$$G(s, X_i(s)) \approx V(s, X_i(s)) = \int_{\mathbb{R}^d} F\big(X_i(s) - y\big)\mu_N(s, y)dy$$
$$= \sum_{j \in I} F\big(X_i(s) - X_j(s)\big)\rho_0(\alpha_j)h^d \tag{8}$$

Hence, we get the random particle method by replacing G by V in (7)

$$X_i(t) = X_i(0) + \int_0^t \sum_{j \in I} F\big(X_i(s) - X_j(s)\big)\rho_0(\alpha_j)h^d ds + \sqrt{2\nu}B_i(t) \quad i \in I$$

Random particle blob method

Introducing a blob function to mollify F, we have the random particle blob method for the KS equation

$$X_{i,\varepsilon}(t) = X_{i,\varepsilon}(0) + \int_0^t \sum_{j \in I} F_\varepsilon\big(X_{i,\varepsilon}(s) - X_{j,\varepsilon}(s)\big)\rho_j h^d ds + \sqrt{2\nu} B_i(t) \quad i \in I \tag{9}$$

with the initial data $X_{i,\varepsilon}(0) = \alpha_i = hi$, where

$$\rho_j = \rho_0(\alpha_j), \quad F_\varepsilon = F * \psi_\varepsilon, \quad \psi_\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x), \quad \varepsilon = h^{\frac{q}{2q-1}} \ (q > 1).$$

Main theorem

Suppose $\rho_0(x)$ satisfies Assumption 1, then there exists two positive constants C and C' such that

$$P\left(\max_{0\leq t\leq T_{\max}} ||X_{h,\varepsilon}(t)-X_h(t)||_{\ell_h^p} < \Lambda h|\ln h|\right) \geq 1-exp(-C\Lambda|\ln h|^2)$$

for any $\Lambda > C'$ and $p > \frac{d(2q-1)}{q-1}$.

- $T_{\max}$ be the largest existence time;
- $X_h(t) = (X_i(t))_{i\in I}$ is the exact path of (7);
- $X_{h,\varepsilon}(t) = (X_{i,\varepsilon}(t))_{i\in I}$ is the solution to the random particle blob method (9);
- Blob size $\varepsilon = h^{\frac{q}{2q-1}}$ $(q > 1)$.

Preliminaries on kernel, sampling, concentration and far field estimates

Notations

$$||v||_{\ell_h^p} = ||(v_i)_{i\in I}||_{\ell_h^p} = \left(\sum_{i\in I} |v_i|^p h^d\right)^{1/p} \quad p > 1;$$

$$G(t,x) := F * \rho = \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy;$$

$$G^h(t,x) := \sum_{j\in I} F_\varepsilon\big(x - X_j(t)\big)\rho_j h^d;$$

$$G_\varepsilon^h(t,x) := \sum_{j\in I} F_\varepsilon\big(x - X_{j,\varepsilon}(t)\big)\rho_j h^d.$$

Kernel estimates

Lemma

(i) $F_\varepsilon(0) = 0$ *and* $F_\varepsilon(x) = F(x)h(\frac{|x|}{\varepsilon})$ *for any* $x \neq 0$*, where* $h(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^r \psi(s)s^{d-1}ds$;

(ii) $F_\varepsilon(x) = F(x)$ *for any* $|x| \geq \varepsilon$ *and* $F_\varepsilon(x) \leq \min\{C\frac{|x|}{\varepsilon^d}, |F(x)|\}$;

(iii) $|\partial^\beta F_\varepsilon(x)| \leq C_\beta \varepsilon^{1-d-|\beta|}$, *for any* $x \in \mathbb{R}^d$;

(iv) $|\partial^\beta F_\varepsilon(x)| \leq C_\beta |x|^{1-d-|\beta|}$, *for any* $|x| \geq \varepsilon$;

(v) $||F_\varepsilon||_{W^{|\beta|,q}(\mathbb{R}^d)} \leq C_\beta \varepsilon^{d/q+1-d-|\beta|}$, *for* $q > 1$.

Sampling estimates

Lemma

Suppose that $f \in W^{d+1,1}(\mathbb{R}^d)$*, then*

$$\left| \sum_{i \in \mathbb{Z}^d} f(hi)h^d - \int_{\mathbb{R}^d} f(x)dx \right| \leq C_d h^{d+1} ||f||_{W^{d+1,1}(\mathbb{R}^d)}.$$

The proof of this lemma is based on the Poisson summation formula, which was given by Anderson and Greengard.

(C. Anderson and C. Greengard. On vortex methods. SIAM journal on numerical analysis, 22(3):413 – 440, 1985.)

Sampling estimates

Lemma

Let $X(t,\alpha)$ be the solution of the following SDE under the Assumption 1

$$X(t;\alpha) = X(0;\alpha) + \int_0^t G\left(s, X(s;\alpha)\right) ds + \sqrt{2\nu} B_\alpha(t)$$

with initial data $X(0;\alpha) = \alpha \in D$ and $B_\alpha(t)$ is the standard Brownian motion. Assume $\{X_i(t)\}$ are solutions of the SDEs

$$X_i(t) = X_i(0) + \int_0^t G\left(s, X_i(s)\right) ds + \sqrt{2\nu} B_i(t) \quad i \in I$$

with initial data $X_i(0) = \alpha_i = hi \in D$ and $\{B_i(t)\}$ are independent standard Brownian motions. For $\mathbb{R}^{d'}$ valued functions $f \in W^{d+1,q}(\mathbb{R}^d)$ and $g \in W_0^{d+1,q'}(\mathbb{R}^d)$ with supp $g = D$ and $1/q + 1/q' = 1$, we have the following estimate for the quadrature error

$$\max_{0\leq t\leq T} \left| \sum_{i\in I} \mathbb{E}\left[f\left(X_i(t)\right)\right] g(\alpha_i) h^d - \int_D \mathbb{E}\left[f\left(X(t;\alpha)\right)\right] g(\alpha) d\alpha \right|$$
$$\leq Ch^{d+1} ||f||_{W^{d+1,q}(\mathbb{R}^d)}$$

where C depends only on $d,\ d',\ T,\ ||\rho_0||_{H^k(\mathbb{R}^d)}$ and $||g||_{W_0^{d+1,q'}(\mathbb{R}^d)}$.

Concentration estimates

(Bennett's inequality)

Let $\{Y_i\}_{i=1}^n$ be independent bounded d-dimensional random vectors with mean zero and $|Y_i| \le M$. We define $Var(Y_i) = \mathbb{E}[|Y_i|^2] - |\mathbb{E}[Y_i]|^2$, and $\sum_i Var(Y_i) \le V$. Let $S = \sum_i Y_i$.

Then for all $\eta > 0$,

$$P(|S| \ge \eta) \le 2d \exp\left[-\frac{1}{2d}\eta^2 V^{-1} B(M\eta V^{-1})\right]$$

where $B(\lambda) = 2\lambda^{-2}[(1+\lambda)\ln(1+\lambda) - \lambda], \lambda > 0, \lim_{\lambda\to 0^+} B(\lambda) = 1$, $\lim_{\lambda\to+\infty} B(\lambda) = 0$ and $B(\lambda)$ is decreasing in $(0, +\infty)$.

Concentration estimates

Lemma

Let $\{Y_i\}_{i=1}^n$ *be n independent bounded d-dimensional random vectors satisfying*

(i) $\mathbb{E}[Y_i] = 0$ *and* $|Y_i| \leq M$ *for all* $i = 1, \cdots, n$;

(ii) $\sum_{i=1}^{n} Var(Y_i) \leq V$ *with* $Var(Y_i) = \mathbb{E}[|Y_i|^2]$.

If $M \leq C\frac{\sqrt{V}}{\eta}$ *with some positive constant C, then we have*

$$P\left(|\sum_{i=1}^{n} Y_i| \geq \eta\sqrt{V}\right) \leq \exp\left(-C'\eta^2\right)$$

for all $\eta > 0$, *where* C' *only depends on C and d.*

Far field estimates

Lemma

Assume that $X_i(t)$ *is the exact solution to* (7)*, for* $R > R_0$*, then we have*

$$P(|X_i(t)| \geq R) \leq \frac{C}{R^2}$$

where C *depends on* $d,\ T,\ R_0$ *and* $||\rho||_{H^k(\mathbb{R}^d)}$.

Consistency error at the fixed time

There exists two constants $C, C' > 0$ such that

$$P\left(\max_{i\in I}\left|G^h(t,X_i(t)) - G(t,X_i(t))\right| < \Lambda h|\ln h|\right) \geq 1 - exp(-C\Lambda|\ln h|^2)$$

for all $\Lambda > C'$, where $X_i(t)$ is the exact path of (7).

$$G(t,x) := F * \rho = \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy;$$

$$G^h(t,x) := \sum_{j\in I} F_\varepsilon\big(x - X_j(t)\big)\rho_j h^d.$$

Sketch of the proof

- Step 1: Decomposing

$$
\begin{aligned}
&|G^h(t,x) - G(t,x)| \\
&\leq \left| \sum_{j\in I} F_\varepsilon(x - X_j(t))\rho_j h^d - \sum_{j\in I} \mathbb{E}[F_\varepsilon(x - X_j(t))]\rho_j h^d \right| \\
&\quad + \left| \sum_{j\in I} \mathbb{E}[F_\varepsilon(x - X_j(t))]\rho_j h^d - \int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho(t,y)dy \right| \\
&\quad + \left| \int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho(t,y)dy - \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy \right| \\
&:= |e_s(t,x)| + |e_d(t,x)| + |e_m(t,x)|.
\end{aligned}
$$

Use the estimate of $||F_\varepsilon||_{W^{|\beta|,q}(\mathbb{R}^d)}$ and take $\varepsilon = h^{\frac{q}{2q-1}}$

- Step 2: $|e_m(t,x)| \le C_1\varepsilon^2$
- Step 3: $|e_d(t,x)| \le C_2 h^{d+1}\varepsilon^{d/q-2d}$
- Step 4: $P\left(|e_s(t,x)| \ge C_3 h|\ln h|\right) \le h^{CC_3|\ln h|}$
- Step 5: $P(|G^h(t,x) - G(t,x)| \ge C_4 h|\ln h|) \le h^{CC_4|\ln h|}$
- Step 6: For the lattice points $z_k = hk$ in ball $B(R)$ with $R = h^{-\gamma|\ln h|}$, we have

$$P\left(\max_k |G^h(t,z_k) - G(t,z_k)| \ge C_4' h|\ln h|\right) \le h^{CC_4'|\ln h|}$$

with some constant $C > 0$.

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- Step 7: For any fixed t, denote the event $U := \{X_i(t) \in B(R)\}$, then we know from far field estimate that $P(U^c) \le \frac{C}{R^2} = Ch^{2\gamma|\ln h|}$. Now, we do the estimate under event U, and suppose z_i is the closest lattice point to $X_i(t)$ with $|X_i(t) - z_i| \le h$. Hence, we have

$$P\left(\max_{i\in I}\left|G^h(t,X_i(t)) - G(t,X_i(t))\right| \ge C_5 h|\ln h|\right) \le h^{CC_5|\ln h|}.$$

- Step 8: Finally, we concludes the proof of this theorem by using $P(A^c) = 1 - P(A)$.

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There exists two constants $C, C' > 0$ such that

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for all $\Lambda > C'$, where $X_i(t)$ is the exact path of (7).

$$G(t,x) := F * \rho = \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy;$$

$$G^h(t,x) := \sum_{j\in I} F_\varepsilon\big(x - X_j(t)\big)\rho_j h^d.$$

Stability estimate

Stability condition:

$$\max_{0\le t\le T}\max_{i\in I}|X_{i,\varepsilon}(t)-X_i(t)|\le\varepsilon,$$

Then there exists two positive constants $C,\ C'$ such that

$$P\left(||G_\varepsilon^h(t,X_{h,\varepsilon}(t))-G^h(t,X_h(t))||_{\ell_h^p}<\Lambda||X_{h,\varepsilon}(t)-X_h(t)||_{\ell_h^p},\ \forall\ t\in[0,T]\right)$$
$$\ge 1-exp(-C\Lambda|\ln h|^2) \qquad (10)$$

for any $\Lambda > C'$.

$$G^h(t,x):=\sum_{j\in I}F_\varepsilon\big(x-X_j(t)\big)\rho_j h^d;$$

$$G_\varepsilon^h(t,x):=\sum_{j\in I}F_\varepsilon\big(x-X_{j,\varepsilon}(t)\big)\rho_j h^d.$$

Sketch of the proof

- Step 1: In order to prove (11), we divide $[0, T]$ into N' subintervals with length $\Delta t = h^r$ for some $r > 2$ and $t_n = nh^r$, $n = 0, \ldots, N'$. If we denote the following events

$$A_n := \left\{ ||G^h_\varepsilon(t, X_{h,\varepsilon}(t)) - G^h(t, X_h(t))||_{\ell^p_h} \geq \Lambda ||X_{h,\varepsilon}(t) - X_h(t)||_{\ell^p_h},\ \exists\ t \in [t_n, t_{n+1}] \right\},$$

$$\tilde{A} := \left\{ ||G^h_\varepsilon(t, X_{h,\varepsilon}(t)) - G^h(t, X_h(t))||_{\ell^p_h} \geq \Lambda ||X_{h,\varepsilon}(t) - X_h(t)||_{\ell^p_h},\ \exists\ t \in [0, T] \right\},$$

then, one has

$$P\left(\tilde{A}\right) = P\left(\bigcup_{n=0}^{N'-1} A_n\right).$$

So our main idea of this proof is to give the estimate of $P(A_n)$ first.

- Step 2: Decomposing

$$
\begin{aligned}
&G_\varepsilon^h(t, X_{i,\varepsilon}(t)) - G^h(t, X_i(t)) \\
&= \sum_{j \in I} \left[F_\varepsilon\big(X_{i,\varepsilon}(t) - X_{j,\varepsilon}(t)\big) - F_\varepsilon\big(X_i(t) - X_j(t)\big)\right] \rho_j h^d \\
&= \sum_{j \in I} \nabla F_\varepsilon\big(X_i(t_n) - X_j(t_n) + \xi_{ij}\big) \cdot \big(X_{i,\varepsilon}(t) - X_i(t) + X_j(t) - X_{j,\varepsilon}(t)\big) \rho_j h^d \\
&= \sum_{j \in I} \nabla F_\varepsilon\big(X_i(t_n) - X_j(t_n) + \xi_{ij}\big) \cdot \big(X_{i,\varepsilon}(t) - X_i(t)\big) \rho_j h^d \\
&\quad + \sum_{j \in I} \nabla F_\varepsilon\big(X_i(t_n) - X_j(t_n) + \xi_{ij}\big) \cdot \big(X_j(t) - X_{j,\varepsilon}(t)\big) \rho_j h^d \\
&:= \mathcal{I}_i + \mathcal{J}_i
\end{aligned}
$$

- Step 3:

$$P\left(||(\mathcal{I}_i)_{i\in I}||_{\ell_h^p} \geq C_1||X_{h,\varepsilon}(t)-X_h(t)||_{\ell_h^p},\ \exists\ t\in[t_n,t_{n+1}]\right) \leq h^{CC_1|\ln h|}$$

- Step 4:

$$P\left(||(\mathcal{J}_i)_{i\in I}||_{\ell_h^p} \geq C_2||(e_j\rho_j)_{j\in I}||_{\ell_h^p},\ \exists\ t\in[t_n,t_{n+1}]\right) \leq h^{CC_2|\ln h|}$$

where $e_j = X_j(t) - X_{j,\varepsilon}(t)$.

- Step 5:

$$P(A_n) \le h^{C\Lambda|\ln h|} \quad n = 0, \cdots, N' - 1$$

$$P\left(\tilde{A}\right) = P\left(\bigcup_{n=0}^{N'-1} A_n\right) \le h^{C\Lambda|\ln h|}$$

Finally, $P\left(\tilde{A}^c\right) = 1 - P\left(\tilde{A}\right)$ concludes our proof.

Stability estimate

Stability condition:

$$\max_{0\le t\le T}\max_{i\in I}|X_{i,\varepsilon}(t)-X_i(t)|\le\varepsilon,$$

Then there exists two positive constants $C,\ C'$ such that

$$P\left(||G_\varepsilon^h(t,X_{h,\varepsilon}(t))-G^h(t,X_h(t))||_{\ell_h^p}<\Lambda||X_{h,\varepsilon}(t)-X_h(t)||_{\ell_h^p},\ \forall\ t\in[0,T]\right)$$
$$\ge 1-exp(-C\Lambda|\ln h|^2) \qquad (11)$$

for any $\Lambda > C'$.

$$G^h(t,x):=\sum_{j\in I}F_\varepsilon\big(x-X_j(t)\big)\rho_j h^d;$$

$$G_\varepsilon^h(t,x):=\sum_{j\in I}F_\varepsilon\big(x-X_{j,\varepsilon}(t)\big)\rho_j h^d.$$

The proof of the main theorem

- We denote the following events

$$A_1^n : \left\{ \max_{i \in I} \left| G^h(t_n, X_i(t_n)) - G(t_n, X_i(t_n)) \right| < \Lambda_1 h |\ln h| \right\}$$

$$A_2 : \left\{ \max_n \max_{t_n \le t \le t_{n+1}} |X_i(t) - X_i(t_n)| < C(h^r + \nu^{1/2} h) \right\}$$

$$A_3 : \left\{ ||G_\varepsilon^h(t, X_{h,\varepsilon}(t)) - G^h(t, X_h(t))||_{\ell_h^p} < \Lambda_3 ||X_{h,\varepsilon}(t) - X_h(t)||_{\ell_h^p}, \ \forall \ t \in [0, T] \right\}$$

with that Λ_1, Λ_3 are bigger than a constant depending only on T, p, d, R_0 and $||\rho_0||_{H^k(\mathbb{R}^d)}$.

$$P((A_1^n)^c) \le h^{C\Lambda_1 |\ln h|}; \quad P(A_3^c) \le h^{C\Lambda_3 |\ln h|};$$

$$P(A_2^c) \le C' h^{\frac{r}{2}-1} \exp(-C'' h^{2-r}).$$

- Therefore

$$\max_{0\le t\le T} ||G^h(t,X_h(t)) - G(t,X_h(t))||_{\ell_h^p} < (C+\Lambda_1)h|\ln h| \quad (12)$$

under the event $\bigcap_{n=0}^{N'} A_1^n \cap A_2$.

- Denote $e(t) = (e_i)_{i\in I} = X_{h,\varepsilon}(t) - X_h(t)$.

$$||\frac{de}{dt}||_{\ell_h^p} < \Lambda_3||e(t)||_{\ell_h^p} + (C+\Lambda_1)h|\ln h| \quad (13)$$

under the event $\bigcap_{n=0}^{N'} A_1^n \cap A_2 \cap A_3$. It follows from (13) and the fact $\frac{d||e||_{\ell_h^p}}{dt} \le ||\frac{de}{dt}||_{\ell_h^p}$, by using Gronwall's inequality with $e(0)=0$ that

$$\max_{0\le t\le T} ||e(t)||_{\ell_h^p} < C(T,\Lambda_1,\Lambda_3)h|\ln h| = \Lambda h|\ln h|$$

- Therefore

$$\max_{0\le t\le T} ||G^h(t,X_h(t)) - G(t,X_h(t))||_{\ell_h^p} < (C+\Lambda_1)h|\ln h| \quad (12)$$

under the event $\bigcap_{n=0}^{N'} A_1^n \cap A_2$.

- Denote $e(t) = (e_i)_{i\in I} = X_{h,\varepsilon}(t) - X_h(t)$.

$$||\frac{de}{dt}||_{\ell_h^p} < \Lambda_3||e(t)||_{\ell_h^p} + (C+\Lambda_1)h|\ln h| \quad (13)$$

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$$\max_{0\le t\le T} ||e(t)||_{\ell_h^p} < C(T,\Lambda_1,\Lambda_3)h|\ln h| = \Lambda h|\ln h|$$

- To complete the proof, we need to justify the stability condition: $|X_{i,\varepsilon}(t) - X_i(t)| \leq \varepsilon$ for all i and $0 \leq t \leq T$

$$\max_{i \in I} |e_i(t)| \leq h^{-d/p} ||e(t)||_{\ell_h^p} < Ch^{1-d/p} |\ln h| < \frac{\varepsilon}{2} \quad \text{for } 0 \leq t \leq T$$

by choosing $p > \frac{d(2q-1)}{q-1}$, $\varepsilon = h^{\frac{q}{2q-1}}$ with $q > 1$, and h small enough. Hence, $\max_{i \in I} |e_i|$ can hardly reach ε.

- From the discussion above, we have

$$
\begin{aligned}
&P\left(\max_{0\le t\le T}||X_{h,\varepsilon}(t)-X_h(t)||_{L_h^p}\ge \Lambda h|\ln h|\right)\\
&\le P\left(\bigcup_{n=0}^{N'-1}(A_1^n)^c\cup A_2^c\cup A_3^c\right)\le \sum_{n=0}^{N'-1}P((A_1^n)^c)+P(A_2^c)+P(A_3^c)\\
&\le Ch^{-r}h^{C\Lambda_1|\ln h|}+C'h^{r/2-1}\exp(-C''h^{2-r})+h^{C\Lambda_3|\ln h|}\le h^{C\Lambda|\ln h|}
\end{aligned}
$$

Thus the proof has been completed.

Main theorem

Suppose $\rho_0(x)$ satisfies Assumption 1, then there exists two positive constants C and C' such that

$$P\left(\max_{0\leq t\leq T_{\max}} ||X_{h,\varepsilon}(t) - X_h(t)||_{\ell_h^p} < \Lambda h|\ln h|\right) \geq 1 - exp(-C\Lambda|\ln h|^2)$$

for any $\Lambda > C'$ and $p > \frac{d(2q-1)}{q-1}$.

- $T_{\max}$ be the largest existence time;
- $X_h(t) = (X_i(t))_{i\in I}$ is the exact path of (7);
- $X_{h,\varepsilon}(t) = (X_{i,\varepsilon}(t))_{i\in I}$ is the solution to the random particle blob method (9);
- Blob size $\varepsilon = h^{\frac{q}{2q-1}}$ $(q > 1)$.

Thanks for your attention